Introduction to super potential (informal evening talk)

Michel Van den Bergh

Boundary conditions for open strings (branes) form a triangulated category. In the B-model, this triangulated category is the derived category $\mathcal{A}$ of coherent sheaves over a Calabi-Yau manifold [6].

It is often useful to consider triangulated subcategories $\mathcal{B} \subset \mathcal{A}$ which are derived equivalent to $D^b(\text{f.l.}A)$ where A is the (completed) path algebra of a quiver with relations. These are the so-called quiver gauge theories (see e.g. [4]).

A standard example is given by the derived category of the canonical bundle on $\mathbb{P}^2$. This is a non-compact Calabi-Yau. The derived category of sheaves supported on the zero section is equivalent to $D^b(k[[x,y,z]] * (\mathbb{Z}/3\mathbb{Z}))$ (see [5]).

It seems therefore interesting to be able to construct A such that $D^b(\text{f.l.}A)$ is Calabi-Yau. Physicists have a construction of such A in terms of so-called super potentials. It is not clear exactly when this construction works, but if it works, then the resulting algebra is Calabi-Yau of dimension 3.

For notational simplicity ,we will explain the construction in the case that A has only one simple module. The general case is entirely similar.

Put $F = k\langle\langle x_1, \dots, x_n \rangle\rangle$. For a general monomial $a \in F$, we define the *circular derivative* of a with respect to x_i as

$$\frac{{}^\circ\partial a}{\partial x_i} = \sum_{a = u x_i v} vu.$$

The circular derivative extends to a linear map

$$\frac{{}^\circ\partial}{\partial x_i} : F/[F,F] \to F.$$

The ordinary partial derivative of a with respect to x_i is defined as

$$\frac{\partial a}{\partial x_i} = \sum_{a = u x_i v} u \otimes v.$$

This extends to a linear map

$$\frac{\partial}{\partial x_i} : F \to F \otimes F.$$

It is convenient to write

$$\frac{\partial^2 a}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \frac{{}^\circ\partial a}{\partial x_i},$$

and it is easy to check that

$$\frac{\partial^2 a}{\partial x_i \partial x_j} = \tau \frac{\partial^2 a}{\partial x_j \partial x_i},$$

where $\tau(p \otimes q) = q \otimes p$.

A *super potential* is an element $w \in F/[F,F]$ containing only monomials of degree ≥ 3. Put

$$A = F/I,$$

where I is the twosided ideal topologically generated by

$$r_i = \frac{{}^{\circ}\partial w}{\partial x_i}.$$

Put $dx_i = x_i \otimes 1 - 1 \otimes x_i$. We consider the following complex of A-bimodules.

$$0 \to A^e \xrightarrow{(\cdot dx_i \cdot)_i} (A^e)^n \xrightarrow{\left(\cdot \frac{\partial^2 w}{\partial x_j \partial x_i} \cdot\right)_{ij}} (A^e)^n \xrightarrow{(\cdot dx_j \cdot)_j} A^e \to A \to 0$$

Here A^e is $A \hat{\otimes} A$ equipped with its outer bimodule structure. If this complex is exact then it represents a resolution of A as an A-bimodule.

Furthermore, from the fact that the resolution is self-dual, one may deduce, using standard homological algebra, that $D^b(\text{f.l.}A)$ is indeed Calabi-Yau.

Remark 1. *I haven't checked the details, but it seems not unlikely that the above construction is reversible and that a* 3*-dimensional Calabi-Yau algebra* A *is always given by a super potential.*

Remark 2. *Super potentials form an (infinite dimensional) affine space. This is reminiscent of the smoothness of the moduli-spaces of compact Calabi-Yau manifolds (Tian, Bogomolov, Ran, and Kawamata). See the talk by Ragnar Buchweitz during this evening seminar.*

Unfortunately, the construction does not always work (take the zero super potential). For a generic super potential, one would expect that the construction works if there are enough variables (or arrows in the quiver case), but this is entirely speculative. The following non-example was communicated to me by Berenstein.

$$\begin{tikzcd} & \bullet \arrow[dr, "y"] & \\ \bullet \arrow[ur, "x"] & & \bullet \arrow[ll, "z"] \end{tikzcd} \qquad\qquad w = xyz$$

In this case, it is easy to see that the resulting algebra is self-injective, but not Calabi-Yau.

Cases that are completely understood are, when there are either three or two variables and the degree of w is 3 or 4 respectively. This follows from the classification of 3-dimensional Artin-Schelter regular algebras [1, 2, 3].

References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension 3*, Adv. in Math. **66** (1987), 171–216.

[2] M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. 1, Birkhäuser (1990), pp. 33–85.

[3] M. Artin, J. Tate, and M. Van den Bergh, *Modules over regular algebras of dimension 3*, Invent. Math. **106** (1991), 335–388.

[4] D. Berenstein and M. Douglas, *Seiberg duality for quiver gauge theories*, available as `arXiv:hep-th/0207027`.
[5] T. Bridgeland, *T-structures on some local Calabi-Yau varieties*, available as `arXiv:math.AG/0502050`.
[6] A. Kapustin and D. Orlov, *Lectures on mirror symmetry, derived categories, and D-branes*, available as `arXiv:math.AG/0308173`.