On the finitistic dimension conjecture

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1. Introduction

Given an artin algebra A, the finitistic dimension of A is defined to be the supremum of the projective dimensions of the finitely generated left A-modules of finite projective dimension. The famous **finitistic dimension conjecture** says that for any artin algebra A the finitistic dimension of A is finite. This conjecture was proposed 45 years ago and still remains open, and has been related to at least five other homological conjectures (see the last 6 conjectures of the total 13 conjectures in the book [4]):

Strong Nakayama conjecture [7]: If M is a non-zero module over an artin algebra A, then there is an integer $n \geq 0$ such that $\mathrm{Ext}^n_A(M, A) \neq 0$.

Generalized Nakayama conjecture [2]: If $0 \to {}_AA \to I_0 \to I_1 \to \dots$ is a minimal injective resolution of an artin algebra A, then any indecomposable injective is a direct summand of some I_j. Equivalently, if M is a finitely generated A-module such that $\mathrm{add}(A) \subseteq \mathrm{add}(M)$ and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i \geq 1$, then M is projective.

Nakayama conjecture [15]: If all I_j in a minimal injective resolution of an artin algebra A, say $0 \to {}_AA \to I_0 \to I_1 \to \dots$, are projective, then A is self-injective.

Gorenstein symmetry conjecture: Let A be an artin algebra. If the injective dimension of ${}_AA$ is finite, then the injective dimension of A_A is finite.

In general, all the above conjectures are still open. They have the following well-known relationship: The finitistic dimension conjecture $\Longrightarrow$ the strong Nakayama conjecture $\Longrightarrow$ the generalized Nakayama conjecture $\Longrightarrow$ the Nakayama conjecture. And, the finitistic dimension conjecture $\Longrightarrow$ the Gorenstein symmetry conjecture.

In this talk I shall report on some new developments attacking the finitistic dimension conjecture. Our idea to approach the conjecture is to use a chain of subalgebras with certain radical conditions. Let us introduce the following notion:

Definition. (1) Given an artin algebra A, we say that the *left representation distance* of A, denoted by $\mathrm{lrep.dis}(A)$, is the minimum of the lengths of chains of subalgebras $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_s$ such that $\mathrm{rad}(A_i)$ is a left ideal in A_{i+1} for all i and that A_s is representation-finite. Here we have denoted the Jacobson radical of A by $\mathrm{rad}(A)$.

(2) A homomorphism $f : B \longrightarrow A$ between two algebras A and B is said to be *radical-full* if $\mathrm{rad}(A) = \mathrm{rad}({}_BA)$.

Note that the left representation distance of an artin algebra is always finite by [19] and invariant under Morita equivalences. Every surjective homomorphism is radical-full. Note that if B is a subalgebra of an artin algebra A, the inclusion map being radical-full does not imply that $\mathrm{rad}(B)$ is a left ideal in A.

2. Main results

In this section we shall summarize some new results in the recent papers [19, 20]. For some known results on finitistic dimension conjecture we refer to [3, 9, 10, 8, 11, 16] and many other papers. (I apologize that I could not display all literature here.)

Our main results are the following.

Theorem 1. *Let A be an artin algebra.*

(1) *If* $\mathrm{lrep.dis}(A) \leq 2$*, then the finitistic dimension conjecture is true for A.*
(2) *Let B be a subalgebra of A such that* $\mathrm{rad}(B)$ *is a left ideal of A and that the inclusion map is radical-full. If the global dimension of A is at most 4, then the finitistic dimension conjecture is true for B.*

We may also use a chain of factor algebras to bound the finitistic dimension. In this direction, we have the following result.

Theorem 2. *Let A be an artin algebra, and let I and J be two ideals in A with* $IJ\,\mathrm{rad}(A) = 0$*. If A/I and A/J are representation-finite, then the finitistic dimension conjecture is true for A.*

The proofs of Theorem 1 and Theorem 2 are based on the following lemmas.

Lemma 1. *Suppose B is a subalgebra of A such that* $\mathrm{rad}(B)$ *is a left ideal in A. Then, for any B-module X and integer $i \geq 2$, there is a projective A-module Q and an A-module Z such that $\Omega_B^i(X) \simeq \Omega_A(Z) \oplus Q$ as A-modules, where Ω_B stands for the first syzygy over the algebra B.*

Lemma 2 ([11])**.** *For any artin algebra A there is a function Ψ from the finitely generated A-modules to the non-negative integers such that*

(1) $\Psi(M) = \mathrm{proj.dim}(M)$ *if M has finite projective dimension.*
(2) *For any natural number n, $\Psi(\bigoplus_{j=1}^n M) = \Psi(M)$.*
(3) *For any A-modules X and Y, $\Psi(X) \leq \Psi(X \oplus Y)$.*
(4) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in A-mod such that the projective dimension of Z is finite, then $\Psi(Z) \leq \Psi(X \oplus Y) + 1$.*

Based on the above results, there are many elementary questions, for example, if the left representation distance of A is 3, could we prove the finitistic dimension conjecture for A? For more information and the details of the proofs of the above main results we refer to the papers [19, 20]. Preprints can be downloaded from http:/math.bnu.edu.cn/∼ccxi/.

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