Rigid Cohen-Macaulay modules over a three dimensional Gorenstein ring

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1. Main theorem

Let k be an algebraically closed field of characteristic zero, and let $S = k[[x, y, z]]$ be a formal power series ring in three variables x, y and z. The cyclic group $G = \mathbb{Z}/3\mathbb{Z}$ of order 3 acts linearly on S in such a way that

$$x^\sigma = \zeta x, \quad y^\sigma = \zeta y, \quad z^\sigma = \zeta z,$$

where σ is a generator of G and $\zeta \in k$ is a primitive cubic root of unity. We denote by R the invariant subring of S by this action of G. It is easy to see that

$$R = k[[\{\text{monomials of degree three in } x, y, z\}]],$$

which is often called (the completion of) the Veronese subring of degree three. It is known and is easy to prove that R is a Gorenstein complete local normal domain that has an isolated singularity.

The action of G gives a G-graded structure on S such as

$$S = S_0 \oplus S_1 \oplus S_2,$$

where each S_j is the R-module of semi-invariants that is defined as

$$S_j = \{f \in S \mid f^\sigma = \zeta^j f\}.$$

Note that $S_0 = R$. It is known that S_j $(0 \leqq j \leqq 2)$ are maximal Cohen-Macaulay modules over R, and in particular they are reflexive R-modules of rank one, whose classes form the divisor class group of R;

$$\mathrm{Cl}(R) = \{[S_0], [S_1], [S_2]\}.$$

In particular, any maximal Cohen-Macaulay module of rank one over R is isomorphic to one of S_j $(0 \leqq j \leqq 2)$.

It is not difficult to see that the category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay modules over R is of wild representation type. Actually, one can construct a family of nonisomorphic classes of indecomposable maximal Cohen-Macaulay modules over R in relation with the representations of the following quiver.

$$Q = \left(\bullet \quad \begin{array}{c}\longrightarrow\\[-6pt]\longrightarrow\\[-6pt]\longrightarrow\end{array} \quad \bullet \right)$$

In this talk I am interested in rigid maximal Cohen-Macaulay modules that are defined as follows:

Definition. An R-module M is called *rigid* if $\mathrm{Ext}^1_R(M, M) = 0$. And we denote the full subcategory of $\mathrm{mod}\, R$ consisting of all rigid maximal Cohen-Macaulay modules by $\mathcal{C}$.

By computation, the modules S_j $(0 \leqq j \leqq 2)$ and any of their syzygies and any of their cosyzygies are rigid (and indecomposable). Our main theorem is the following:

Theorem. *Let $\mathcal{S}$ be a sequence of indecomposable rigid maximal Cohen-Macaulay modules defined as follows:*

$$\mathcal{S} = (\cdots, \Omega^{-2}S_1,\ \Omega^{-2}S_2,\ \Omega^{-1}S_1,\ \Omega^{-1}S_2,\ S_1,\ S_2,\ \Omega^{1}S_1,\ \Omega^{1}S_2,\ \Omega^{2}S_1,\ \Omega^{2}S_2, \cdots).$$

Then any object in $\mathcal{C}$ is isomorphic to a module of the following form:

$$P^a \oplus Q^b \oplus R^c,$$

where a, b, c are nonnegative integers and $\{P, Q\}$ is a pair of two adjacent modules in the sequence $\mathcal{S}$.

2. Outline of Proof

The proof of the theorem is divided into the following four steps.

2.1. **First Step (Approximation).**

Let $\mathcal{E}$ be the full subcategory of $\operatorname{mod} R$ consisting of modules M which can be embedded in an exact sequence of the following type:

$$(*) \qquad 0 \longrightarrow S_1^n \longrightarrow S_2^m \oplus R^\ell \longrightarrow M \longrightarrow 0$$

If $M \in \mathcal{E}$, then the sequence $(*)$ gives a right $\operatorname{add}_R S$-approximation of M that is, of course, right minimal.

Claim 1. *Let M be an indecomposable object in $\mathcal{C}$. Suppose that M is isomorphic neither to S_1 nor $\Omega^{-1}S_2$. Then M belongs to $\mathcal{E}$.*

The claim means that $\operatorname{Ind}(\mathcal{C}) = \operatorname{Ind}(\mathcal{C} \cap \mathcal{E}) \cup \{S_1, \Omega^{-1}S_2\}$.

2.2. **Second Step (Rigidity).**

Claim 2. *Let M and M' be objects in $\mathcal{C} \cap \mathcal{E}$. Suppose there are exact sequences:*

$$0 \longrightarrow S_1^n \xrightarrow{\ f\ } S_2^m \oplus R^\ell \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow S_1^{n'} \xrightarrow{\ f'\ } S_2^{m'} \oplus R^{\ell'} \longrightarrow M' \longrightarrow 0.$$

If $n = n'$ and $m = m'$, then M and M' are stably isomorphic to each other.

2.3. **Third Step (Tate-Vogel cohomology).**

The Tate-Vogel cohomology for maximal Cohen-Macaulay modules is defined as follows:

$$\check{\operatorname{Ext}}_R^i(M, N) = \underline{\operatorname{Hom}}_R(\Omega^i M, N),$$

for any $i \in \mathbb{Z}$ and $M, N \in \operatorname{CM}(R)$. We define

$$e_j^i(M) = \dim_k\ \check{\operatorname{Ext}}_R^i(S_j, M)$$

for any $i \in \mathbb{Z}$, $j \in G$ and $M \in \operatorname{CM}(R)$.

Claim 3. *Let M be an object in $\mathcal{C} \cap \mathcal{E}$, and suppose there is an exact sequence:*

$$0 \longrightarrow S_1^n \longrightarrow S_2^m \oplus R^\ell \longrightarrow M \longrightarrow 0.$$

Then $n = e_1^1(M)$ and $m = e_2^0(M)$.

Now we define a mapping e from the isomorphism classes of modules in $\mathcal{C} \cap \mathcal{E}$ to nonnegative integral vectors $\mathbb{Z}^2_{\geqq 0}$ by

$$e(M) = (e_1^1(M),\ e_2^0(M)).$$

Note that it follows from Claim 2 that the mapping

$$e : \mathcal{C} \cap \mathcal{E}/\cong \ \longrightarrow \ \mathbb{Z}^2_{\geqq 0}$$

is an injection. Hence, to classify the objects in $\mathcal{C} \cap \mathcal{E}$, it is enough to determine the image of the mapping e.

Remark. *Note that the Auslander-Reiten-Serre duality says that*

$$\mathrm{Ext}_R^3(\check{\mathrm{E}}\mathrm{xt}_R^i(M, N), R) \cong \check{\mathrm{E}}\mathrm{xt}_R^{2-i}(N, M),$$

for any $i \in \mathbb{Z}$ and $M, N \in \mathrm{CM}(R)$. Therefore, the triangulated category $\underline{\mathrm{CM}}(R)$ is 2-Calabi-Yau.

2.4. **Fourth Step (Root system).**

Let H be the set of nonnegative integral vectors (x, y) with $x^2 - 3xy + y^2 \geqq 1$:

$$H = \{(x, y) \in \mathbb{Z}^2_{\geqq 0} \mid x^2 - 3xy + y^2 \geqq 1\}$$

It is easy to see that $H = H_+ \cup H_-$ where

$$H_+ = \{(x, y) \in \mathbb{Z}^2_{\geqq 0} \mid 2x - (3+\sqrt{5})y \geqq 0\}, \quad H_- = \{(x, y) \in \mathbb{Z}^2_{\geqq 0} \mid 2x - (3-\sqrt{5})y \leqq 0\},$$

each of which is a semigroup. We can prove the following claim.

Claim 4. *The image of the mapping $e : \mathcal{C} \cap \mathcal{E} \to \mathbb{Z}^2_{\geqq 0}$ is exactly H.*

The main theorem follows from this claim with a little observation.

References

[1] Y. Yoshino, *Rigid Cohen-Macaulay modules over a three dimensional Gorenstein ring*, Preprint (in preparation), 2005.