Quantum cluster algebras

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Cluster algebras were introduced and studied by S. Fomin and A. Zelevinsky in [3, 5, 1]. This is a family of commutative rings designed to serve as an algebraic framework for the theory of total positivity and canonical bases in semisimple groups and their quantum analogs. Here we report on a joint work with A. Berenstein [2], where we introduce and study quantum deformations of cluster algebras.

We start by recalling the definition of cluster algebras (of geometric type). Let m and n be two positive integers with $m \geq n$. Let $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in m independent (commuting) variables.

Definition 1. A *seed* in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}}, \tilde{B})$, where

- $\tilde{\mathbf{x}} = \{x_1, \ldots, x_m\}$ is a free (i.e., algebraically independent) generating set for $\mathcal{F}$.
- $\tilde{B}$ is an $m \times n$ integer matrix with rows labeled by $[1, m] = \{1, \ldots, m\}$ and columns labeled by an n-element subset $\mathbf{ex} \subset [1, m]$, such that, for some positive integers d_j $(j \in \mathbf{ex})$, we have $d_i b_{ij} = -d_j b_{ji}$ for all $i, j \in \mathbf{ex}$.

The subset $\mathbf{x} = \{x_j : j \in \mathbf{ex}\} \subset \tilde{\mathbf{x}}$ (resp. $\mathbf{c} = \tilde{\mathbf{x}} - \mathbf{x}$) is called the *cluster* (resp. the *coefficient set*) of a seed $(\tilde{\mathbf{x}}, \tilde{B})$. The seeds are defined up to a relabeling of elements of $\tilde{\mathbf{x}}$ together with the corresponding relabeling of rows and columns of $\tilde{B}$.

Definition 2. Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed in $\mathcal{F}$. For any $k \in \mathbf{ex}$, the *seed mutation* in direction k transforms $(\tilde{\mathbf{x}}, \tilde{B})$ into a seed $(\tilde{\mathbf{x}}', \tilde{B}')$ given by:

- $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}} - \{x_k\} \cup \{x'_k\}$, where $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_k = x_k^{-1} \Big(\prod_{\substack{i \in [1,m] \\ b_{ik} > 0}} x_i^{b_{ik}} + \prod_{\substack{i \in [1,m] \\ b_{ik} < 0}} x_i^{-b_{ik}} \Big) . \tag{1}$$

- The entries of $\tilde{B}'$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \dfrac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{2}$$

The seed mutations generate an equivalence relation: we say that two seeds $(\tilde{\mathbf{x}}, \tilde{B})$ and $(\tilde{\mathbf{x}}', \tilde{B}')$ are *mutation-equivalent* if $(\tilde{\mathbf{x}}', \tilde{B}')$ can be obtained from $(\tilde{\mathbf{x}}, \tilde{B})$ by a sequence of seed mutations.

Fix a mutation-equivalence class $\mathcal{S}$ of seeds. Let $\mathcal{X} \subset \mathcal{F}$ denote the union of clusters, and $\mathbf{c}$ the common coefficient set of all seeds from $\mathcal{S}$. The *cluster algebra* $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ is the $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$.

We now define a family of q-deformations of $\mathcal{A}(\mathcal{S})$. The following setup is a simplified version of that in [2]. The main idea is to deform each extended cluster $\tilde{\mathbf{x}}$ to a quasi-commuting family $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ satisfying

$$X_i X_j = q^{\lambda_{ij}} X_j X_i \tag{3}$$

for some skew-symmetric integer $m \times m$ matrix $\Lambda = (\lambda_{ij})$. Let $\mathcal{F}_q$ denote the skew-field of fractions of the ring $\mathbb{Z}[q^{\pm 1/2}, X_1, \ldots, X_m]$, where $X_1, \ldots, X_m$ are algebraically independent variables satisfying (3). For any $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, we set

$$X^a = q^{\frac{1}{2}\sum_{i>j} \lambda_{ij} a_i a_j} X_1^{a_1} \cdots X_m^{a_m} \ . \tag{4}$$

Definition 3. A *free generating set* for $\mathcal{F}_q$ is a subset $\{Y_1, \ldots, Y_m\} \subset \mathcal{F}_q$ of the following form: $Y_j = \varphi(X^{c_j})$, where φ is a $\mathbb{Q}(q^{\pm 1/2})$-linear automorphism of $\mathcal{F}_q$, and $\{c_1, \ldots, c_m\}$ is a basis of the lattice $\mathbb{Z}^m$.

Note that the subset $\{Y_1, \ldots, Y_m\}$ can be used instead of $\{X_1, \ldots, X_m\}$ in the definition of the ambient field $\mathcal{F}_q$, with the matrix Λ replaced by $C^T \Lambda C$, where C is the matrix with columns $c_1, \ldots, c_m$.

Definition 4. A *quantum seed* in $\mathcal{F}_q$ is a pair $(\tilde{\mathbf{X}}, \tilde{B})$, where

- $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ is a free generating set for $\mathcal{F}_q$.
- $\tilde{B}$ is a $m \times n$ integer matrix with rows labeled by $[1, m]$ and columns labeled by an n-element subset $\mathbf{ex} \subset [1, m]$, which is *compatible* with the matrix Λ given by (3), in the following sense: for some positive integers d_j $(j \in \mathbf{ex})$, we have

$$\sum_{k=1}^{m} b_{kj} \lambda_{ki} = \delta_{ij} d_j \quad (j \in \mathbf{ex},\ i \in [1, m]) \ . \tag{5}$$

As in Definition 1, the quantum seeds are defined up to a relabeling of elements of $\tilde{\mathbf{X}}$ together with the corresponding relabeling of rows and columns of $\tilde{B}$.

Note that (5) implies that $d_i b_{ij} = -d_j b_{ji}$ for all $i, j \in \mathbf{ex}$, i.e., $\tilde{B}$ is as in Definition 1.

Example. *Let $m = 2n$, $\mathbf{ex} = [1, n]$, and let $\tilde{B}$ be of the form*

$$\tilde{B} = \begin{pmatrix} B \\ I \end{pmatrix} ,$$

where I is the identity $n \times n$ matrix. Here B is an arbitrary integer $n \times n$ matrix satisfying $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$: in other words, B is skew-symmetrizable*, that is, DB is skew-symmetric, where D is the diagonal matrix with diagonal entries $d_1, \ldots, d_n$. An easy calculation shows that the skew-symmetric matrices Λ compatible with $\tilde{B}$ in the sense of* (5) *are those of the form*

$$\Lambda = \begin{pmatrix} \Lambda_0 & -D - \Lambda_0 B \\ D - B^T \Lambda_0 & -DB + B^T \Lambda_0 B \end{pmatrix} , \tag{6}$$

where Λ_0 is an arbitrary skew-symmetric integer $n \times n$ matrix.

Definition 5. Let $(\tilde{\mathbf{X}}, \tilde{B})$ be a quantum seed in $\mathcal{F}_q$. For any $k \in \mathbf{ex}$, the *quantum seed mutation* in direction k transforms $(\tilde{\mathbf{X}}, \tilde{B})$ into a quantum seed $(\tilde{\mathbf{X}}', \tilde{B}')$ given by:

- $\tilde{\mathbf{X}}' = \tilde{\mathbf{X}} - \{X_k\} \cup \{X'_k\}$, where $X'_k \in \mathcal{F}_q$ is given by

$$X'_k = X^{-e_k + \sum_{b_{ik}>0} b_{ik} e_i} + X^{-e_k - \sum_{b_{ik}<0} b_{ik} e_i}, \tag{7}$$

where the terms on the right are defined via (4), and $\{e_1, \ldots, e_m\}$ is the standard basis in $\mathbb{Z}^m$.
- The matrix entries of $\tilde{B}'$ are given by (2).

The fact that $(\tilde{\mathbf{X}}', \tilde{B}')$ is a quantum seed is not automatic: for the proof see [2, Proposition 4.7].

Based on definitions (4) and (5), one defines the *quantum cluster algebra* associated with a mutation-equivalence class of quantum seeds, in exactly the same way as the ordinary cluster algebra. It is shown in [2] that practically all the structural results on cluster algebras obtained in [3, 5, 1] extend to the quantum setting. This includes the Laurent phenomenon obtained in [3, 4, 1] and the classification of cluster algebras of finite type given in [5].

References

[1] A. Berenstein, S. Fomin and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.

[2] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, to appear in Adv. in Math.

[3] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.

[4] S. Fomin and A. Zelevinsky, *The Laurent phenomenon*, Adv. in Applied Math. **28** (2002), 119–144.

[5] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121.