# SOME EXAMPLES OF BOUNDED RESOLUTIONS OVER RINGS WITH RADICAL CUBE (OR TO THE FOURTH POWER) ZERO

**Some Questions and Some Answers.** Compiled by D. Jorgensen for the Seminar Darstellungstheorie, Bielefeld, Juli 2009.

**Question 1.** (Eisenbud ∼1976) Assume that $R$ is commutative Noetherian local,

$$F: \quad \cdots \to R^n \to R^n \to \cdots \to R^n \to 0$$

a minimal free resolution with constant Betti numbers. Then is $F$ periodic of period $\leq 2$?

**Answer:** NO, due to Gasharov–Peeva [1] (∼1990). Examples 1 and 1′ are their original examples. See also Examples 2 and 2′ below.

**Example 1.** Choose $0 \neq \alpha \in k$ ($k$ a field) and let

$$B = B_\alpha = k[X_1, X_2, X_3, X_4]/I_\alpha$$

where $I_\alpha$ is generated by the 7 homogeneous quadratics

$$\alpha X_1X_3 + X_2X_3,\ X_1X_4 + X_2X_4,\ X_3^2,\ X_4^2,\ X_1^2,\ X_2^2,\ X_3X_4$$

Then $B_\alpha$ has Hilbert polynomial $1 + 4t + 3t^2$ and

$$\cdots \to B^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^i x_3 + x_4 \\ 0 & x_2 \end{pmatrix}} B^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^{i-1} x_3 + x_4 \\ 0 & x_2 \end{pmatrix}} B^2 \to \cdots$$

is a resolution with constant Betti numbers 2, and of period $\circ(\alpha)$ if $\circ(\alpha) < \infty$, and aperiodic otherwise.

A $k$-basis for $B_2$ is given by $x_1x_2,\ x_1x_3,\ x_1x_4$.

**Example 1′.** Choose $0 \neq \alpha \in k$ ($k$ a field) and let

$$A = A_\alpha = k[X_1, X_2, X_3, X_4, X_5]/I_\alpha$$

where $I_\alpha$ is generated by the 10 homogeneous quadratics

$$\alpha X_1X_3 + X_2X_3,\ X_1X_4 + X_2X_4,\ X_3^2 + \alpha X_1X_5 - X_2X_5,$$
$$X_4^2 + X_1X_5 - X_2X_5,\ X_1^2,\ X_2^2,\ X_3X_4,\ X_3X_5,\ X_4X_5,\ X_5^2$$

Then $A_\alpha$ is selfinjective and has Hilbert polynomial $1 + 4t + 4t^2 + t^3$. The same facts hold for the resolution

$$\cdots \to A^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^i x_3 + x_4 \\ 0 & x_2 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^{i-1} x_3 + x_4 \\ 0 & x_2 \end{pmatrix}} A^2 \to \cdots$$

as for the resolution in Example 1. Note that $B = A/(x_5)$, and a $k$-basis for $A_{\geq 2}$ is given by $x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_2x_5, x_1x_2x_5$.

**Question 2.** (Auslander ∼19??) Assume that $\Lambda$ is a finite dimensional algebra over a field $k$. For finitely generated ${}_\Lambda M$, does there exist an $n = n(M)$ such that $\mathrm{Ext}^i_\Lambda(M, N) = 0$ for all $i \gg 0$ implies $\mathrm{Ext}^i_\Lambda(M, N) = 0$ for all $i \geq n$?

**Answer:** NO, due to Jorgensen–Şega [3] (2003).

**Example 2.** Take the same $B = B_\alpha$ as in Example 1, but consider the resolution

$$C : \cdots \to B^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^i x_3 \\ x_4 & x_2 \end{pmatrix}} B^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^{i-1} x_3 \\ x_4 & x_2 \end{pmatrix}} B^2 \to \cdots$$

This a complete resolution, meaning that the dual resolution $C^* = \mathrm{Hom}_A(C, A)$ is also acyclic: $\mathrm{H}(C^*) = 0$. Letting $N = B/(x_1 - x_2, x_1 - \alpha^q x_3, x_1 - x_4)$, the complex $C \otimes_B N$ has homology in precisely two consecutive degrees, depending on $q$, if $C$ is aperiodic, and two consecutive periodic degrees if $C$ is periodic. Letting $M$ be an image in $C$ or the dual of $C$, we get examples answering Auslander's question, as well as the analogous question for $\mathrm{Tor}(M, N)$.

**Example 2′.** Take the same $A = A_\alpha$ as in Example 1′ and the matrices as in Example 2. Then the same conclusions as in Example 2 hold, this time using $N' = B/(x_1 - x_2, x_1 - \alpha^q x_3, x_1 - x_4, x_5)$.

*Note.* The resolutions in Examples 1 and 1′ will not work to answer Question 2. There subsequently is a simpler non-commutative example answering Auslander's question in the negative due to Smalø [6], in which the Betti numbers are constant and equal to 1.

**Question 3.** (Avramov–Martsinkovsky, L. W. Christensen, et al ∼1969) Assume $R$ is Noetherian. In the definition of a module $M$ of G-dimension zero ($M$ is finitely generated with

(1) the natural map $M \to M^{**}$ is an isomorphism
(2) $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$
(3) $\mathrm{Ext}^i_R(M^*, R) = 0$ for all $i > 0$

here $-^* = \mathrm{Hom}_R(-, R)$) are the conditions 1–3 independent?

**Answer:** YES, due to Jorgensen–Şega [4] (2004).

**Example 3.** Let $0 \neq \alpha \in k$ ($k$ a field) be of infinite multiplicative order, and

$$A = A_\alpha = k[X_1, X_2, X_3, X_4]/I_\alpha$$

with $I_\alpha$ generated by the 7 homogeneous quadratics

$$X_1X_2 + \alpha X_2X_4,\ X_1X_3 + X_3X_4,\ X_1^2,\ X_1X_3 - X_2^2,\ X_1X_2 + X_3^2,\ X_4^2,\ X_2X_3$$

Then we have a minimal acyclic complex $C$ of free modules of constant rank 2 to the right, and which blows up (grows exponentially) to the left:

$$\cdots \to A^7 \xrightarrow{\begin{pmatrix} x_1 & \alpha^{-2}x_2 & -x_3 & 0 & 0 & 0 & 0 \\ x_3 & x_4 & \alpha x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \end{pmatrix}}$$

$$A^3 \xrightarrow{\begin{pmatrix} x_1 & \alpha^{-1}x_2 & x_3x_4 \\ x_3 & x_4 & 0 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha x_2 \\ x_3 & x_4 \end{pmatrix}}$$

$$\cdots \to A^2 \xrightarrow{\begin{pmatrix} x_1 & \alpha^i x_2 \\ x_3 & x_4 \end{pmatrix}} A^2 \to \cdots$$

We have $\mathrm{H}_n(C^*) = 0$ only for $n$ in degrees occurring in the flat half of the complex $C^*$. Thus the answer to Question 3 comes by letting $M$ be the image of any square matrices of sufficiently large $i$ (any $i \geq 1$ actually works).

A basis for $A_2$ is given by $x_1x_2,\ x_1x_3,\ x_1x_4$.

**Question 4.** (Avramov–Buchweitz ∼1998) Assume that $R$ is commutative Noetherian local. In a minimal totally acyclic complex of finitely generated free modules (i.e., a minimal complete resolution)

$$F \qquad \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

do the sequences $\{\operatorname{rank} F_i\}_{i\geq 0}$ and $\{\operatorname{rank} F^i\}_{i\geq 0}$ grow symmetrically?

**Answer:** NO, due to Jorgensen–Şega [5] (2005).

**Example 4.** Let $0 \neq \alpha \in k$ ($k$ a field) be of infinite multiplicative order, and

$$A = A_\alpha = k[X_1, X_2, X_3, X_4, X_5, X_6]/I_\alpha$$

where $I_\alpha$ is the ideal generated by the following 15 homogeneous quadratic polynomials:

$$X_6^2,\ X_2^2,\ X_2X_5,\ X_3^2,\ X_2X_4,\ X_1X_2,\ X_4X_5,\ X_5X_6 + X_3X_5,\ X_4X_6 + \alpha X_3X_4,$$
$$X_2X_6 - X_1X_4 - \alpha X_2X_3,\ X_5^2 - X_1X_4 - (\alpha-1)X_2X_3,\ X_4^2 - X_1X_4 - X_1X_3,$$
$$X_1X_6 + X_1X_5 + \alpha X_3X_4,\ X_1X_5 - X_3X_4 + X_1X_3,\ X_1^2 + (\alpha+1)X_2X_3 - X_3X_5.$$

Then $A$ is a selfinjective $k$-aglebra with Hilbert polynomial $1 + 6t + 6t^2 + t^3$.

We have a minimal totally acyclic complex $C$ (so that $\mathrm{H}(C^*) = 0$) of free modules of constant rank 2 to the right, and which blows up (grows exponentially) to the left:

$$\cdots \to A^3 \xrightarrow{\begin{pmatrix} x_3 & x_5 & 0 \\ x_4 & x_6 & x_1x_3 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x_3 & x_5 \\ \alpha x_4 & x_6 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x_3 & x_5 \\ \alpha^2 x_4 & x_6 \end{pmatrix}}$$

$$\cdots \to A^2 \xrightarrow{\begin{pmatrix} x_3 & x_5 \\ \alpha^i x_4 & x_6 \end{pmatrix}} A^2 \to \cdots$$

Of course the behavior is reversed for the totally acyclic complex $C^*$.

A basis for $A_{\geq 2}$ is given by $x_1x_3,\ x_1x_4,\ x_2x_3,\ x_3x_4,\ x_3x_5,\ x_3x_6,\ x_1x_3x_4$.

**Question 5.** (Christensen–Veliche ∼2006) Assume that $R$ a commutative Noetherian local ring. In a minimal acyclic complex of finitely generated free modules

$$F \quad \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

does one always have $\mathrm{H}_i \operatorname{Hom}_R(F, R) = 0$ for all $i \gg 0$?

**Answer:** NO, due to Hughes–Jorgensen–Şega [2] (2008).

**Example 5.** Let $k$ be a field and

$$A = k[X_1, X_2, X_3, X_4, X_5]/I$$

where $I$ is the ideal generated by the following 11 homogeneous quadratic relations:

$$X_1^2,\ X_4^2,\ X_2X_3,\ X_1X_2 + X_2X_4,\ X_1X_3 + X_3X_4,$$
$$X_2^2,\ X_2X_5 - X_1X_3,\ X_3^2 - X_1X_5,\ X_4X_5,\ X_5^2, X_3X_5$$

Then $A$ has Hilbert polynomial $1 + 5t + 4t^2$, and

$$C: \qquad \cdots \to A^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}} A^2 \to \cdots$$

Is a minimal acyclic complex of free modules of rank 2, and we have $\mathrm{H}_n(C^*) \neq 0$ for all $n \in \mathbb{Z}$.

A $k$-basis of $A$ is given by $x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5$.

*Remark.* All of the algebras in the above examples are Koszul.

## References


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[4] D. A. Jorgensen, L. M. Şega, *Independence of the total reflexivity conditions for modules*, Algebr. Represent. Theory **9** (2006), 217–226.

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