Lattice theory: Two approaches, Hasse diagrams and a version of Krull-Remak-Schmidt

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15th of December 2014

Lattices (as in order theory) have been of interest to many mathematicians for quite some time. As there are two different approaches of understanding lattices, we will take a look at both of them.

Considering a set with a partial order $(L, \leq)$ and requiring any two-element subset $\{a, b\} \subseteq L$ to have a least upper bound (i.e. among all $s \in L$ with $s \geq a$ and $s \geq b$ there exists a minimal element with such property), also called join or supremum, and a greatest lower bound (dually), also called meet or infimum, we may call L a lattice. We shall denote the join of a and b by $a \vee b$ and their meet as $a \wedge b$. Natural examples include the natural numbers $\mathbb{N}$, which may be viewed with the regular min and max operations or those induced by the least common multiple or the greatest common divisor.

We might also consider a lattice as an algebraic structure $(L, \vee, \wedge)$ consisting of a given set L and two binary operations $\vee$ and $\wedge$, which are commutative and associative for all elements of L and for which the following identities

$$a \vee (a \wedge b) = a, \quad a \wedge (a \vee b) = a,$$

hold for all $a, b \in L$. For any set A, one may view the power set $\mathcal{P}(A)$ along with usual intersection and union as such a lattice.

We shall see that those two definitions given can be used equivalently. As lattices are partially ordered set, we may use Hasse diagrams to visualize those with finitely many elements.

In the next step, we shall call a lattice L modular, if it satisfies the following condition:

$$u \leq v \Rightarrow u \vee (v \wedge w) = (u \vee v) \wedge w, \quad \forall u, v, w \in L.$$

As union and intersection are distributive in the usual sense, and hence modular, we have $\mathcal{P}(A)$ as a (special) example for modular lattices. Another example is given by the subspaces of a given vector space (or in general the submodules of a given module over a ring).

This motivates the final part of the talk, focusing on the Krull-Remak-Schmidt(-Azumaya) theorem, which one might be familiar with for groups or modules. We shall see that we have a similar statement for elements of a modular lattice, if we ask for a specific chain condition on lattice elements.