EFFECTIVE ZERO-CYCLES AND THE BLOCH–BEILINSON FILTRATION

OLIVIER MARTIN AND CHARLES VIAL

Abstract. A conjecture of Voisin states that two points on a smooth projective complex variety whose algebra of holomorphic forms is generated in degree 2 are rationally equivalent to each other if and only if their difference lies in the third step of the Bloch–Beilinson filtration. In this note, we formulate a generalization that allows for rational equivalence of effective zero-cycles of higher degree, at the expense of looking deeper in the Bloch–Beilinson filtration. In the first half, we provide evidence in support of this conjecture in the case of abelian varieties and projective hyper-Kähler manifolds. Notably, we give explicit criteria for rational equivalence of effective zero-cycles on moduli spaces of semistable sheaves on K3 surfaces, generalizing that of Marian–Zhao. In the second half, in an effort to explain our main conjecture, we formulate a second conjecture predicting when the diagonal of a smooth projective variety belongs to a subalgebra of the ring of correspondences generated in low degree.

1. Introduction

We work throughout over the field of complex numbers and with Chow rings with rational coefficients. The class in $\mathrm{CH}_0(X)$ of a closed point $x \in X$ will be denoted by $[x]$.

1.1. **Effective zero-cycles on moduli spaces of sheaves on K3 surfaces.** Building on work of O'Grady [O'G13] and Shen–Yin–Zhao [SYZ20], Marian and Zhao established the following criterion for two points on a smooth projective moduli space of semistable sheaves to be rationally equivalent.

Theorem 1.1 (Marian–Zhao [MZ20] ; see Theorem 2.5)**.** *Let $\mathcal{M}$ be a smooth projective moduli space of semistable sheaves on a projective K3 surface S and let $\mathcal{F}$ and $\mathcal{G}$ be closed points of $\mathcal{M}$. Then*

$$[\mathcal{F}] = [\mathcal{G}] \text{ in } \mathrm{CH}_0(\mathcal{M}) \iff c_2(\mathcal{F}) = c_2(\mathcal{G}) \text{ in } \mathrm{CH}_0(S).$$

A first aim of this paper is to establish the following generalization to effective zero-cycles of arbitrary degree:

Theorem 1.2 (Theorem 2.6)**.** *Let $\mathcal{M}$ be as in Theorem 1.1 and let $2n$ be its dimension. Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and $\mathcal{G}_1, \ldots, \mathcal{G}_m$ be closed points of $\mathcal{M}$. Then*

$$\sum_{i=1}^m [\mathcal{F}_i] = \sum_{i=1}^m [\mathcal{G}_i] \text{ in } \mathrm{CH}_0(\mathcal{M}) \iff \sum_{i=1}^m c_2(\mathcal{F}_i)^{\times k} = \sum_{i=1}^m c_2(\mathcal{G}_i)^{\times k} \text{ in } \mathrm{CH}_0(S^k) \text{ for all } k \leq \min(m, n).$$

We also refer to Theorem 2.7 and Theorem 2.8 for versions of Theorem 1.2 concerned with other types of hyper-Kähler varieties, namely generalized Kummer varieties and Fano varieties of lines on smooth cubic fourfolds.

1.2. **Rational equivalence of effective zero-cycles.** As outlined in [Voi22, §2.2], motivated by the criterion of Marian–Zhao, Voisin formulated the following:

Voisin's Conjecture 1.3 ([Voi22, Conj. 2.11])**.** *Let X be a smooth projective variety whose algebra of holomorphic forms is generated in degree ≤ 2. Then, given closed points x, y of X,*

$$[x] = [y] \text{ in } \mathrm{CH}_0(X) \iff [x] = [y] \text{ in } \mathrm{CH}_0(X)/F^3_{\mathrm{BB}}\mathrm{CH}_0(X), \tag{1.1}$$

where $F^\bullet_{\mathrm{BB}}$ denotes the (conjectural) Bloch–Beilinson filtration.

In turn, motivated by Theorem 1.2, we propose the following generalization of Voisin's Conjecture 1.3, by allowing for rational equivalence of effective zero-cycles of higher degree, at the expense of looking deeper into the Bloch–Beilinson filtration.

The research of Ch.V. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB-TRR 358/1 2023 – 491392403.

Main Conjecture 1.4. *Let X be a smooth projective variety whose algebra of holomorphic forms is generated in degree $\leq d$. Then, for closed point $x_1,\dots,x_m,y_1,\dots,y_m \in X$,*

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(X) \quad \Longleftrightarrow \quad \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(X)/F_{\mathrm{BB}}^{md+1}\mathrm{CH}_0(X). \tag{1.2}$$

Equivalently, the image of the following map intersects $F_{\mathrm{BB}}^{md+1}\mathrm{CH}_0(X)$ only in $\{0\}$:

$$\begin{array}{rccc} \Phi_m : & \mathrm{Sym}^m X \times \mathrm{Sym}^m X & \longrightarrow & \mathrm{CH}_0(X) \\ & (x_1+\cdots+x_m, y_1+\cdots+y_m) & \longmapsto & \sum_{i=1}^m [x_i] - \sum_{i=1}^m [y_i]. \end{array}$$

Remark 1.5. Since $F_{\mathrm{BB}}^r\mathrm{CH}_0(X) = 0$ for $r \geq \dim X$, Main Conjecture 1.4 is only interesting for small m and l. For instance, it holds trivially for curves or varieties with an indecomposable top form e.g., Calabi-Yau varieties, complete intersections of general type, etc.

In practice, we will consider Main Conjecture 1.4 with respect to candidate filtrations for the Bloch–Beilinson filtration. As a first example, using Beauville's filtration $F_{\mathrm{B}}^\bullet$ as the candidate Bloch–Beilinson filtration for abelian varieties, we have:

Theorem 1.6 (Theorem 3.3)**.** *Let A be an abelian g-fold and let $x_1,\dots,x_m,y_1,\dots,y_m \in A$,*

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(A) \quad \Longleftrightarrow \quad \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(X)/F_{\mathrm{B}}^{m+1}\mathrm{CH}_0(X). \tag{1.3}$$

In (1.3) we write

$$F_{\mathrm{B}}^{m+1}\mathrm{CH}_0(X) =_{\mathrm{def}} \bigoplus_{s=m+1}^{g} \mathrm{CH}^g_{(s)}(A),$$

where $\mathrm{CH}^g_{(s)}(A)$ are the graded pieces of the Beauville filtration on $\mathrm{CH}_0(A)$:

$$\mathrm{CH}^g_{(s)}(A) \ =_{\mathrm{def}} \ \{\alpha \in \mathrm{CH}^g(A) : [k]^*(\alpha) = k^{2g-s}\alpha \text{ for all } k \in \mathbb{Z}\},$$

where $[k] : A \longrightarrow A$ is the multiplication by k isogeny.

There are two further candidates for the Bloch–Beilinson filtration on $\mathrm{CH}_0(X)$ that we will consider in this work. The first, $F_{\mathrm{V}}^\bullet\mathrm{CH}_0(X)$, was proposed by Voisin in [Voi04], and is defined by

$$F_{\mathrm{V}}^i\mathrm{CH}_0(X) \ = \ \bigcap_{\Gamma,Y} \ker\big(\Gamma_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(Y)\big),$$

where Y ranges over all smooth projective $(i-1)$-folds and Γ over all correspondences in the group $\mathrm{CH}^{i-1}(X \times Y)$. In Proposition 3.2 and in Proposition 3.6, we adapt arguments of Voisin from [Voi22] to deduce Main Conjecture 1.4 for abelian varieties and for hyper-Kähler varieties with respect to the candidate filtration $F_{\mathrm{V}}^\bullet$ from well-known conjectures on algebraic cycles.

The second candidate, which is particularly useful to provide unconditional evidence in favor of Main Conjecture 1.4, is inspired by Murre's filtration [Mur93] and uses the language of birational motives as introduced by Kahn–Sujatha [KS16], see [Via22, §2] for an overview. Briefly, a birational correspondence between two connected smooth projective varieties X and Y over a field k is a cycle

$$\gamma \in \lim_{U \subseteq X} \mathrm{CH}^{\dim Y}(U \times_k Y) = \mathrm{CH}_0(Y_{k(X)}),$$

where the limit runs through all Zariski open subsets of X and where $k(X)$ is the function field of X. Birational correspondences can be composed and there is a well-defined induced action

$$\gamma_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y),$$

which in fact determines γ if k is a universal domain [Via22, Lem. 2.1]. A *birational motive* is a pair (X, ϖ), also denoted $\mathfrak{h}^\circ_\varpi(X)$, consisting of a smooth projective variety X and a birational idempotent correspondence $\varpi \in \mathrm{CH}_0(X_{k(X)})$.

Let now $\varpi \in \mathrm{End}(\mathfrak{h}^\circ(X))$ be any birational idempotent correspondence and denote by $\delta^{k-1} : X \hookrightarrow X^k$ the diagonal embedding, where by convention δ^{-1} is the structure morphism. These induce a morphism

$$\bigoplus_{k\geq 0} \varpi^{\otimes k} \circ \delta^{k-1} \colon \mathfrak{h}^\circ(X) \longrightarrow \mathrm{Sym}^* \mathfrak{h}^\circ_\varpi(X).$$

We define for all j and all $1 \leq i \leq d$ the following descending filtration

$$F^{dj+i}_\varpi \mathrm{CH}_0(X) =_{\mathrm{def}} \ker \left(\bigoplus_{k=0}^{j} \varpi^{\otimes k} \circ \delta^{k-1}_* \ : \ \mathrm{CH}_0(X) \longrightarrow \bigoplus_{k=0}^{j} \mathrm{CH}_0(X^k) \right).$$

If X is a smooth projective variety whose algebra of holomorphic forms is generated in degree d and if ϖ acts on $H^0(X, \Omega^{nd}_X)$ as the identity for $n = 1$ and as zero otherwise, then this filtration is a candidate for the Bloch–Beilinson filtration (see [Via22]).

In this language, we have

Proposition 1.7 (Proposition 2.2)**.** *Let X be a smooth projective variety and $\mathfrak{h}^\circ_\varpi(X) =_{\mathrm{def}} (X, \varpi)$ a direct summand of the birational motive $\mathfrak{h}^\circ(X)$ such that $\mathfrak{h}^\circ(X)$ is co-generated by $\mathfrak{h}^\circ_\varpi(X)$, i.e., such that the morphism $\mathfrak{h}^\circ(X) \longrightarrow \mathrm{Sym}^* \mathfrak{h}^\circ_\varpi(X)$ is split injective. If $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are closed points of X, then*

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in}\ \mathrm{CH}_0(X) \iff \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in}\ \mathrm{CH}_0(X)/F^{dm+1}_\varpi \mathrm{CH}_0(X).$$

The link to Main Conjecture 1.4 is provided by the following. Assume that $H^0(X, \Omega^\bullet_X)$ is generated by $\bigoplus_{i\leq d} H^0(X, \Omega^i_X)$. A combination of the standard conjectures and of the Bloch–Beilinson conjecture implies the existence of a birational idempotent correspondence ϖ such that $\varpi_* H^0(X, \Omega^\bullet_X) = \bigoplus_{0<i\leq d} H^0(X, \Omega^i_X)$ and, for any choice of such ϖ, $\mathfrak{h}^\circ(X)$ is co-generated by $\mathfrak{h}^\circ_\varpi(X)$ (this is [Via22, Conj. 5.1]) and the filtration $F^\bullet_\varpi$ defined above is the Bloch–Beilinson filtration on $\mathrm{CH}_0(X)$. Hence, one may formulate the following variant of Main Conjecture 1.4:

Conjecture 1.8. *Let X be a smooth projective variety whose algebra of holomorphic forms is generated in degree $\leq d$. Then there exists a birational idempotent correspondence ϖ such that $\varpi_* H^0(X, \Omega^\bullet_X) = \bigoplus_{0<i\leq d} H^0(X, \Omega^i_X)$ and such that, for $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ closed points on X,*

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in}\ \mathrm{CH}_0(X) \iff \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in}\ \mathrm{CH}_0(X)/F^{dm+1}_\varpi \mathrm{CH}_0(X).$$

The following gives evidence for Conjecture 1.8 and therefore Main Conjecture 1.4 in the case of hyper-Kähler varieties.

Theorem 1.9 (Theorem 2.3)**.** *Let X be a hyper-Kähler variety. Assume that X is one of the following:*

(1) $\mathrm{Hilb}^n(S)$, the Hilbert scheme of length-n closed subschemes on a K3 surface S ;
(2) $\mathrm{M}_\sigma(v)$, a moduli space of stable objects on a K3 surface ;
(3) $K_n(A)$, the generalized Kummer variety associated to an abelian surface A ;
(4) $F(Y)$, the Fano variety of lines on a smooth cubic fourfold Y ;
(5) $\widetilde{K}_v(A)$, O'Grady's six-dimensional example.

Then there exists a birational idempotent correspondence ϖ such that $\varpi_* H^0(X, \Omega^\bullet_X) = H^0(X, \Omega^2_X)$ and such that, for any closed points $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ on X,

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in}\ \mathrm{CH}_0(X) \iff \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in}\ \mathrm{CH}_0(X)/F^{dm+1}_\varpi \mathrm{CH}_0(X).$$

From there and the work carried out in [Via22], we derive explicit criteria as in Theorem 1.2 for rational equivalence of effective zero-cycles; we refer to Theorem 2.6 for case (2), Theorem 2.7 for case (3) and Theorem 2.8 for case (4).

1.3. **Polynomial decomposition of the diagonal.** In the last part of the paper, in an attempt to further explain and motivate Main Conjecture 1.4, we introduce in Definition 4.2 the notion of polynomial decomposition of the diagonal up to coniveau c. A special instance of the definition is the following:

Definition 1.10. A smooth projective n-fold X admits a degree l polynomial decomposition of the diagonal if

$$\Delta_X = Z_1 + Z_2 \in \mathrm{CH}^n(X \times X),$$

where Z_1 belongs to the subalgebra of $\mathrm{CH}^\bullet(X \times X)$ generated in degree $\leq l$ and Z_2 is supported on $D \times X$ for some divisor $D \subset X$.

In Proposition 4.8, we observe that if X has a degree l polynomial decomposition of the diagonal, then its algebra of holomorphic forms is generated in degrees $\leq l$. We show in Proposition 4.10 that if X has a degree l polynomial decomposition of the diagonal and satisfies Nori's Conjecture 3.1, then for $x_1, \ldots, x_m, y_1, \ldots, y_m \in X$,

$$\sum_{i=1}^m x_i \;=\; \sum_{i=1}^m y_i \text{ in } \mathrm{CH}_0(X) \quad \Longleftrightarrow \quad \sum_{i=1}^m x_i = \sum_{i=1}^m y_i \text{ in } \mathrm{CH}_0(X)/F_{\mathrm{V}}^{ml+1}\mathrm{CH}_0(X).$$

Accordingly, the following conjecture in conjunction with Nori's Conjecture 3.1 implies Main Conjecture 1.4:

Conjecture 1.11 (Special instance of Conjecture 4.6)**.** *A smooth projective variety X admits a degree l polynomial decomposition of the diagonal if and only if the algebra $H^0(X, \Omega^\bullet)$ is generated in degree $\leq l$.*

Finally, we will show in Proposition 4.9 how Conjecture 1.11 easily implies the generalized Bloch conjecture in coniveau 1.

Acknowledgments. We thank Salvatore Floccari for mentioning Remark 2.4 and its proof, which led to a proof of Theorem 2.3(5). Ch.V. is grateful to Stony Brook University for a pleasant stay in March 2023.

2. Effective zero-cycles on hyper-Kähler varieties

In this section we show that Main Conjecture 1.4 holds unconditionally for certain hyper-Kähler varieties, with respect to a certain Bloch–Beilinson candidate filtration induced by some birational correspondence. We also give explicit criteria for the rational equivalence of effective zero-cycles on some hyper-Kähler varieties.

2.1. **Hyper-Kähler varieties satisfying Main Conjecture 1.4.** We start by considering the general situation of an arbitrary smooth projective variety X over an algebraically closed field. We take on the definitions and notation from [Via22] concerning birational motives and their co-algebra structure. Our main tool used to prove Theorem 1.2 and its variants for generalized Kummer varieties and Fano varieties of lines on smooth cubic fourfolds is the observation that [Via22, Prop. 5.2] can be extended to the following:

Proposition 2.1. *Let X be a smooth projective variety over an algebraically closed field and denote $\delta^{k-1} : X \hookrightarrow X^k$ the diagonal embedding, where by convention δ^{-1} is the structure morphism. Let $\mathfrak{h}^\circ_\varpi(X) =_{\mathrm{def}} (X, \varpi)$ be a direct summand of the birational motive $\mathfrak{h}^\circ(X)$ and assume that there exists $r \geq 0$ such that the morphism*

$$\bigoplus_{k=0}^r \varpi^{\otimes k} \circ \delta^{k-1}_* : \; \mathfrak{h}^\circ(X) \longrightarrow \mathrm{Sym}^{\leq r}\, \mathfrak{h}^\circ_\varpi(X)$$

is split injective (we say $\mathfrak{h}^\circ(X)$ is co-generated *by $\mathfrak{h}^\circ_\varpi(X)$).*

If $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are closed points on X, then

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(X) \iff \sum_{i=1}^m (\varpi_*[x_i])^{\times k} = \sum_{i=1}^m (\varpi_*[y_i])^{\times k} \text{ in } \mathrm{CH}_0(X^k) \text{ for all } k \leq \min(m, r).$$

Proof. Under the morphism $\mathfrak{h}^\circ(X) \to \mathrm{Sym}^{\leq r}\,\mathfrak{h}^\circ_\varpi(X)$, the class of a closed point x is mapped to $1 + (\varpi)_*[x] + \cdots + (\varpi^{\otimes r})_*\delta_*^{r-1}[x]$. Since $\delta_*^{k-1}[x] = [x] \times \cdots \times [x]$ in $\mathrm{CH}_0(X^k)$, we find that

$$\sum_{i=1}^m [x_i] \mapsto m + \sum_{i=1}^m \varpi_*[x_i] + \cdots + \sum_{i=1}^m (\varpi_*[x_i])^{\times r}.$$

Now, the basic point is that if the morphism $\mathfrak{h}^\circ(X) \to \mathrm{Sym}^{\leq r}\,\mathfrak{h}^\circ_\varpi(X)$ is split injective, then the induced map on Chow groups of zero-cycles is injective. Hence

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(X) \iff \sum_{i=1}^m (\varpi_*[x_i])^{\times k} = \sum_{i=1}^m (\varpi_*[y_i])^{\times k} \text{ in } \mathrm{CH}_0(X^k) \text{ for all } k \leq r.$$

On the other hand, the fundamental theorem on symmetric polynomials provides the equivalence

$$\sum_{i=1}^m (\varpi_*[x_i])^{\times k} = \sum_{i=1}^m (\varpi_*[y_i])^{\times k}\ \forall k \leq r \iff \sum_{i=1}^m (\varpi_*[x_i])^{\times k} = \sum_{i=1}^m (\varpi_*[y_i])^{\times k} \ \text{ for all } k \leq \min(m, r).$$

This concludes the proof. □

Let now $\varpi \in \mathrm{End}(\mathfrak{h}^\circ(X))$ be any birational idempotent correspondence. Recall that for all j and all $1 \leq i \leq d$ we consider the descending filtration

$$F^{dj+i}_\varpi \mathrm{CH}_0(X) =_{\mathrm{def}} \ker\Big(\bigoplus_{k=0}^{j} \varpi^{\otimes k} \circ \delta_*^{k-1} \ :\ \mathrm{CH}_0(X) \longrightarrow \bigoplus_{k=0}^{j} \mathrm{CH}_0(X^k)\Big).$$

Proposition 2.1 can be reformulated as:

Proposition 2.2. *Let X be a smooth projective variety over an algebraically closed field. Let $\mathfrak{h}^\circ_\varpi(X) =_{\mathrm{def}} (X, \varpi)$ be a direct summand of the birational motive $\mathfrak{h}^\circ(X)$ and assume that $\mathfrak{h}^\circ(X)$ is co-generated by $\mathfrak{h}^\circ_\varpi(X)$, i.e., that the morphism $\mathfrak{h}^\circ(X) \longrightarrow \mathrm{Sym}^*\,\mathfrak{h}^\circ_\varpi(X)$ is split injective. If $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are closed points on X, then*

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in } \mathrm{CH}_0(X) \iff \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in } \mathrm{CH}_0(X)/F^{dm+1}_\varpi \mathrm{CH}_0(X).$$

The following gives evidence for Conjecture 1.8 and Main Conjecture 1.4 in the case of hyper-Kähler varieties.

Theorem 2.3. *Let X be a hyper-Kähler variety. Assume that X is one of the following:*

(1) $\mathrm{Hilb}^n(S)$*, the Hilbert scheme of length-n closed subschemes on a K3 surface S ;*
(2) $\mathrm{M}_\sigma(v)$*, a moduli space of stable objects on a K3 surface ;*
(3) $K_n(A)$*, the generalized Kummer variety associated to an abelian surface A ;*
(4) $F(Y)$*, the Fano variety of lines on a smooth cubic fourfold Y ;*
(5) $\widetilde{K}_v(A)$*, O'Grady's six-dimensional example* [O'G03].

Then there exists a birational idempotent correspondence ϖ such that $\varpi_* H^0(X, \Omega^\bullet_X) = H^0(X, \Omega^2_X)$ and such that, for any closed points $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ on X,

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in } \mathrm{CH}_0(X) \iff \sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \ \textit{in } \mathrm{CH}_0(X)/F^{dm+1}_\varpi \mathrm{CH}_0(X).$$

Proof. For a hyper-Kähler variety X, the existence of a birational idempotent correspondence ϖ such that $\varpi_* H^0(X, \Omega^\bullet_X) = H^0(X, \Omega^2_X)$ and such that the morphism $\mathfrak{h}^\circ(X) \to \mathrm{Sym}^{\leq n}\mathfrak{h}^\circ_\varpi(X)$ is an isomorphism (in particular, $\mathfrak{h}^\circ(X)$ is co-generated by $\mathfrak{h}^\circ_\varpi(X)$) is conjectured in [Via22, Conj. 2] (see also [Via22, Prop. 5.3]) and is established in cases (1), (2), (3), and (4) in [Via22, Thm. 5.5]. Regarding case (5), we use below the work of Mongardi–Rapagnetta–Saccà [MRS18] and Floccari [Flo23] to reduce to case (1) by showing that the birational motive of $\widetilde{K}_v(A)$ is isomorphic to that of a Hilbert scheme of length-3 closed subschemes on a K3 surface as co-algebra objects. The theorem in all cases listed then follows from Proposition 2.2.

O'Grady's hyper-Kähler sixfold is obtained as follows. Let A be an abelian surface and let $v = 2v_0$ be a Mukai vector on A such that v_0 is primitive, effective, and of square 2. Given a v-generic polarization H, let $\mathrm{M}_v(A)$ be the corresponding moduli space of H-semistable sheaves on A and denote $K_v(A)$ its Albanese fiber. Then $K_v(A)$ admits a crepant resolution $\widetilde{K}_v(A)$ which is a hyper-Kähler sixfold. By [MRS18], there exists a variety $Y_v(A)$, which by [Flo23, Prop. 3.3] is birational to a moduli space $\mathrm{M}(A,v)$ of stable sheaves on the Kummer surface associated to A, and a rational map $f : Y_v(A) \dashrightarrow \widetilde{K}_v(A)$ of degree 2. From the results of [Flo23], $f_*\colon \mathrm{CH}_0(Y_v(A)) \to \mathrm{CH}_0(\widetilde{K}_v(A))$ is an isomorphism, and consequently by [Via22, Prop. 2.3] gives an isomorphism $\mathfrak{h}^\circ(\mathrm{M}(A,v)) \cong \mathfrak{h}^\circ(\widetilde{K}_v(A))$ of co-algebra objects. ☐

Remark 2.4 (The Voisin filtration $S_\bullet$ and the co-radical filtration agree in case (5))**.** The proof of Theorem 2.3 in Case (5) was made possible after Salvatore Floccari communicated to us the following. Let $\widetilde{K}_v(A)$ be O'Grady's six-dimensional example. Together with [Via22, Thm. 1(i)], the arguments in the proof of Theorem 2.3 in that case show that there exists a point $o \in \widetilde{K}_v(A)$ such that, for all $k \geq 0$ and for all $x \in \widetilde{K}_v(A)$,

$$[x] \in S_k\mathrm{CH}_0(\widetilde{K}_v(A)) \iff ([x]-[o])^{\times k+1} = 0 \text{ in } \mathrm{CH}_0(\widetilde{K}_v(A)^{k+1}),$$

or equivalently, such that

$$S_k\mathrm{CH}_0(\widetilde{K}_v(A)) = R_k\mathrm{CH}_0(\widetilde{K}_v(A))$$

for all $k \geq 0$. Here, $S_\bullet$ is Voisin's filtration [Voi16] and $R_\bullet$ is the co-radical filtration introduced in [Via22].

2.2. **Explicit criteria for rational equivalence of effective zero-cycles on hyper-Kähler varieties.** In this paragraph we exploit the so-called *strictness* of the explicit co-multiplicative birational Chow–Künneth decompositions on the birational motive of certain hyper-Kähler varieties, which were constructed in [Via22], to derive criteria for the coincidence of effective zero-cycles on these hyper-Kähler varieties.

2.2.1. *Moduli spaces of stable objects on K3 surfaces.* Let S be a smooth projective complex K3 surface. For a primitive $v \in H^\bullet(S,\mathbb{Z})$, and a v-generic stability condition σ, let $\mathrm{M}_\sigma(v)$ be the moduli space of σ-stable complexes on S of Mukai vector v; this defines a hyper-Kähler variety and we denote by $2n$ its dimension. Marian and Zhao [MZ20] have established the following:

Theorem 2.5 (Marian–Zhao [MZ20])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be closed points of $\mathrm{M}_\sigma(v)$. Then*

$$[\mathcal{F}] = [\mathcal{G}] \text{ in } \mathrm{CH}_0(\mathrm{M}_\sigma(v)) \iff c_2(\mathcal{F}) = c_2(\mathcal{G}) \text{ in } \mathrm{CH}_0(S).$$

We have the following generalization to effective zero-cycles of arbitrary degree:

Theorem 2.6. *Let $\mathcal{F}_1,\ldots,\mathcal{F}_m$ and $\mathcal{G}_1,\ldots,\mathcal{G}_m$ be closed points of $\mathrm{M}_\sigma(v)$. Then*

$$\begin{aligned}&\sum_{i=1}^m [\mathcal{F}_i] = \sum_{i=1}^m [\mathcal{G}_i] \text{ in } \mathrm{CH}_0(\mathrm{M}_\sigma(v))\\ &\iff \sum_{i=1}^m c_2(\mathcal{F}_i)^{\times k} = \sum_{i=1}^m c_2(\mathcal{G}_i)^{\times k} \text{ in } \mathrm{CH}_0(S^k) \text{ for all } k \leq \min(m,n).\end{aligned}$$

Proof. Denote by $[o_S]$ the Beauville–Voisin class on S and let $c \in \mathbb{Z}$ be the constant (which depends only on the Mukai vector v) such that $\deg c_2(\mathcal{F}) = c$ for all $\mathcal{F} \in \mathrm{M}_\sigma(v)$. We denote $\mathfrak{h}^\circ_2(S)$ the image of the birational idempotent correspondence $(\Delta_S - S \times [o_S])|_{\eta_S \times S}$ acting on $\mathfrak{h}^\circ(S)$. The birational motive $\mathfrak{h}^\circ(\mathrm{M}_\sigma(v))$ is then canonically isomorphic to $\mathrm{Sym}^{\leq n}\mathfrak{h}^\circ_\varpi(\mathrm{M}_\sigma(v))$ for a birational idempotent correspondence ϖ factorizing as $\varpi : \mathfrak{h}^\circ(\mathrm{M}_\sigma(v)) \twoheadrightarrow \mathfrak{h}^\circ_2(S) \hookrightarrow \mathfrak{h}^\circ(\mathrm{M}_\sigma(v))$ with the left arrow satisfying $[\mathcal{F}] \mapsto c_2(\mathcal{F}) - c[o_S]$ for all $\mathcal{F} \in \mathrm{M}_\sigma(v)$. See the proofs of [Via22, Thm. 3.1] and [Via22, Thm. 5.5], which are based on the theorem of Marian–Zhao. From Proposition 2.1 it follows that

$$\begin{aligned}&\sum_{i=1}^m [\mathcal{F}_i] = \sum_{i=1}^m [\mathcal{G}_i] \text{ in } \mathrm{CH}_0(\mathrm{M}_\sigma(v))\\ &\iff \sum_{i=1}^m (c_2(\mathcal{F}_i) - c[o_S])^{\times k} = \sum_{i=1}^m (c_2(\mathcal{G}_i) - c[o_S])^{\times k} \text{ in } \mathrm{CH}_0(S^k) \text{ for all } k \leq \min(m,n).\end{aligned}$$

The latter is easily seen to be further equivalent to $\sum_{i=1}^m c_2(\mathcal{F}_i)^{\times k} = \sum_{i=1}^m c_2(\mathcal{G}_i)^{\times k}$ in $\mathrm{CH}_0(S^k)$ for all $k \leq \min(m,n)$. ☐

2.2.2. *Generalized Kummer varieties.* Let A be an abelian surface. Recall that the $2n$-dimensional generalized Kummer variety $K_n(A)$ is the fiber over 0 of the composition of the Hilbert–Chow morphism with the sum map $\mathrm{Hilb}^{n+1}(A) \to A^{n+1}/\mathfrak{S}_{n+1} \to A$. We thus have a natural morphism $K_n(A) \to A_0^{n+1}/\mathfrak{S}_{n+1}$, where $A_0^{n+1} =_{\mathrm{def}} \ker(\Sigma : A^{n+1} \to A)$. Let us denote by $\{x_1, \dots, x_{n+1}\}$ the closed points of $A_0^{n+1}/\mathfrak{S}_{n+1}$; these are unordered $(n+1)$-tuple of closed point of A such that $x_1 + \cdots + x_{n+1} = 0$ in A. By [FTV19, §6], the pushforward map $\mathrm{CH}_0(K_n(A)) \to \mathrm{CH}_0(A_0^{n+1}/\mathfrak{S}_{n+1})$ is an isomorphism. Therefore the rational equivalence class of a point in $K_n(A)$ only depends on its image in $A_0^{n+1}/\mathfrak{S}_{n+1}$ and we have a canonical isomorphism $\mathfrak{h}^\circ(K_n(A)) \cong \mathfrak{h}^\circ(A_0^{n+1}/\mathfrak{S}_{n+1})$.

Theorem 2.7. *Let $p_1, \dots, p_m$ and $q_1, \dots, q_m$ be closed points of $K_n(A)$ with respective images $\{x_{1,1}, \dots, x_{1,n+1}\}, \dots, \{x_{m,1}, \dots, x_{m,n+1}\}$ and $\{y_{1,1}, \dots, y_{1,n+1}\}, \dots, \{y_{m,1}, \dots, y_{m,n+1}\}$ in $A_0^{n+1}/\mathfrak{S}_{n+1}$. Then*

$$\sum_{i=1}^m [p_i] = \sum_{i=1}^m [q_i] \text{ in } \mathrm{CH}_0(K_n(A))$$
$$\iff \sum_{i=1}^m \Big(\sum_{j=1}^{n+1} [x_{i,j}]\Big)^{\times k} = \sum_{i=1}^m \Big(\sum_{j=1}^{n+1} [y_{i,j}]\Big)^{\times k} \text{ in } \mathrm{CH}_0(A^k) \text{ for all } k \le \min(m,n).$$

Proof. We equip the Chow motive $\mathfrak{h}(A)$ of A with its Deninger–Murre Chow–Künneth decomposition (which we consider covariantly)

$$\mathfrak{h}(A) = \bigoplus_{i=0}^4 \mathfrak{h}_i(A), \quad \text{with } \mathfrak{h}_i(A) =_{\mathrm{def}} (A, \varpi_i^A).$$

By construction, this decomposition is an eigenspace decomposition for the multiplication by r maps $[r] : A \to A, a \mapsto ra$. The action on zero-cycles satisfies

$$(\varpi_0^A)_*[a] = [0], \quad (\varpi_1^A)_*[a] = \frac{1}{2}([a] - [-a]) \quad \text{and} \quad (\varpi_2^A)_*[a] = \frac{1}{2}([a] + [-a]) - [0]$$

for all closed points $a \in A$. The product Chow–Künneth decomposition on A^l (which coincides with the Deninger–Murre Chow–Künneth decomposition on A^l) then also provides an eigenspace decomposition for the multiplication by r maps on A^l and hence provide a Chow–Künneth decomposition for $\mathfrak{h}(A^l/\mathfrak{S}_l)$. Likewise, identifying A_0^{l+1} with A^l and noting that the sum map commutes with the multiplication by r maps on A^l, the product Chow–Künneth decomposition on A^l provides a Chow–Künneth decomposition for $\mathfrak{h}(A_0^{l+1}/\mathfrak{S}_{l+1})$. Via the identification $\mathfrak{h}^\circ(K_n(A)) \cong \mathfrak{h}^\circ(A_0^{n+1}/\mathfrak{S}_{n+1})$, this provides a birational Chow–Künneth decomposition $\mathfrak{h}^\circ(K_n(A)) = \bigoplus_{i=0}^n \mathfrak{h}^\circ_{2i}(K_n(A))$ and it is shown in [Via22, Thm. 5.5(iii)] that the canonical morphism

$$\mathfrak{h}^\circ(K_n(A)) \xrightarrow{\simeq} \mathrm{Sym}^{\le n}\mathfrak{h}_2^\circ(K_n(A))$$

is a graded isomorphism. On the other hand, the natural embedding $\iota : A_0^{n+1}/\mathfrak{S}_{n+1} \hookrightarrow A^{n+1}/\mathfrak{S}_{n+1}$ commutes with the multiplication by r maps and thus induces a graded morphism of Chow motives

$$\iota_* \colon \mathfrak{h}(A_0^{n+1}/\mathfrak{S}_{n+1}) \longrightarrow \mathfrak{h}(A^{n+1}/\mathfrak{S}_{n+1}).$$

Moreover, its restriction to the generic point

$$\iota_* \colon \mathfrak{h}^\circ(K_n(A)) \simeq \mathfrak{h}^\circ(A_0^{n+1}/\mathfrak{S}_{n+1}) \longrightarrow \mathfrak{h}^\circ(A^{n+1}/\mathfrak{S}_{n+1})$$

is split injective. Indeed, if

$$\Gamma =_{\mathrm{def}} A^{(n,n+1)} =_{\mathrm{def}} \{(\{x_1, \dots, x_n\}, \{x_1, \dots, x_n, x_{n+1}\}) \mid x_1, \dots, x_{n+1} \in A\} \subset A^n/\mathfrak{S}_n \times A^{n+1}/\mathfrak{S}_{n+1}$$

denotes the incidence correspondence, we have $\frac{1}{n+1}\Gamma^*\iota_*([p]) = [p]$ in $\mathrm{CH}_0(K_n(A))$ for all closed points $p \in K_n(A)$.

Therefore, by Proposition 2.1, $\sum_{i=1}^m [p_i] = \sum_{i=1}^m [q_i]$ in $\mathrm{CH}_0(K_n(A))$ if and only if

$$\sum_{i=1}^m \big((\varpi_2^{A^{n+1}})_*\iota_*[p_i]\big)^{\times k} = \sum_{i=1}^m \big((\varpi_2^{A^{n+1}})_*\iota_*[q_i]\big)^{\times k} \text{ in } \mathrm{CH}_0((A^{n+1})^k) \text{ for all } k \le \min(m,n),$$

where

$$\varpi_2^{A^{n+1}} = \underbrace{\left(\varpi_2^A \otimes \varpi_0^A \otimes \cdots \otimes \varpi_0^A + (\mathrm{sym})\right)}_{\varpi_{2,0}} + \underbrace{\left(\varpi_1^A \otimes \varpi_1^A \otimes \varpi_0^A \otimes \cdots \otimes \varpi_0^A + (\mathrm{sym})\right)}_{\varpi_{1,1}}.$$

Since $(\varpi_{2,0})_*\iota_*[p_i] = c\sum_{j=1}^{n+1}[\{x_{ij}, 0, \dots, 0\}]$ for some non-zero combinatorial constant c only depending on n (and similarly for q_i in place of p_i), we only need to show that, for any point $x = \{x_1, \dots, x_{n+1}\}$ in $A_0^{n+1}/\mathfrak{S}_{n+1}$, $(\varpi_2^{A^{n+1}})_*[x]$ vanishes if and only if $(\varpi_{2,0})_*[x]$ vanishes. The "only if" part is clear since the idempotents $\varpi_{2,0}$ and $\varpi_{1,1}$ are orthogonal. For the "if" part, we first note that $(\varpi_{2,0})_*[x]$ is equal to the symmetrization of $d(\sum_{i=1}^{n+1}[x_i] - (n+1)[0]) \times [0]^{\times n}$ in $\mathrm{CH}_0(A^{n+1})$ for some non-zero combinatorial constant d. In particular, if $(\varpi_{2,0})_*[x]$ vanishes, then $\sum[x_i] = (n+1)[0]$ in $\mathrm{CH}_0(A)$. Second, the identity $\sum_i x_i = 0$ in A implies [Lin18] that $\sum_i[x_i] = \sum_i[-x_i]$ in $\mathrm{CH}_0(A)$, i.e., that $(\varpi_1^A)_* \sum_i[x_i] = 0$ in $\mathrm{CH}_0(A)$. Now $(\varpi_{1,1})_*[x]$ is a non-zero multiple of the symmetrization of

$$\Big(\sum_i(\varpi_1^A)_*[x_i] \times \sum_i(\varpi_1^A)_*[x_i] - \sum_i(\varpi_1^A)_*[x_i] \times (\varpi_1^A)_*[x_i]\Big) \times [0]^{\times(n-1)}$$

in $\mathrm{CH}_0(A^{n+1})$. Thus if $x_1, \dots, x_{n+1}$ are closed points in A such that $x_1 + \cdots + x_{n+1} = 0$ in A, then $(\varpi_{1,1})_*[x]$ is a non-zero multiple of the symmetrization of

$$\sum_i(\varpi_1^A)_*[x_i] \times (\varpi_1^A)_*[x_i] \times [0]^{\times(n-1)} = \Big((\varpi_1^A \times \varpi_1^A)_*\delta_* \sum_i[x_i]\Big) \times [0]^{\times(n-1)},$$

where $\delta : A \hookrightarrow A \times A$ is the diagonal embedding. It then clearly follows that if $\sum_i[x_i] = (n+1)[0]$, then $(\varpi_{1,1})_*[x]$ vanishes and hence that $(\varpi_2^{A^{n+1}})_*[x]$ vanishes. $\square$

2.2.3. *Fano varieties of lines on cubic fourfolds.* Let Y be a smooth cubic hypersurface in $\mathbb{P}^5_{\mathbb{C}}$ and let $X = F(Y)$ be the Fano variety of lines on Y; it is a hyper-Kähler variety of dimension 4 polarized by the restriction g of the Plücker polarization on the Grassmannian $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^5)$. By abuse we will denote both by l a line in Y and the corresponding point in X.

Theorem 2.8. *Let $X = F(Y)$ be the Fano variety of lines on a smooth cubic fourfold Y. Then*

$$[l] = [l'] \text{ in } \mathrm{CH}_0(F(Y)) \iff [l] = [l'] \text{ in } \mathrm{CH}_1(Y)$$

and, for $m > 1$,

$$\sum_{i=1}^m[l_i] = \sum_{i=1}^m[l'_i] \text{ in } \mathrm{CH}_0(F(Y)) \iff \begin{cases} \sum_{i=1}^m[l_i] = \sum_{i=1}^m[l'_i] \text{ in } \mathrm{CH}_1(Y), \text{ and} \\ \sum_{i=1}^m[l_i] \times [l_i] = \sum_{i=1}^m[l'_i] \times [l'_i] \text{ in } \mathrm{CH}_2(Y \times Y). \end{cases}$$

Proof. Recall from [SV16, Thm. 21.9] that the action of Voisin's birational map $\varphi : X \dashrightarrow X$ on $\mathrm{CH}_0(X)$ diagonalizes. We then consider the birational idempotent correspondence ϖ on $\mathfrak{h}^\circ(X)$ given by the projector with respect to the eigenspace decomposition of φ on the eigenspace corresponding to the eigenvalue -2. By [Via22, Thm. 5.5(iv)], the canonical morphism $\mathfrak{h}^\circ(X) \to \mathrm{Sym}^{\leq 2}\mathfrak{h}^\circ_\varpi(X)$ is an isomorphism.

Let now $Z =_{\mathrm{def}} \{(y, l) \in Y \times X \mid y \in l\}$ be the universal line. By the Franchetta conjecture for $X \times X$ [FLV19], ϖ is in fact the restriction to the generic point of the idempotent correspondence

$$\mathfrak{h}^6(X)_{\mathrm{prim}} \xrightarrow{Z_*} \mathfrak{h}^4(Y)_{\mathrm{prim}}(-1) \xrightarrow{-\frac{1}{6}(p_2^*g^2)\circ Z^*} \mathfrak{h}^6(X)_{\mathrm{prim}}.$$

Moreover both arrow are isomorphisms of Chow motives. Here $\mathfrak{h}^4(Y)_{\mathrm{prim}}$ is the direct summand of the Chow motive $\mathfrak{h}(Y)$ cut out by the idempotent $p =_{\mathrm{def}} \Delta_Y - \frac{1}{3}\sum_{i=0}^4 h^i \times h^{4-i}$ with $h \in \mathrm{CH}^1(Y)$ the hyperplane class, and $\mathfrak{h}^6(X)_{\mathrm{prim}}$ is the direct summand of $\mathfrak{h}(X)$ cut out by the idempotent correspondence $Z_* \circ p \circ \big(-\frac{1}{6}(p_2^*g^2) \circ Z^*\big)$. We can then conclude by Proposition 2.1 after noting that $p \circ Z_*[l] = [l] - [l_0]$ where l_0 is any line on Y with class $\frac{1}{3}h^3$. $\square$

3. The Voisin filtration

In this section we show that Main Conjecture 1.4 holds unconditionally with respect to the Beauville candidate for the Bloch–Beilinson filtration for abelian varieties, and present some cases in which Main Conjecture 1.4 is satisfied conditional on well-known conjectures on algebraic cycles with respect to the Voisin candidate for the Bloch–Beilinson filtration.

3.1. **Main Conjecture 1.4 for abelian varieties.** In [Voi22], Voisin shows that given an abelian variety A, a desingularization of the quotient $A/\pm$ satisfies Voisin's Conjecture 1.3. This is essentially equivalent to the fact that abelian varieties satisfy Main Conjecture 1.4 for $m = 2$, and the proof can be adapted to show that the following conjecture of Nori implies Main Conjecture 1.4 for abelian varieties.

Nori's Conjecture 3.1 ([Nor93])**.** *Let X be a smooth projective variety and $w \in \mathrm{CH}^i(X)$ a cycle such that $w|_T = 0 \in \mathrm{CH}_0(T)$ for any i-fold $T \subset X$. Then $w = 0 \in \mathrm{CH}^i(X)$.*

Proposition 3.2. *Main Conjecture 1.4 holds for abelian varieties and $m = 2$. Moreover, Nori's Conjecture 3.1 for an abelian g-fold A implies Main Conjecture 1.4 for A.*

Proof. Suppose
$$[a] + [a'] = [b] + [b'] \text{ in } \mathrm{CH}_0(A)/F^3_{\mathrm{V}}\mathrm{CH}_0(A).$$
Then $a + a' = b + b'$ so, translating by $-a - a'$ if needed, we can assume that $a' = -a$, $b' = -b$. Voisin shows in [Voi22, Prop. 2.17] that
$$[a] + [-a] = [b] + [-b] \text{ in } \mathrm{CH}_0(A)/F^3_{\mathrm{BB}}\mathrm{CH}_0(A) \iff [a] + [-a] = [b] + [-b] \text{ in } \mathrm{CH}_0(A).$$

For the general case we can follow a similar argument. The main differences are the use of the fundamental theorem on symmetric polynomials and the substitution of Nori's Conjecture 3.1 in place of a theorem of Joshi [Jos95]. Suppose that
$$\sum_{i=1}^{m}[x_i] = \sum_{i=1}^{m}[y_i] \text{ in } \mathrm{CH}_0(A)/F^{m+1}_{\mathrm{BB}}\mathrm{CH}_0(A). \tag{3.1}$$
Let Θ be an ample divisor that gives an isogeny
$$\begin{array}{rcl} A & \longrightarrow & \widehat{A} \\ x & \longmapsto & D_x =_{\mathrm{def}} \Theta_x - \Theta. \end{array}$$
The map $x \mapsto D_x^j \in \mathrm{CH}^j(A)$ is a given by a correspondence in $\mathrm{CH}^j(A \times A)$. Hence, assuming Nori's Conjecture 3.1, we can use Lemma 3.5 and (3.1) to deduce that
$$\sum_{i=1}^{m} D^j_{x_i} = \sum_{i=1}^{m} D^j_{y_i}, \qquad \forall j \in \{1, \dots, m\}.$$
The fundamental theorem on symmetric polynomials then implies that
$$\sum_{i=1}^{m} D^j_{x_i} = \sum_{i=1}^{m} D^j_{y_i} \text{ in } \mathrm{CH}^j(A), \qquad \forall j \in \mathbb{N}.$$
The Chow ring of A has two ring structures, one given by intersection and the other by the Pontryagin product:
$$\begin{array}{rcl} \mathrm{CH}^\bullet(A) \times \mathrm{CH}^\bullet(A) & \longrightarrow & \mathrm{CH}^\bullet(A) \\ (\alpha, \beta) & \longmapsto & \sigma_*(\alpha \times \beta), \end{array}$$
where $\sigma : A \times A \longrightarrow A$ is the summation map. A formula of Beauville then gives
$$\sum_{i=1}^{m} \frac{\Theta^{g-j}}{(g-j)!} D^j_{x_i} = \sum_{i=1}^{m} \frac{\Theta^g}{g!} * \gamma(x_i)^{*j} \text{ in } \mathrm{CH}_0(A), \qquad \forall j \in \mathbb{N},$$
where
$$\gamma(x) = \sum_{k=1}^{n} \frac{1}{k}([x] - [0_A])^{*k}$$

is the logarithm of $[x]$. Since $\exp(\gamma(x)) = [x]$ and we can assume that $\Theta^g/g! = d[0_A]$ for some positive integer d, we get

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m \exp(\gamma(x_i)) = \sum_{i=1}^m \exp(\gamma(y_i)) = \sum_{i=1}^m [y_i] \in \mathrm{CH}_0(A).$$

□

Instead of considering the filtration $F^\bullet_{\mathrm{V}}$, we can consider the Beauville filtration $F^\bullet_{\mathrm{B}}$ on the Chow ring of an abelian variety (see [Bea86]) to obtain an unconditional proof of an analogue of Main Conjecture 1.4:

Theorem 3.3. *Let A be an abelian g-fold and let $x_1, \dots, x_m, y_1, \dots, y_m \in A$,*

$$\sum_{i=1}^m [x_i] = \sum_{i=1}^m [y_i] \text{ in } \mathrm{CH}_0(A) \iff \sum_{i=1}^m [x_i] - \sum_{i=1}^m [y_i] \text{ lies in } \bigoplus_{s=m+1}^{g} \mathrm{CH}^g_{(s)}(A).$$

Proof. The map $x \mapsto D^j_x \in \mathrm{CH}^j(A)$ as above is a given by a correspondence $\Gamma_j \in \mathrm{CH}^j(A \times A)$. Since $\mathrm{CH}^i_{(s)}(A) = 0$ for all $s > i$,

$$\Gamma_{j*}(\mathrm{CH}^g_{(s)}(A)) = 0, \qquad \forall s \geq j.$$

Thus, if $\sum_{i=1}^m [x_i] - \sum_{i=1}^m [y_i] \in \bigoplus_{s=m+1}^g \mathrm{CH}^g_{(s)}(A)$,

$$\sum_{i=1}^m D^j_{x_i} = \sum_{i=1}^m D^j_{y_i}, \qquad \forall j \in \{1, \dots, m\}.$$

We can then apply the same reasoning as in the proof of Proposition 3.2. □

Remark 3.4. Let $\mathfrak{h}(A) = \bigoplus_{i=0}^{2g} \mathfrak{h}_i(A)$, with $\mathfrak{h}_i(A) = (A, \varpi_i)$, be the Deninger–Murre decomposition of the Chow motive of A. If one takes ϖ to be $\varpi_1|_{\eta_A \times A}$, where η_A is the generic point of A, then by Künnemann [Kün94] $\mathfrak{h}^\circ(A)$ is co-generated by $\mathfrak{h}^\circ_\varpi(A)$ and the filtration $F^\bullet_\varpi$ is the Beauville filtration, so that Proposition 2.2 recovers Theorem 3.3.

3.2. **Deducing Main Conjecture 1.4 for hyper-Kähler varieties from well-known conjectures on algebraic cycles.** In [Voi22], the author shows how the nilpotence conjecture implies Voisin's Conjecture 1.3 for hyper-Kähler varieties satisfying the Lefschetz standard conjecture in degree 2. In [Voi22, Rem. 2.14] she specifies that her proof does not imply a stronger version of the statement of Main Conjecture 1.4, where the depth in the Bloch–Beilinson does not depend on the degree m of the effective zero-cycles. In short the reason is that $(s-t) \in \mathbb{Z}[s,t]$ divides $(s^n - t^n)$ for all $n > 0$, whereas

$$\sum_{i=1}^m s_i - \sum_{i=1}^m t_i \text{ in } \mathbb{Q}[s_1, \dots, s_m, t_1, \dots, t_m]$$

need not divide

$$\sum_{i=1}^m s_i^n - \sum_{i=1}^m t_i^n.$$

Nonetheless, one can adapt Voisin's argument using the fundamental theorem on symmetric polynomials and Nori's Conjecture 3.1. The main consequence of Nori's Conjecture 3.1 we will need is the following:

Lemma 3.5 ([Voi22, Lem. 2.8])**.** *If Nori's Conjecture 3.1 holds, then for any smooth projective varieties X, Y, correspondence $\Gamma \in \mathrm{CH}^i(X \times Y)$, and $w \in F^{i+1}_{\mathrm{BB}}\mathrm{CH}_0(X)$, we have $\Gamma_* w = 0$.*

Proposition 3.6. *Let X be a hyper-Kähler variety which satisfies Nori's Conjecture 3.1 and the Lefschetz standard conjecture in degree 2. Then the nilpotence conjecture implies Main Conjecture 1.4 for X.*

Proof. This proof is a simple adaptation of Voisin's argument in Section 2.2 of [Voi22] and we use the same notation. X is a smooth projective hyper-Kähler variety of dimension $2n$ and we fix a polarizing class $h_X \in \mathrm{Pic}(X)$. We let $Z_{\mathrm{lef}} \in \mathrm{CH}^2(X \times X)$ be a cycle whose existence is predicted by the Lefschetz standard conjecture in degree 2, namely such that $[Z_{\mathrm{lef}}]^* : H^{4n-2}(X) \longrightarrow H^2(X)$ is the inverse of the cup product map with h_X^{2n-2}. As shown in [Voi22], there is a polynomial in Z_{lef} and $\mathrm{pr}_2^* h_X$

$$P \;=\; \sum_{i=0}^{n} \mu_i Z_{\mathrm{lef}}^i \cdot \mathrm{pr}_2^* h_X^{2n-2i} \text{ in } \mathrm{CH}^{2n}(X \times X)$$

such that $[P]^*$ acts as the identity on holomorphic forms. Reasoning as in [Voi22] and assuming the nilpotence conjecture, one sees that the map $P_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(X)$ is the identity.

Suppose that $x_1, \ldots, x_m, y_1, \ldots, y_m \in X$ and

$$\sum_{i=1}^{m} [x_i] - \sum_{i=1}^{m} [y_i] \in F_{\mathrm{V}}^{2m+1} \mathrm{CH}_0(X).$$

Then by Nori's Conjecture 3.1

$$\sum_{i=1}^{m} (Z_{\mathrm{lef}}^j)_{x_i} \;=\; \sum_{i=1}^{m} (Z_{\mathrm{lef}}^j)_{y_i} \text{ in } \mathrm{CH}^{2i}(X), \qquad j = 1, \ldots, m.$$

Here, for a correspondence $Z \in \mathrm{CH}^i(X \times X)$ and a closed point $x \in X$, we write Z_x for $Z_*[x]$. Denoting by $\iota_x : X \longrightarrow X \times X$ the embedding given by $\iota_x(y) = (x, y)$, we have

$$\begin{aligned} Z_{\mathrm{lef}}^j \cdot \{x\} \times X = Z_{\mathrm{lef}}^j \cdot \iota_{x*}(X) &= \iota_{x*}(\iota_x^*(Z_{\mathrm{lef}}^j)) = \iota_{x*}(\iota_x^*(Z_{\mathrm{lef}})^j) \\ &= \iota_{x*}\big((Z_{\mathrm{lef},x})^j\big) = \mathrm{pr}_1^*(\{x\}) \cdot \mathrm{pr}_2^*\big((Z_{\mathrm{lef},x})^j\big). \end{aligned}$$

Accordingly,

$$(Z_{\mathrm{lef}}^j)_x = (Z_{\mathrm{lef},x})^j, \qquad j = 0, \ldots, n.$$

This gives

$$\sum_{i=1}^{m} (Z_{\mathrm{lef},x_i})^j = \sum_{i=1}^{m} (Z_{\mathrm{lef},y_i})^j, \qquad j = 0, \ldots, m.$$

By the fundamental theorem on symmetric polynomials the same equality holds for all j and thus

$$\sum_{i=1}^{m} (Z_{\mathrm{lef}}^j)_{x_i} = \sum_{i=1}^{m} (Z_{\mathrm{lef}}^j)_{y_i}, \qquad j = 0, \ldots, n.$$

Finally, we conclude that

$$\sum_{i=1}^{m} [x_i] = P_*\left(\sum_{i=1}^{m} [x_i]\right) = P_*\left(\sum_{i=1}^{m} [y_i]\right) = \sum_{i=1}^{m} [y_i] \text{ in } \mathrm{CH}_0(X).$$

□

4. Polynomial decompositions of the diagonal

4.1. **Definition and examples.** In [Voi22], Voisin formulates the following conjecture, which implies Voisin's Conjecture 1.3 for hyper-Kähler varieties with respect to the Voisin filtration; see [Voi22, Prop. 2.15].

Conjecture 4.1 ([Voi22, Conj. 2.16]). *Consider a smooth projective hyper-Kähler $2n$-fold X, a polarization h_X, and a cycle $Z_{\mathrm{lef}} \in \mathrm{CH}^2(X \times X)$ such that $[Z_{\mathrm{lef}}]^*$ is the inverse of the cup product with h_X^{2n-2}. There are cycles $\gamma_i \in \mathrm{CH}^{2n-2i}(X)$, $i = 0, \ldots, n$, a divisor $D \subset X$, and a cycle $W \in \mathrm{CH}^{2n}(X \times X)$ supported on $D \times X$ such that*

$$\Delta_X = \sum_{i=0}^{n} Z_{\mathrm{lef}}^i \cdot \mathrm{pr}_2^*(\gamma_i) + W \in \mathrm{CH}^{2n}(X \times X).$$

This conjecture suggests the following definition:

Definition 4.2. A smooth projective variety X admits a *degree l polynomial decomposition of the diagonal up to coniveau c* if
$$\Delta_X = Z_1 + Z_2 \in \mathrm{CH}^n(X \times X),$$
where Z_1 belongs to the subalgebra of $\mathrm{CH}^\bullet(X \times X)$ generated in degree at most l and Z_2 is supported on $Y \times X$ for some closed $Y \subset X$ of codimension c. If c can be taken to be positive, we say that X has a *degree l polynomial decomposition of the diagonal.*

Examples 4.3.

(i) $\mathbb{P}^n$ has a degree 1 polynomial decomposition of the diagonal. Indeed, writing h for $c_1(\mathcal{O}(1))$, we have
$$\Delta_{\mathbb{P}^n} = \sum_{i=0}^{n} \mathrm{pr}_1^*(h^i) \cdot \mathrm{pr}_2^*(h^{n-i}) \in \mathrm{CH}^n(\mathbb{P}^n \times \mathbb{P}^n).$$

(ii) A variety with a rational decomposition of the diagonal in the sense of Bloch-Srinivas [BS83] has a polynomial decomposition of the diagonal in degree 1 up to coniveau 1.

(iii) Curves have a degree 1 polynomial decomposition of the diagonal. More generally, any n-fold has a degree n polynomial decomposition of the diagonal.

(iv) A surface X with $p_g(X) = 0$ has a degree 1 polynomial decomposition of the diagonal if and only if it satisfies Bloch's conjecture. Murre [Mur90] defines a decomposition of the motive $h(X)$
$$h(X) = \sum_{i=0}^{4} h^i(X).$$
The motives $h^i(X),\ i \neq 2$ are cut out by idempotent correspondences which are products of divisors. The motive $h^2(X)$ further breaks up as a sum $h^2_{\mathrm{alg}}(X) + h^2_{\mathrm{tr}}(X)$ [KMP07]. If Bloch's conjecture holds for X, then $h^2_{\mathrm{tr}}(X) = 0$ and thus Δ is a polynomial in divisor classes. We will explain the converse in Proposition 4.9.

(v) An abelian variety A has a degree 1 polynomial decomposition of the diagonal. Indeed, the diagonal of A is a rational multiple of $f^*(\Theta)^{\dim A}$, where $f : A^2 \longrightarrow A$ is given by $f(a, b) = a - b$, and Θ is a symmetric ample divisor.

(vi) If $X_1, \dots, X_r$ have polynomial decompositions of degree $l_1, \dots, l_r$ up to coniveau $c_1, \dots, c_r$ then $X_1 \times \cdots \times X_r$ has a degree $\max_{1 \leq i \leq r}(l_i)$ polynomial decomposition of the diagonal up to coniveau $\max_{1 \leq i \leq r}(c_i)$.

Example 4.4. Let X be a smooth projective variety with $H^0(X, \Omega^1) = 0$. Since
$$\mathrm{Pic}(X \times X) = \mathrm{pr}_1^*\mathrm{Pic}(X) \oplus \mathrm{pr}_2^*\mathrm{Pic}(X),$$
X has a degree 1 polynomial decomposition decomposition of the diagonal up to coniveau 1 if and only if X has a rational decomposition of the diagonal. By Proposition 1 of [BS83] this is the case if and only if the degree map $\mathrm{CH}_0(X) \longrightarrow \mathbb{Q}$ is an isomorphism.

The generalized Bloch conjecture predicts when the degree map $\mathrm{CH}_0(X) \longrightarrow \mathbb{Q}$ is an isomorphism.

Conjecture 4.5 (generalized Bloch conjecture, see e.g. [Voi02])**.** *Let X be a smooth projective n-fold such that $H^{p,q}(X) = 0$ for all $p \neq q$ and $p < c$. Then the cycle class map*
$$cl : \mathrm{CH}_i(X) \longrightarrow H^{2n-2i}(X, \mathbb{Q})$$
is injective for all $i < c$.

4.2. **Conjecture and relationship with the generalized Bloch conjecture.** Example 4.4 explains how the generalized Bloch conjecture predicts that a variety with $H^0(X, \Omega^1) = 0$ has a degree 1 polynomial decomposition of the diagonal up to coniveau 1 if and only if
$$H^k(X, \mathbb{Q}) = N^1_H H^k(X, \mathbb{Q}) \qquad \forall k > 0,$$
where $N^\bullet_H$ denotes the Hodge coniveau filtration. Recall that $N^r_H H^\bullet(X, \mathbb{Q})$ is by definition the largest Hodge substructure $V \subset H^\bullet(X, \mathbb{Q})$ which has coniveau at least r, namely such that $V^{p,q}_{\mathbb{C}} = 0$ if $p < r$.

This provides evidence that a polynomial decomposition of the diagonal can be detected by Hodge theory.

Conjecture 4.6. *Let X be a smooth projective n-fold. Then X admits a degree l polynomial decomposition of the diagonal up to coniveau c if and only if*

$$N^r_H H^\bullet(X,\mathbb{Q})/N^{r+1}_H H^\bullet(X,\mathbb{Q})$$

is generated in degree at most $l+r$ for all $r<c$.

Remark 4.7. It suffices to check this in degree $\leq n$ as the hard Lefschetz theorem then implies the same generation statement in high degree. Example 4.3 (4) illustrates why the shift in degree of generation according to coniveau is necessary. For example, a smooth cubic surface has a degree 1 polynomial decomposition of the diagonal but its primitive cohomology is not generated in degree 1.

As is often the case, it is easy to deduce Hodge-theoretic information from cycle-theoretic information:

Proposition 4.8. *If X has a degree l decomposition of the diagonal up to coniveau c the algebra*

$$N^r_H H^\bullet(X,\mathbb{Q})/N^{r+1}_H H^\bullet(X,\mathbb{Q})$$

is generated in degrees at most $l+r$ for all $r<c$.

Proof. Observe that the subspace of $H^d(X,\mathbb{Q})$ generated in degrees at most $l+r$ is a sub-Hodge structure whose complexification is the subspace of $H^d(X,\mathbb{C})$ generated in degrees at most $l+r$. Consider a simple Hodge structure

$$V\subset N^r_H H^d(X,\mathbb{Q})$$

which is not contained in $N^{r+1}_H H^d(X,\mathbb{Q})$ and cycles $W_1,\dots,W_k\subset X\times X$ of codimension $d_1,\dots,d_k\leq l$ with

$$n=_{\mathrm{def}}\dim X=d_1+\cdots+d_k.$$

For each i, decompose the cycle class of W_i into Künneth components

$$[W_i]=\sum_{a_i,b_i=0}^{d_i} v_i^{a_i,b_i}\otimes w_i^{d_i-a_i,d_i-b_i},$$

where

$$v_i^{a_i,b_i}\otimes w_i^{d_i-a_i,d_i-b_i}\in H^{a_i,b_i}(X)\otimes H^{d_i-a_i,d_i-b_i}(X).$$

Given a non-zero class $\alpha\in V^{r,d-r}_{\mathbb{C}}$, we have

$$[W_1\cdots W_k]^*\alpha=\sum_{|\mathbf{a}|=r,|\mathbf{b}|=d-r}\left[\prod_{i=1}^k(v_i^{a_i,b_i}\otimes w_i^{d_i-a_i,d_i-b_i})\right]^*\alpha$$

where the sum is over tuples $\mathbf{a}=(a_1,\dots,a_k)$, $\mathbf{b}=(b_1,\dots b_k)$ with sums r and $d-r$ respectively. Since

$$(v_1^{a_1,b_1}\otimes w_1^{d_1-a_1,d_1-b_1}\cdots v_k^{a_k,b_k}\otimes w_k^{d_k-a_k,d_k-b_k})^*\alpha$$

is a multiple of $v_1^{a_1,b_1}\cdots v_k^{a_k,b_k}$ and $a_i\leq r$, $b_i\leq d_i\leq l$, we must have $a_i+b_i\leq l+r$. This shows that $[W_1\cdots W_k]^*\alpha$ is in the subspace of $H^d(X,\mathbb{C})$ generated in degree $\leq l+r$, and thus that V is contained in the subspace of $H^d(X,\mathbb{Q})$ generated in degree $\leq l+r$. □

Proposition 4.9. *Let X be a smooth projective n-fold with $H^0(X,\Omega^p)=0$ for all $p\geq 2$ and set $F^2\mathrm{CH}_0(X)$ to be either $F^2_{\mathrm{V}}\mathrm{CH}_0(X)$ or the kernel of the albanese map on zero-cycles of degree 0. If X has a degree 1 polynomial decomposition of the diagonal up to coniveau 1 then $F^2\mathrm{CH}_0(X)=0$. In particular, Conjecture 4.6 for $c=1$ implies the generalized Bloch conjecture for $c=1$.*

Proof. Since X has a degree 1 polynomial decomposition of the diagonal up to coniveau 1 we can write

$$\Delta_X=Z_1+Z_2\in\mathrm{CH}^n(X\times X),$$

where Z_1 is a polynomial in divisors and Z_2 is supported on $Y\times X$ for some divisors $Y\subset X$. Accordingly,

$$\mathrm{id}_{\mathrm{CH}_0(X)}=\Delta_{X*}=Z_{1*}:\mathrm{CH}_0(X)\longrightarrow\mathrm{CH}_0(X).$$

We will show that any monomial $D_1\cdots D_n$, where $D_i\in\mathrm{Pic}(X\times X)$, gives a zero map on $F^2\mathrm{CH}_0(X)$ which will imply that $F^2\mathrm{CH}_0(X)=0$. Observe that

$$\mathrm{Pic}(X\times X)=\mathrm{pr}_1^*\mathrm{Pic}(X)\oplus\mathrm{pr}_2^*\mathrm{Pic}(X)\oplus H,$$

where H is the group that consists of pullbacks of divisors classes on $\mathrm{Alb}(X)\times\mathrm{Alb}(X)$ which have trivial restriction to the factors.

Given a monomial $D_1\cdots D_n$, if one of the D_i is in $\mathrm{pr}_1^*\mathrm{Pic}(X)$ then the map

$$(D_1\cdots D_n)_*:\mathrm{CH}_0(X)\longrightarrow\mathrm{CH}_0(X)$$

is identically zero. Similarly, if at least $n-1$ of the D_i belong to $\mathrm{pr}_2^*\mathrm{Pic}(X)$ then the same map factors through the Chow group of zero-cycles on a variety of dimension 1 so that the induced map

$$(D_1\cdots D_n)_*:F^2\mathrm{CH}_0(X)\longrightarrow F^2\mathrm{CH}_0(X)$$

is identically zero.

Finally, suppose that $D_1,\dots,D_i$ belong to $\mathrm{pr}_2^*\mathrm{Pic}(X)$ and that $D_{i+1},\cdots,D_n$ are pulled back from divisors $D'_{i+1},\dots,D'_n$ on $\mathrm{Alb}(X)\times\mathrm{Alb}(X)$ under $\alpha\times\alpha$, where $\alpha:X\longrightarrow\mathrm{Alb}(X)$ is the Albanese morphism. Since $H^0(X,\Omega^2)=0$, the Albanese dimension of X is at most 1. We can assume that the image of X in its Albanese is a curve C, or else $H=0$ and we are done by the considerations above. Since $D_{i+1}\cdot D_n=0$ if $i<n-2$, it suffices to consider the case $i=n-2$. Then $D_{n-1}\cdot D_n$ is the pullback of a zero cycle on $C\times C$. Since the pullback of a point on $C\times C$ is the product of a divisor in $\mathrm{pr}_1^*\mathrm{Pic}(X)$ and a divisor in $\mathrm{pr}_2^*\mathrm{Pic}(X)$, we see that $D_1\cdots D_n$ induces the zero map on $\mathrm{CH}_0(X)$. □

4.3. **Relation between Conjecture 4.6 and Main Conjecture 1.4.** The following proposition shows that Conjecture 4.6 for $c=1$ along with Nori's Conjecture 3.1 implies Main Conjecture 1.4 using Voisin's filtration as a candidate Bloch–Beilinson filtration.

Proposition 4.10. *Assume that X has a degree l polynomial decomposition of the diagonal and that X satisfies Nori's Conjecture 3.1. Then, for closed point $x_1,\dots,x_m,y_1,\dots,y_m\in X$,*

$$\sum_{i=1}^m[x_i]=\sum_{i=1}^m[y_i]\ \textit{in}\ \mathrm{CH}_0(X)\quad\Longleftrightarrow\quad\sum_{i=1}^m[x_i]=\sum_{i=1}^m[y_i]\ \textit{in}\ \mathrm{CH}_0(X)/F_{\mathrm{V}}^{ml+1}\mathrm{CH}_0(X).$$

Proof. By assumption $\Delta_X=Z_1+Z_2$, where Z_1 is in the subalgebra of $\mathrm{CH}^\bullet(X\times X)$ generated in degrees $\le l$, and Z_2 is supported on some proper closed subset $Y\subset X$. Hence,

$$\Delta_{X*}=Z_{1*}:\mathrm{CH}_0(X)\longrightarrow\mathrm{CH}_0(X).$$

Consider cycles $W_1,\dots,W_n$ in $\bigoplus_{i\le l}\mathrm{CH}^i(X\times X)$ generating a subalgebra containing Z_1. Consider $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{N}^n\setminus\{\mathbf{0}\}$ with $a_1+\cdots+a_n\le m$. The cycle

$$P_{\mathbf{a}}=_{\mathrm{def}}W_1^{a_1}\cdots W_n^{a_n}$$

has codimension at most ml. Given $x_1,\dots,x_m,y_1,\dots,y_m\in X$ such that

$$\sum_{i=1}^m[x_i]-\sum_{i=1}^m[y_i]\ \text{ lies in }F_{\mathrm{V}}^{ml+1}\mathrm{CH}_0(X),$$

Nori's Conjecture 3.1 and Lemma 3.5 imply that

$$\sum_{i=1}^m P_{\mathbf{a},x_i}=\sum_{i=1}^m P_{\mathbf{a},y_i}.$$

The same argument as in the proof of Proposition 3.6 shows that

$$P_{\mathbf{a},x_i}=W_{1,x_i}^{a_1}\cdots W_{n,x_i}^{a_n}\in\mathrm{CH}^\bullet(X),$$

so that

$$\sum_{i=1}^m W_{1,x_i}^{a_1}\cdots W_{n,x_i}^{a_n}=\sum_{i=1}^m W_{1,y_i}^{a_1}\cdots W_{n,y_i}^{a_n}\in\mathrm{CH}^\bullet(X).\tag{4.1}$$

In order to proceed, we will need a generalization of the fundamental theorem on symmetric polynomials. Consider variables $w_j^{(i)}$ where $1\le j\le n$ and $1\le i\le m$. The symmetric group $\mathfrak{S}_m$ acts on the ring $\mathbb{Q}[w_j^{(i)}]$ by permuting the superscript and the subalgebra $\mathbb{Q}[w_j^{(i)}]^{\mathfrak{S}_m}$ is called the subalgebra of multi-symmetric polynomials. Given an element $\mathbf{a} =_{\mathrm{def}} (a_1, \ldots, a_n) \in \mathbb{N}^n \setminus \mathbf{0}$, the corresponding power sum multisymmetric polynomial is

$$p_{\mathbf{a}}(w_j^{(i)}) = \sum_{s=1}^{m} {w_1^{(s)}}^{a_1} \cdots {w_n^{(s)}}^{a_n} \in \mathbb{Q}[w_j^{(i)}]^{\mathfrak{S}_m}.$$

It is a classical fact that $\mathbb{Q}[w_j^{(i)}]^{\mathfrak{S}_m}$ is generated by elementary multisymmetric power sums (see the references in the introduction of [Bri04]).

Proposition 4.11 ([Bri04, Cor. 5])**.** *All power sums are in the ideal generated by the powers sums $p_{\mathbf{a}}(w_j^{(i)})$ with $a_1 + \cdots + a_n \leq m$.*

Now consider two morphisms $f, g : \mathbb{Q}[w_j^{(i)}] \longrightarrow \mathrm{CH}^{\bullet}(X)$ given respectively by $f(w_j^{(i)}) = W_{j,x_i}$ and $g(w_j^{(i)}) = W_{j,y_i}$. The equality (4.1) amounts to $f(p_{\mathbf{a}}(w_j^{(i)})) = g(p_{\mathbf{a}}(w_j^{(i)}))$ for all $\mathbf{a} = (a_1, \ldots, a_n)$ in $\mathbb{N}^n \setminus \{\mathbf{0}\}$ satisfying $a_1 + \cdots + a_n \leq m$. Proposition 4.11 then implies that $f(p_{\mathbf{a}}(w_j^{(i)})) = g(p_{\mathbf{a}}(w_j^{(i)}))$, namely

$$\sum_{i=1}^{m} P_{\mathbf{a},x_i} = \sum_{i=1}^{m} P_{\mathbf{a},y_i} \in \mathrm{CH}^{\bullet}(X),$$

for any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n \setminus \{\mathbf{0}\}$. Since Z_1 is a polynomial in the W_i, it follows that

$$\sum_{i=1}^{m} [x_i] = \sum_{i=1}^{m} Z_{1,x_i} = \sum_{i=1}^{m} Z_{1,y_i} = \sum_{i=1}^{m} [y_i] \ \text{ in } \mathrm{CH}_0(X),$$

thereby concluding the proof of the proposition. □

References

[Bea86] A. Beauville. Sur l'anneau de Chow d'une variété abélienne. *Math. Ann.*, 273:647–651, 1986.

[Bri04] Emmanuel Briand. When is the algebra of multisymmetric polynomials generated by the elementary multisymmetric polynomials? *Beiträge Algebra Geom.*, 45(2):353–368, 2004.

[BS83] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105 (5):1235–1253, 1983.

[Flo23] Salvatore Floccari. On the motive of O'Grady's six dimensional hyper-Kähler varieties. *Épijournal de Géométrie Algébrique*, Volume 7, February 2023.

[FLV19] Lie Fu, Robert Laterveer, and Charles Vial. The generalized Franchetta conjecture for some hyper-Kähler varieties. *J. Math. Pures Appl. (9)*, 130:1–35, 2019. With an appendix by the authors and Mingmin Shen.

[FTV19] Lie Fu, Zhiyu Tian, and Charles Vial. Motivic hyper-Kähler resolution conjecture, I: generalized Kummer varieties. *Geom. Topol.*, 23(1):427–492, 2019.

[Jos95] K. Joshi. A Noether-Lefschetz theorem and applications. *J. Algebraic Geom.*, 4:105–135, 1995.

[KMP07] Bruno Kahn, Jacob P. Murre, and Claudio Pedrini. On the transcendental part of the motive of a surface. In *Algebraic cycles and motives. Vol. 2*, volume 344 of *London Math. Soc. Lecture Note Ser.*, pages 143–202. Cambridge Univ. Press, Cambridge, 2007.

[KS16] B. Kahn and S. Sujatha. Birational motives, I: pure birational motives. *Ann. K-Theory*, 1(14):379–440, 2016.

[Kün94] Klaus Künnemann. On the Chow motive of an abelian scheme. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 189–205. Amer. Math. Soc., Providence, RI, 1994.

[Lin18] Hsueh-Yung Lin. Corrigendum to "On the Chow group of zero-cycles of a generalized Kummer variety" [Adv. Math. 298 (2016) 448–472] [MR3505747]. *Adv. Math.*, 331:1016–1021, 2018.

[MRS18] Giovanni Mongardi, Antonio Rapagnetta, and Giulia Saccà. The Hodge diamond of O'Grady's six-dimensional example. *Compos. Math.*, 154(5):984–1013, 2018.

[Mur90] J.P. Murre. On the motive of an algebraic surface. *J. Reine Angew. Math.*, 409:190–204, 1990.

[Mur93] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.

[MZ20] A. Marian and X. Zhao. On the group of zero-cycles of holomorphic symplectic varieties. *Épijournal Géom. Algébrique*, 4, 2020.

[Nor93] M. V. Nori. Algebraic cycles and Hodge theoretic connectivity. *Invent. Math.*, 111(2):349–374, 1993.

[O'G03] Kieran G. O'Grady. A new six-dimensional irreducible symplectic variety. *J. Algebraic Geom.*, 12(3):435–505, 2003.

[O'G13] Kieran G. O'Grady. Moduli of sheaves and the Chow group of $K3$ surfaces. *J. Math. Pures Appl. (9)*, 100(5):701–718, 2013.

[SV16] Mingmin Shen and Charles Vial. The Fourier transform for certain hyperkähler fourfolds. *Mem. Amer. Math. Soc.*, 240(1139):vii+163, 2016.

[SYZ20] Junliang Shen, Qizheng Yin, and Xiaolei Zhao. Derived categories of $K3$ surfaces, O'Grady's filtration, and zero-cycles on holomorphic symplectic varieties. *Compos. Math.*, 156(1):179–197, 2020.

[Via22] Charles Vial. On the birational motive of hyper-Kähler varieties. *J. Math. Pures Appl.*, 163:577–624, 2022.

[Voi02] Claire Voisin. *Théorie de Hodge et géométrie algébrique complexe*, volume 10 of *Cours Spécialisés*. Société Mathématique de France, 2002.

[Voi04] Claire Voisin. Remarks on filtrations on Chow groups and the Bloch conjecture. *Ann. Mat. Pura Appl. (4)*, 183:421–438, 2004.

[Voi16] Claire Voisin. Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties. In *K3 Surfaces and Their Moduli, Proceedings of the Schiermonnikoog conference 2014*, volume 315 of *Progress in Math 315*, pages 365–399, 2016.

[Voi22] Claire Voisin. On the Lefschetz standard conjecture for Lagrangian covered hyper-Kähler varieties. *Adv. Math.*, 396, 2022.

Stony Brook University, NY, USA
Email address: olivier.martin@stonybrook.edu

Universität Bielefeld, Germany
Email address: vial@math.uni-bielefeld.de