J. reine angew. Math. [redacted] (2007), 1—21
DOI 10.1515/CRELLE.2007.[redacted]

Journal für die reine und angewandte Mathematik

Berlin · New York 2007

T-spectra and Poincaré duality

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Abstract. Frank Adams introduced the notion of a complex oriented cohomology theory represented by a commutative ring spectrum and proved the Poincaré Duality Theorem for this general case. In the current paper we consider oriented cohomology theories on algebraic varieties represented by symmetric commutative ring T-spectra and prove the Duality Theorem, which mimics the result of Adams. This result is held, in particular, for Motivic Cohomology and Algebraic Cobordism of Voevodsky.

0. Introduction

In certain cases a commutative ring spectrum E can be equipped with a distinguished element $c \in E^2(\mathbb{P}^\infty)$ called a complex orientation of E (see [1]). The pair (E, c) is called a complex oriented ring spectrum. Given a complex orientation c of E, every smooth complex projective variety X can be equipped with a homological class $[X] \in E_{2d}(X)$ called the fundamental class of X (here d stays for the complex dimension of X). This class has the property that the cap-product

$$\frown [X] : E^*(X) \to E_{2d-*}(X)$$

conducts an isomorphism of cohomology and homology groups of X. This isomorphism is often called the Poincaré Duality isomorphism.

From the modern point of view it looks pretty interesting to obtain an analogue of this result in the context of Algebraic Geometry. It is reasonable in this case to choose and fix a field k and consider a symmetric commutative ring T-spectrum $\mathscr{A}$ in the sense of Voevodsky [13] (for the concept of symmetric T-spectrum see Jardine [4]). The T-spectrum $\mathscr{A}$ determines bi-graded cohomology and homology theories ($A^{*,*}$ and $A_{*,*}$) on the category of algebraic varieties (see [13], p. 595). (We also assume the spectrum A to be a ring spectrum i.e. be endowed with a multiplication $\mu : \mathscr{A} \wedge \mathscr{A} \to \mathscr{A}$, which induces product

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The first author was supported in part by the Ellentuck Fund and by the Presidium of RAS Program "Fundamental Research". Both authors were partially supported by RTN Network HPRN-CT-2002-00287, INTAS, and the Russian Academy of Sciences research grants from the "Support Fund of National Science" 2001-3 (the first author) and for 2004-5 (the second one).

structures in (co)homology.) In some cases $\mathscr{A}$ can be equipped with a distinguished element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$, which Morel calls an orientation of $\mathscr{A}$. Following him, the pair $(\mathscr{A}, \gamma)$ is called an oriented symmetric commutative ring T-spectrum. The orientation γ equips both cohomology $A^{*,*}$ and homology $A_{*,*}$ with trace structures ([8], [11]). The latter means that for every projective morphism $f : Y \to X$ of smooth irreducible varieties over k with $d = \dim(X) - \dim(Y)$ there are two operators $f_! : A^{*,*}(Y) \to A^{*+2d,*+d}(X)$ and $f^! : A_{*,*}(X) \to A_{*-2d,*-d}(Y)$ satisfying a list of natural properties. Define now a fundamental class of a smooth projective equi-dimensional variety X/k of dimension d as $[X] := \pi^!(1) \in A_{2d,d}(X)$, where $\pi : X \to \mathrm{pt}$ is the structure morphism. Our main result claims that the map

$$\frown [X] : A^{*,*}(X) \xrightarrow{\simeq} A_{2d-*,d-*}(X)$$

is a grade-preserving isomorphism (Poincaré Duality isomorphism).

There are at least two interesting examples of oriented symmetric commutative ring T-spectra. The first one is a symmetric model $\mathbb{MGL}$ of the algebraic cobordism T-spectrum MGL of Voevodsky [13], p. 601. This symmetric commutative ring T-spectrum $\mathbb{MGL}$ together with an orientation $\gamma \in \mathbb{MGL}^{2,1}(\mathbb{P}^\infty)$ is described in Proposition B.4. So that, every smooth irreducible projective variety X/k of dimension d has the fundamental class $[X] \in \mathbb{MGL}_{2d,d}(X)$ and the cap-product with this class

$$\frown [X] : \mathbb{MGL}^{*,*}(X) \xrightarrow{\simeq} \mathbb{MGL}_{2d-*,d-*}(X)$$

is an isomorphism.

The second example is the Eilenberg-Mac Lane T-spectrum H (it is intrinsically a symmetric T-spectrum representing the motivic cohomology). This T-spectrum H is constructed in [13], p. 598, and we briefly describe its orientation here. Recall that for a smooth variety X/k the first Chern class of a line bundle with value in the motivic cohomology determines a functorial isomorphism $\mathrm{Pic}(X) = \mathrm{H}^{2,1}_{\mathscr{M}}(X)$. Thus, $\mathbb{Z} = \mathrm{H}^{2,1}_{\mathscr{M}}(\mathbb{P}^\infty)$ and the class of the line bundle $\mathscr{O}(1)$ over $\mathbb{P}^\infty$ is a free generator of $\mathrm{H}^{2,1}_{\mathscr{M}}(\mathbb{P}^\infty)$. This class provides the required orientation of H. Similarly to the case of algebraic cobordism, one has the fundamental class $[X] \in \mathrm{H}^{\mathscr{M}}_{2d,d}(X)$ in Motivic homology and the isomorphism:

$$\frown [X] : \mathrm{H}^{*,*}_{\mathscr{M}}(X) \xrightarrow{\simeq} \mathrm{H}^{\mathscr{M}}_{2d-*,d-*}(X).$$

To embellish this result, let us mention that unlike the topological context in the algebraic-geometrical case the canonical pairing $\mathrm{H}^{*,*}_{\mathscr{M}}(X) \otimes \mathrm{H}^{\mathscr{M}}_{*,*}(X) \to \mathrm{H}^{*,*}_{\mathscr{M}}(\mathrm{pt})$ is generally degenerated even with rational coefficients [14].

The paper is organized as follows. Section 1 is devoted to product structures in extraordinary cohomology and homology theories. In section 2 we formulate Poincare Duality Theorem and derive it from two projection formulas, which are proven in sections 3 and 4. Finally, in Appendices A and B we display some useful properties of orientable theories.

Acknowledgements. The first author is in debt to A. Merkurjev for inspiring discussions at the initial stage of the work. He is especially grateful to the Institute for Advanced Study (Princeton) for excellent working conditions.

The main result of the paper was obtained during the stay of the second author at Universität Essen and the current text was mostly written during his short-time visits to Universität Bielefeld and IHÉS (Bures-sur-Yvette). The second author is very grateful to all of these institutes for shown hospitality and excellent working possibilities during the visits.

Notation. Throughout the paper we use Greek letters to denote elements of cohomology groups and Latin for homological ones;

- Sm/k is a category of smooth quasi-projective algebraic varieties over a field k.
- Δ always denotes a diagonal morphism.
- Symbol $\mathbf{1}$ denotes trivial one-dimensional bundle.
- For a vector bundle $\mathscr{E}$ over X we write $s(\mathscr{E})$ for its section sheaf.
- For a vector bundle $\mathscr{E}$ over X we write $\mathscr{E}^\vee$ for the dual to $\mathscr{E}$.
- $\mathbb{P}(\mathscr{E}) := \mathbf{Proj}\big(\mathrm{Symm}^*\big(s(\mathscr{E}^\vee)\big)\big)$ is the projective bundle of lines in $\mathscr{E}$.
- Typically $\mathbb{P}^n$ is regarded as a hyperplane in $\mathbb{P}^{n+1}$.
- $T := \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ in the category *Spc* of [13].
- $\mathbb{P}^\infty := \operatorname{colim}_n(\mathbb{P}^n)$ in the category *Spc* of [13].
- $\mathrm{pt} := \operatorname{Spec} k$.

For the convenience of perception we usually move indexes up and down oppositely to the predefined positions of $*$ or $!$.

1. Some products in (co)homology

Consider a symmetric T-spectrum $\mathscr{A}$ ([4], p. 505), endowed with a multiplication $\mu : \mathscr{A} \wedge \mathscr{A} \to \mathscr{A}$ making $\mathscr{A}$ a symmetric commutative ring T-spectrum. Then the spectrum $\mathscr{A}$ determines bigraded cohomology and homology theories on the category of algebraic varieties ([13], p. 595). A ring structure in cohomology is then given by the cup-product satisfying the following commutativity law. For $\alpha \in A^{p,q}$ and $\beta \in A^{p',q'}$, one has:

$$\alpha \smile \beta = (-1)^{pp'} \varepsilon^{qq'} (\beta \smile \alpha), \tag{1.1}$$

where $\varepsilon : A^{*,*} \to A^{*,*}$ is the involution described in Appendix B.

Definition 1.1. Let $\mathscr{A}$ be endowed with an element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$ satisfying the following two conditions:

(i) $\gamma|_{\mathbb{P}^0} = 0 \in A^{2,1}(\mathbb{P}^0)$.

(ii) $\gamma|_{\mathbb{P}^1} = \Sigma_T(1) \in A^{2,1}_{\{0\}}(\mathbb{P}^1)$ is the T-suspension of the unit element $1 \in A^{0,0}(\mathrm{pt})$.

Then the pair $(\mathscr{A}, \gamma)$ is called *an oriented symmetric commutative ring T-spectrum*. If $\mathscr{A}$ can be endowed with an element $\gamma \in A^{2,1}(\mathbb{P}^\infty)$ satisfying the conditions (i) and (ii) then $\mathscr{A}$ is called *an orientable symmetric commutative ring T-spectrum*.

For an orientable T-spectrum $\varepsilon = \mathrm{id}$ by Lemma B.3 and the commutativity law is reduced to $\alpha \smile \beta = (-1)^{pp'}(\beta \smile \alpha)$. In this case it is convenient to set $A^0 = \bigoplus_{p,q} A^{2p,q}$, $A^1 = \bigoplus_{p,q} A^{2p-1,q}$, $A_0 = \bigoplus_{p,q} A_{2p,q}$, and $A_1 = \bigoplus_{p,q} A_{2p-1,q}$, where $A^{*,*}$ (resp. $A_{*,*}$) are (co)homology theories represented by the T-spectrum $\mathscr{A}$. The functors $A^* = A^0 \oplus A^1 : Sm/k \to \mathbb{Z}/2\text{-}\mathscr{A}b$ and $A_* = A_0 \oplus A_1 : Sm/k \to \mathbb{Z}/2\text{-}\mathscr{A}b$ are (co)homology theories taking values in the category of $\mathbb{Z}/2$-graded abelian groups. Although all our duality results hold for bigraded (co)homology groups, we shall work, for simplicity, with the $\mathbb{Z}/2$-grading just introduced.

Multiplicativity of the T-spectrum $\mathscr{A}$ gives a canonical way ([12], 13.50) to supply the functors A^* and A_* (contravariant and covariant, respectively) with a product structure consisting of two cross-products

$$\underline{\times} : A_p(X) \otimes A_q(Y) \to A_{p+q}(X \times Y),$$
$$\overline{\times} : A^p(X) \otimes A^q(Y) \to A^{p+q}(X \times Y)$$

and two slant-products

$$/ : A^p(X \times Y) \otimes A_q(Y) \to A^{p-q}(X),$$
$$\backslash : A^p(X) \otimes A_q(X \times Y) \to A_{q-p}(Y).$$

One also defines two inner products

$$\smile : A^p(X) \otimes A^q(X) \to A^{p+q}(X),$$
$$\frown : A^p(X) \otimes A_q(X) \to A_{q-p}(X),$$

as $\alpha \smile \beta := \Delta^*(\alpha \,\overline{\times}\, \beta)$ and $\alpha \frown a := \alpha \backslash \Delta_*(a)$, correspondingly. The cup-product makes the group $A^*(X)$ an associative skew-commutative $\mathbb{Z}/2$-graded unitary ring and this structure is functorial. (Skew-commutativity is not obvious and implied by the orientability of $\mathscr{A}$ as it is shown in Appendix B). The cap-product makes the group $A_*(X)$ a unital $A^*(X)$-module ($1 \frown a = a$ for every $a \in A_*(X)$) and this structure is functorial in the sense that $\alpha \frown f_*(a) = f_*\big(f^*(\alpha) \frown a\big)$.

Below we shall need the following associativity relations, which are completely analogous to ones existing in the topological context (see, for example, [12], 13.61). For $\alpha \in A^*(X \times Y)$, $\beta \in A^*(Y)$, $\eta \in A^*(X)$, $a \in A_*(Y)$, and $b \in A_*(X)$, we have:

(AR.1) $\alpha/(\beta \frown a) = \big(\alpha \smile p_Y^*(\beta)\big)/a$,

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(AR.2) $\eta \smile (\alpha/a) = \big(p_X^*(\eta) \smile \alpha\big)/a$,

(AR.3) $(\alpha/a) \frown b = p_*^X\big(\alpha \frown (a \underline{\times} b)\big)$,

where p_X and p_Y denote the corresponding projections.

We shall also need the following functoriality property of the /-product (comp. [12], 13.52.iii). For morphisms $f : X \to X'$, $g : Y \to Y'$, and elements $\alpha \in A^*(X' \times Y')$ and $a \in A_*(Y)$, one has: $(f \times g)^*(\alpha)/a = f^*\big(\alpha/g_*(a)\big)$.

For the final object pt in Sm/k one, clearly, has $A^*(\mathrm{pt}) = A_*(\mathrm{pt})$. This provides us with a distinguished element $[\mathrm{pt}] \in A_0(\mathrm{pt})$ (fundamental class of the point) such that for any smooth X and arbitrary $\alpha \in A^*(X)$, one has: $\alpha/[\mathrm{pt}] = \alpha$. (Here we assume the standard identification $X \times \mathrm{pt} = X$.) One can easily verify that the canonical isomorphism $A^*(\mathrm{pt}) = A_*(\mathrm{pt})$ may be written as $\alpha \mapsto \alpha \frown [\mathrm{pt}]$. Throughout the paper we implicitly use this construction and usually denote [pt] by 1.

2. Poincaré Duality Theorem

Let $(\mathscr{A}, \gamma)$ be an oriented symmetric commutative ring T-spectrum. Then the involution ε from (1.1) coincides with the identity as explained in Appendix B. So that the commutativity law is reduced to $\alpha \smile \beta = (-1)^{pp'}(\beta \smile \alpha)$. Setting $A^0 = \bigoplus\limits_{p,q} A^{2p,q}$, $A^1 = \bigoplus\limits_{p,q} A^{2p-1,q}$, we see that the functor $A^* := A^0 \oplus A^1$ takes value in the category of skew-commutative $\mathbb{Z}/2$-graded rings. The orientation γ assigns a Chern structure in the cohomology theory A^* in the sense of [9], Definition 3.2, and a commutative Chern structure in the homology theory A_* (see [11], Definitions 2.1.1, 2.2.12).

To describe this Chern structure, consider a functor isomorphism

$$\varphi : \mathrm{Pic}(-) \xrightarrow{\simeq} \mathrm{Mor}_{H^{\mathbb{A}^1}(k)}(-, \mathbb{P}^\infty)$$

on the category of smooth varieties, produced in [6], Proposition 4.3.8. Here $\mathrm{Pic}(-)$ is the Picard functor and $H^{\mathbb{A}^1}(k)$ is the $\mathbb{A}^1$-homotopy category of [6]. For a line bundle $\mathscr{L}$ over a smooth variety X one sets

$$c(\mathscr{L}) := \varphi(\mathscr{L})^*(\gamma) \in A^0(X). \tag{2.1}$$

We claim that the assignment $\mathscr{L} \mapsto c(\mathscr{L})$ is a Chern structure on A^*. In fact, the element $c(\mathscr{L})$ depends only on the isomorphism class of $\mathscr{L}$, it is functorial with respect to pull-backs of line bundles, and $c(\mathbf{1})$ vanishes, since $\gamma|_{\mathbb{P}^0} = 0$. Finally, by Lemma B.1, for a smooth variety X and the projection $p : \mathbb{P}^1 \times X \to \mathbb{P}^1$ the elements 1 and $p^*(\gamma|_{\mathbb{P}^1}) \in A^0(\mathbb{P}^1 \times X)$ form a free basis of the $A^*(X)$-bimodule $A^*(\mathbb{P}^1 \times X)$. Hence, the assignment $\mathscr{L} \mapsto c(\mathscr{L})$ is a Chern structure. It is also worth to notice that $\gamma = c\big(\mathscr{O}_{\mathbb{P}^\infty}(1)\big)$ in $A^0(\mathbb{P}^\infty)$.

Any Chern structure in A^* (resp. on A_*) determines a trace structure in the cohomology (resp. homology), see [8], Theorem 4.1.2 (resp. [11], Theorem 5.1.4). Namely, to every projective morphism $f : Y \to X$ of smooth varieties over k one assigns two grade-preserving operators $f_! : A^*(Y) \to A^*(X)$ and $f^! : A_*(X) \to A_*(Y)$ satisfying a list of natural properties. Precise definitions of trace structures in a ring (co)homology theory is given in [8], [11]. The operators $f_!$ and $f^!$ are called trace operators. (For historical reasons they called them *integrations* in [8].) The trace structures $f \mapsto f_!$ and $f \mapsto f^!$ are explicit and unique up to the following normalization condition. For a smooth divisor $i : D \hookrightarrow X$:

$$i_! i^* = i_!(1) \smile : A^*(X) \to A^*(X), \tag{2.2}$$

$$i_* i^! = i_!(1) \frown : A_*(X) \to A_*(X), \tag{2.3}$$

and $i_!(1) = c\big(\mathcal{L}(D)\big)$.

For a projective morphism $f : Y \to X$ the map $f_! : A^*(Y) \to A^*(X)$ is a two-side $A^*(X)$-module homomorphism, i.e.

$$\begin{aligned} f_!\big(f^*(\alpha) \smile \beta\big) &= \alpha \smile f_!(\beta), \\ f_!\big(\alpha \smile f^*(\beta)\big) &= f_!(\alpha) \smile \beta. \end{aligned} \tag{2.4}$$

Definition 2.1. Let $(\mathscr{A}, \gamma)$ be an oriented symmetric commutative ring T-spectrum. For a smooth projective variety X with the structure morphism $\pi : X \to \mathrm{pt}$ we call $\pi^!(1) \in A_0(X)$ the *fundamental class* of X in A_* and denote it by $[X]$.

Remark 2.2. Definitely, the class $[X]$ depends on the pair (A_*, γ) rather than on the T-spectrum $\mathscr{A}$ itself. However, we often omit mentioning the orientation, since one chosen and fixed orientation γ is always kept in mind for the spectrum $\mathscr{A}$.

With the notion of fundamental class in hands, one can define duality maps

$$\mathscr{D}^\bullet : A^*(X) \to A_*(X) \quad \text{as} \quad \mathscr{D}^\bullet(\alpha) = \alpha \frown [X] \tag{2.5}$$

and

$$\mathscr{D}_\bullet : A_*(X) \to A^*(X) \quad \text{as} \quad \mathscr{D}_\bullet(a) = \Delta_!(1)/a. \tag{2.6}$$

Theorem 2.3 (Poincaré Duality). *Let $(\mathscr{A}, \gamma)$ be an oriented symmetric commutative ring T-spectrum. Then for every smooth projective variety X the maps $\mathscr{D}^\bullet$ and $\mathscr{D}_\bullet$ are mutually inverse isomorphisms.*

If X is equi-dimensional of dimension d then $[X] \in A_{2d,d}(X)$. In this case the isomorphism $\mathscr{D}^\bullet$ identifies $A^{p,q}$ with $A_{2d-p,d-q}$. One can extract the following nice consequence of the Poincaré Duality Theorem, which enables us to interpret trace maps in a way topologists like to do.

Corollary 2.4. *For projective varieties $X, Y \in Sm/k$ and a morphism $f : X \to Y$, one has*:

$$f_! = \mathscr{D}_\bullet^Y f_* \mathscr{D}_X^\bullet \quad \textit{and} \quad f^! = \mathscr{D}_X^\bullet f^* \mathscr{D}_\bullet^Y,$$

where $\mathscr{D}_X$ and $\mathscr{D}_Y$ are the above introduced duality operators for varieties X and Y, respectively.

Proof. To proof the first equality, one should just check that $f_*\mathscr{D}^\bullet_X = \mathscr{D}^\bullet_Y f_!$. Taking into account that $[X] = f^![Y]$, one immediately derives the desired relation from the First Projection Formula below (Theorem 2.5). The second statement can be proven in a similar way, but requires the "dual" projection formula that we do not consider here. □

The proof of Theorem 2.3 is based on two projection formulae for cap- and slant-products.

Theorem 2.5 (First projection formula). *For $X, Y \in Sm/k$, a projective morphism $f : Y \to X$, and any elements $\alpha \in A^*(Y)$ and $a \in A_*(X)$, the relation*

$$f_*\big(\alpha \frown f^!(a)\big) = f_!(\alpha) \frown a \tag{2.7}$$

holds in the group $A_*(X)$.

We need a few simple corollaries of this theorem.

Corollary 2.6. *Let $\tau : X \times X \to X \times X$ be the permutation morphism. Then for any elements $\alpha \in A^*(X)$, $\beta \in A^*(X \times X)$, and $a \in A_*(X \times X)$, we have*:

(a) $$\Delta_!(\alpha) \frown a = \Delta_!(\alpha) \frown \tau_*(a),$$

(b) $$\Delta_!(\alpha) \smile \beta = \Delta_!(\alpha) \smile \tau^*(\beta)$$

in $A_*(X \times X)$ ($A^*(X \times X)$, respectively).

Proof. Consider the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ X & \xrightarrow{\Delta} & X \times X. \end{array} \tag{2.8}$$

Since the map τ is flat, the square is transversal due to [2], B.7.4. By the base change property A.2, one has: $\Delta^!\tau_* = \Delta^!$. By Theorem 2.5, one has:

$$\Delta_!(\alpha) \frown a = \Delta_*\big(\alpha \frown \Delta^!(a)\big) = \Delta_*\big(\alpha \frown \Delta^!\big(\tau_*(a)\big)\big) = \Delta_!(\alpha) \frown \tau_*(a)$$

that implies (a). To get (b) one uses cohomological projection formula (2.4) instead. □

Theorem 2.7 (Second projection formula). *Let $f : Y \to X$ be a projective morphism of smooth varieties. Let also $W \in Sm/k$. Then for every $\alpha \in A^*(W \times Y)$ and $a \in A_*(X)$, one has* (*in $A^*(W)$*):

$$\alpha/f^!(a) = F_!(\alpha)/a, \tag{2.9}$$

where $F = \mathrm{id} \times f$.

Corollary 2.8. *Let X be a smooth projective variety. Then in $A^*(X)$, we have*:

$$\Delta_!(1)/[X] = 1. \tag{2.10}$$

Proof. Denote by $p : X \to \mathrm{pt}$ the structure morphism and let

$$P = \mathrm{id} \times p : X \times X \to X$$

be the projection. By Theorem 2.7, one has:

$$\Delta_!(1)/[X] = \Delta_!(1)/p^!(1) = P_!\big(\Delta_!(1)\big)/1 = 1. \quad \square \tag{2.11}$$

Now we derive the main result as an easy consequence of Corollaries 2.8 and 2.6.

Proof of Theorem 2.3. Let $p_1, p_2 : X \times X \to X$ denote corresponding projections. Observe that for every $\beta \in A^*(X \times X)$ one has the relation $\Delta_!(1) \smile \beta = \beta \smile \Delta_!(1)$. (In fact, the element $\Delta_!(1)$ is of degree zero, because the map $\Delta_!(1)$ is grade-preserving.) Thus, one has:

$$\begin{aligned}\Delta_!(1)/(\alpha \frown [X]) &\overset{\text{(AR.1)}}{=} \big(\Delta_!(1) \smile p_2^*(\alpha)\big)/[X] \overset{2.6(\text{b})}{=} \big(\Delta_!(1) \smile p_1^*(\alpha)\big)/[X] \\ &= \big(p_1^*(\alpha) \smile \Delta_!(1)\big)/[X] \overset{\text{(AR.2)}}{=} \alpha \smile \big(\Delta_!(1)/[X]\big) = \alpha.\end{aligned} \tag{2.12}$$

On the other hand, using 2.6(a), one has:

$$\begin{aligned}\big(\Delta_!(1)/a\big) \frown [X] &\overset{\text{(AR.3)}}{=} p_*\big(\Delta_!(1) \frown (a \,\underline{\times}\, [X])\big) = p_*\big(\Delta_!(1) \frown ([X] \,\underline{\times}\, a)\big) \\ &\overset{\text{(AR.3)}}{=} \big(\Delta_!(1)/[X]\big) \frown a = a. \quad \square\end{aligned} \tag{2.13}$$

To complete the prove of Theorem 2.3 one needs to check formulas (2.7) and (2.9).

3. Proof of the first projection formula

It is convenient to introduce a class $\mathfrak{V}$ of projective morphisms $f : Y \to X$ for which the relation

$$f_*\big(\alpha \frown f^!(a)\big) = f_!(\alpha) \frown a \tag{3.1}$$

holds in $A_*(X)$ for every elements $\alpha \in A^*(Y)$ and $a \in A_*(X)$.

Obviously, this class is closed with respect to composition.

We prove Theorem 2.5 in several stages showing consequently that the following classes of morphisms are contained in the class $\mathfrak{V}$.

- Zero-section morphisms of line bundles: $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathscr{L})$.
- Closed embeddings $i : D \hookrightarrow X$ of smooth divisors.

- Zero-sections of a finite sum of line bundles:

$$s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_n).$$

- Zero-sections of arbitrary vector bundles: $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$.

- Closed embeddings $i : Y \hookrightarrow X$.

- Projections $p : X \times \mathbb{P}^n \to X$.

Lemma 3.1. *Let $\mathcal{L}$ be a line bundle over a smooth variety Y. Then the zero-section $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{L})$ belongs to $\mathfrak{B}$.*

Proof. The map s is a section of the projection map $p : \mathbb{P}(\mathbf{1} \oplus \mathcal{L}) \to Y$. Let $\alpha \in A^*(Y)$ and $a \in A_*\big(\mathbb{P}(\mathbf{1} \oplus \mathcal{L})\big)$. The desired relation follows from (2.3) and (2.2):

$$\begin{aligned} (3.2) \qquad s_*\big(\alpha \frown s^!(a)\big) &= s_*\big(s^*p^*(\alpha) \frown s^!(a)\big) = p^*(\alpha) \frown s_*s^!(a) \\ &= p^*(\alpha) \frown \big(s_!(1) \frown a\big) = s_!\big(s^*p^*(\alpha)\big) \frown a = s_!(\alpha) \frown a. \quad \square \end{aligned}$$

Proposition 3.2. *Let $X, Y \in Sm/k$, $i : Y \hookrightarrow X$ be a closed embedding with a normal bundle $\mathcal{N}$. If the zero-section morphism $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{N})$ belongs to $\mathfrak{B}$ then i belongs to $\mathfrak{B}$.*

Proof. Consider the following deformation diagram, in which B is the blowup of $X \times \mathbb{A}^1$ at $Y \times \{0\}$. This diagram has transversal squares.

$$(3.3) \qquad \begin{tikzcd} & B - Y \times \mathbb{A}^1 \arrow[d, "k_B"] & \\ \mathbb{P}(\mathbf{1} \oplus \mathcal{N}) \arrow[r, hook, "k_0"'] \arrow[d, bend left, "p"] & B & X \arrow[l, hook', "k_1"'] \\ Y \arrow[u, hook, "s"] \arrow[r, hook, "j_0"'] & Y \times \mathbb{A}^1 \arrow[u, hook, "t"] & Y \arrow[l, hook', "j_1"'] \arrow[u, hook, "i"'] \end{tikzcd}$$

One can easily see that the left-hand part of our diagram satisfies the conditions of Lemma A.5.

First, we shall show that the morphism t in diagram (3.3) belongs to the class $\mathfrak{B}$. Let $\alpha \in A^*(Y \times \mathbb{A}^1)$ and $a \in A_*(B)$. Using Lemma A.5 we can rewrite a as $k_*^B(a_B) + k_*^0(a_0)$, where $a_0 \in A_*\big(\mathbb{P}(\mathbf{1} \oplus \mathcal{N})\big)$ and $a_B \in A_*(B - Y \times \mathbb{A}^1)$. From the Gysin exact sequence, we have:

$$(3.4) \qquad t^! k_*^B = 0$$

and

$$(3.5) \qquad k_B^* t_! = 0.$$

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Therefore, $t_*\big(\alpha \frown t^! k_*^B(a_B)\big) = 0$ and $t_!(\alpha) \frown k_*^B(a_B) = 0$. (The second relation yields from (3.5): $t_!(\alpha) \frown k_*^B(a_B) = k_*^B\big(k_B^* t_!(\alpha) \frown a\big) = 0$.) Thus, one has:

$$t_*\big(\alpha \frown t^!(a)\big) = t_*\big(\alpha \frown t^! k_*^0(a_0)\big). \tag{3.6}$$

Applying Lemma A.3 to the left-hand-side square of diagram (3.3) and denoting $j_0^*(\alpha)$ by α_0, one has:

$$t_*\big(\alpha \frown t^! k_*^0(a_0)\big) = k_*^0 s_*\big(\alpha_0 \frown s^!(a_0)\big). \tag{3.7}$$

Similarly

$$t_!(\alpha) \frown a = k_*^0\big(s_!(\alpha_0) \frown a_0\big). \tag{3.8}$$

By the proposition assumption, we have the relation $s_*\big(\alpha_0 \frown s^!(a_0)\big) = s_!(\alpha_0) \frown a_0$. Combining this with equalities (3.6), (3.7), and (3.8), one gets:

$$t_*\big(\alpha \frown t^!(a)\big) = t_!(\alpha) \frown a. \tag{3.9}$$

We now move the desired relation one more step further to the right in diagram (3.3) and show that $i \in \mathfrak{V}$. Observe that k_*^1 is a monomorphism. Therefore, it suffices to check that for every elements $\alpha_1 \in A^*(Y)$ and $a_1 \in A_*(X)$ we have:

$$k_*^1 i_*\big(\alpha_1 \frown i^!(a_1)\big) = k_*^1\big(i_!(\alpha_1) \frown a_1\big). \tag{3.10}$$

Setting $\alpha = (j_1^*)^{-1}(\alpha_1) \in A^*(Y \times \mathbb{A}^1)$, $a = k_*^1(a_1) \in A_*(B)$, and applying Lemma A.3 to the right-hand side square of diagram (3.3), one has: $k_*^1 i_*\big(\alpha_1 \frown i^!(a_1)\big) = t_*\big(\alpha \frown t^!(a)\big)$. In the same way: $k_*^1\big(i_!(\alpha_1) \frown a_1\big) = t_!(\alpha) \frown k_*^0(a_0) = t_!(\alpha) \frown a$. Combining these two relations with (3.9), one sees that $i \in \mathfrak{V}$. □

Corollary 3.3. *For a smooth divisor $i : D \hookrightarrow X$ the morphism i lies in $\mathfrak{V}$.*

Corollary 3.4. *Let $\mathscr{W} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_n$ be an n-dimensional vector bundle over a variety Y which splits in the sum of line bundles. Then the zero-section morphism $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathscr{W})$ belongs to the class $\mathfrak{V}$.*

Proof. Apply Corollary 3.3 to each step of the filtration

$$Y \overset{i_1}{\hookrightarrow} \mathbb{P}(\mathbf{1} \oplus \mathscr{L}_1) \overset{i_2}{\hookrightarrow} \cdots \overset{i_n}{\hookrightarrow} \mathbb{P}(\mathbf{1} \oplus \mathscr{W}), \tag{3.11}$$

where the morphisms i_j are zero-sections of $\mathscr{L}_j$. □

In order to proceed with the case of an arbitrary vector-bundle, we need the homological analogue of the splitting principle. Consider a vector bundle $\mathscr{E} \to Y$ of constant rank n over a smooth irreducible variety Y. Let $\mathscr{GL}_n$ be the corresponding principal GL_n-bundle over Y, $T_n \subset GL_n$ be the diagonal tori, and $Y' = \mathscr{GL}_n/T_n$ be the orbit variety with the projection morphism $p : Y' \to Y$. Finally, we denote by $\mathscr{E}' = \mathscr{E} \times_Y Y'$ the pull-back of the vector bundle $\mathscr{E}$.

Proposition 3.5. *The bundle $\mathscr{E}'$ splits in a direct sum of line bundles and the map $p_* : A_*(Y') \to A_*(Y)$ is a universal splitting epimorphism* (*i.e. for any base-change $Z \to Y$ the induced map $A_*(Z \times_Y Y') \to A_*(Z)$ is a splitting epimorphism*).

Proof. The projection $\mathscr{GL}_n \to Y'$ and the natural T_n-action on $\mathscr{GL}_n$ makes it a principal T_n-bundle over Y'. Moreover, if $\mathscr{GL}'_n = \mathscr{GL}_n \times_Y Y'$ is the pull-back of $\mathscr{GL}_n$, there is a natural isomorphism of principal GL_n-bundles

$$\mathscr{GL}_n \times_{T_n} GL_n \to \mathscr{GL}'_n \tag{3.12}$$

over Y'. The bundle $\mathscr{E}'$ over Y' corresponds exactly to the principal GL_n-bundle $\mathscr{GL}'_n$. Thus, the mentioned isomorphism of principal GL_n-bundles over Y' shows that the bundle $\mathscr{E}'$ splits in a direct sum of line bundles (say corresponding to the fundamental characters $\chi_1, \chi_2, \dots, \chi_n$ of the tori T_n). This proves the first assertion of the proposition.

To prove the second one, consider a Borel subgroup B_n in GL_n (say the subgroup of all upper triangle matrices) and let U_n be the maximal unipotent subgroup of B_n (the group of upper triangle matrices with 1's on the diagonal). Let $\mathscr{F} = \mathscr{GL}_n/B_n$ (this is just the flag bundle over Y associated to $\mathscr{E}$). The bundle $\mathscr{F}$ comes equipped with projections $q : \mathscr{F} \to Y$ and $r : Y' \to \mathscr{F}$, where the projection r is induced by the inclusion $T_n \subset B_n$. Using the natural U_n-action on $\mathscr{GL}_n$, it is easy to check that there is a tower of morphisms:

$$\mathscr{GL}_n = S_m \to S_{m-1} \to \cdots \to S_1 = \mathscr{F}, \tag{3.13}$$

which has a principal $\mathbb{G}_a$-bundle on each level (each level is a torsor over the trivial rank one vector bundle). By the strong homotopy invariance property [9], 2.2.6, the induced map on homology $r_* : A_*(Y') \to A_*(\mathscr{F})$ is an isomorphism.

As it was already mentioned, $\mathscr{F}$ is a full flag bundle over Y associated to the bundle $\mathscr{E}$. Thus, there is a tower of morphisms

$$\mathscr{F} = Z_s \to Z_{s-1} \to \cdots \to Z_1 = Y \tag{3.14}$$

in which each level is a projective bundle associated to a vector bundle. By the Projective Bundle Theorem (PBT) A.6, we have a split epimorphism in homology induced on each floor. Therefore, the map $q_* : A_*(\mathscr{F}) \to A_*(Y)$ is a split epimorphism as well.

These proves that the map $p_* : A_*(Y') \to A_*(Y)$ is also an epimorphism.

One can easily check that all necessary properties of the morphisms p, q, and r are base-change invariant. Therefore, the constructed splitting epimorphism is universal. $\square$

Proposition 3.6. *Let $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathscr{V})$ be the zero-section of the finite-dimensional vector bundle $\mathscr{V}$. Then $s \in \mathfrak{V}$.*

Proof. Letting Y' be as above, denote by $\mathscr{V}'$ the pull-back of the bundle $\mathscr{V}$ with respect to the morphism p. Then by Proposition 3.5 the bundle $\mathscr{V}'$ splits in a direct sum of line bundles and the induced map

$$\bar{p}_* : A_*(\mathbb{P}(\mathbf{1} \oplus \mathscr{V}')) \to A_*(\mathbb{P}(\mathbf{1} \oplus \mathscr{V})) \tag{3.15}$$

is an epimorphism.

Let $s : Y \to \mathbb{P}(\mathbf{1} \oplus \mathscr{V})$ and $\bar{s} : Y' \to \mathbb{P}(\mathbf{1} \oplus \mathscr{V}')$ be morphisms induced by zero-sections of the corresponding vector bundles. Then the diagram

$$\begin{tikzcd} \mathbb{P}(\mathbf{1} \oplus \mathscr{V}') \arrow[r, "\bar{p}"] & \mathbb{P}(\mathbf{1} \oplus \mathscr{V}) \\ Y' \arrow[u, "\bar{s}"] \arrow[r, "p"] & Y \arrow[u, "s"'] \end{tikzcd} \tag{3.16}$$

is transversal.

Let $\alpha \in A^*(Y)$ and $a \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathscr{V}))$. Choosing $b \in A_*(\mathbb{P}(\mathbf{1} \oplus \mathscr{V}'))$ such that $a = \bar{p}_*(b)$ and applying Lemma A.3, one gets:

$$s_*(\alpha \frown s^!(a)) = \bar{p}_* \bar{s}_*(p^*(\alpha) \frown \bar{s}^!(b)) \tag{3.17}$$

and

$$s_!(\alpha) \frown a = \bar{p}_*(\bar{s}_! p^*(\alpha) \frown b). \tag{3.18}$$

Two expressions on the right-hand sides coincide by Proposition 3.4. □

Corollary 3.7. *Let $i : Y \hookrightarrow X$ be a closed embedding. Then $i \in \mathfrak{V}$.*

Proof. Applying Proposition 3.2 we reduce the question to the case of the zero-section morphism $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathscr{N})$ of the normal bundle $\mathscr{N} = \mathscr{N}_{X/Y}$. The morphism s belongs to $\mathfrak{V}$ by Proposition 3.6. □

In order to check that projection morphisms $p : X \times \mathbb{P}^* \to X$ belong to $\mathfrak{V}$ we need a few auxiliary results (3.9–3.11).

Notation 3.8. For a projective morphism f we denote, from now on, the map $f_* f^!$ by $f^\diamond$ and $f_! f^*$ by $f_\diamond$.

Lemma 3.9. (a) $\mathrm{id}^\diamond = \mathrm{id}$.

(b) (*Left distributivity*) *Let a, b, c, and p be projective morphisms. If $a^\diamond = b^\diamond + c^\diamond$ then $(pa)^\diamond = (pb)^\diamond + (pc)^\diamond$, provided that both sides of the equality are well defined.*

(c) *Given a transversal square with projective morphisms f and g*

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "F"'] \arrow[dr, dotted, "h"] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

one has the following equalities: $h^\diamond = g^\diamond f^\diamond = f^\diamond g^\diamond$.

(d) *In the square above*: $g_* F^\diamond = f^\diamond g_*$.

(e) *Let* s_i *be the standard embedding* $\mathbb{P}^{n-i} \hookrightarrow \mathbb{P}^n$ *and* $p_n : \mathbb{P}^n_X \to X$ *be the projection map. Let* ψ_i *be the same as in the Projective Bundle Theorem* (*see* A.6). *Then* $p^n_* s_i^\diamond = \psi_i$.

Proof. Parts (a), (b) immediately follow from the definition of the operation $\diamond$, (c) and (d) are trivial corollaries of the transversal base-change property, (e) easily follows from the PBT. $\square$

Fix now a variety $X \in Sm/k$ and take the n-dimensional projective space $\mathbb{P}^n_X$ over X. (Up to the end of the proof of Lemma 3.10 all the schemes are considered over the base scheme X and the product is implicitly taken over X. In particular, $\mathbb{P}^n$ means $\mathbb{P}^n_X$ and $\mathbb{P}^0$ means X.) Due to the PBT, the element $\Delta_!(1) \in A^*(\mathbb{P}^n \times \mathbb{P}^n)$ may be decomposed as

$$\Delta_!(1) = 1 \boxtimes \zeta^n + \zeta^n \boxtimes 1 + \sum_{i,j=1}^{n} a_{ij} \zeta^i \boxtimes \zeta^j, \tag{3.19}$$

where $\zeta = e(\mathcal{O}(1))$ is the canonical generator of $A^*(\mathbb{P}^n)$ as an $A^*(X)$-algebra and $a_{ij} \in A^*(X)$ (see [7], Lemma 1.9.3).

This equality together with the previous lemma gives us the following decomposition of the identity operator $\mathrm{id}_{\mathbb{P}^n}$. Taking into account the relation $s_{ij}^\diamond(x) = (\zeta^i \boxtimes \zeta^j) \frown x$, where $s_{ij} : \mathbb{P}^{n-i} \times \mathbb{P}^{n-j} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n$ is the standard embedding, we can rewrite the cap-product with $\Delta_!(1)$ operator in the form:

$$\Delta^\diamond = \big(\Delta_!(1) \frown\big) = s_{0n}^\diamond + s_{n0}^\diamond + \sum_{i,j=1}^{n} a_{ij} s_{ij}^\diamond. \tag{3.20}$$

Consider the transversal squares

$$\begin{tikzcd} \mathbb{P}^{n-i} \times \mathbb{P}^{n-j} \arrow[r, hook] \arrow[d] \arrow[dr, dotted, "p_{1,n}s_{ij}"'] & \mathbb{P}^n \times \mathbb{P}^{n-j} \arrow[d, "p_{1,n-j}"] \\ \mathbb{P}^{n-i} \arrow[r, hook, "s_i"'] & \mathbb{P}^n \end{tikzcd} \tag{3.21}$$

(where we denote by $p_{1,k}$ the projection map $\mathbb{P}^n \times \mathbb{P}^k \to \mathbb{P}^n$). Applying $p_{1,n}$ to (3.20), by Lemma 3.9(a), (b), one gets the following equality:

$$\mathrm{id} = (p_{1,n}\Delta)^\diamond = (p_{1,n}s_{0n})^\diamond + (p_{1,n}s_{n0})^\diamond + \sum_{i,j=1}^{n} a_{ij}(p_{1,n}s_{ij})^\diamond. \tag{3.22}$$

Once again, by Lemma 3.9(c), taking into account that $(p_{1,n}s_{0n})^\diamond = p_{1,0}^\diamond s_0^\diamond = \mathrm{id}$, one has:

$$0 = p_{1,n}^\diamond s_n^\diamond + \sum_{i,j=1}^{n} a_{ij} p_{1,n-j}^\diamond s_i^\diamond. \tag{3.23}$$

(AutoPDF V7 11/10/07 16:32) WDG Tmath J-1708 CRELLE, PMU: A1(A1) 30/09/2007 pp. 1–21 1708_5499 (p. 13)

Lemma 3.10. *For the projection morphism $p_n : \mathbb{P}^n \to \mathbb{P}^0$, we have*:

(a) $p_n^\diamond = -\sum_{j=1}^{n} a_{nj} p_{n-j}^\diamond$.

(b) $p_\diamond^n = -\sum_{j=1}^{n} a_{nj} p_\diamond^{n-j}$.

Proof. Let us check the first statement. For $n = 0$ we, trivially, have $p_0^\diamond = \mathrm{id}$. Applying the map p_*^n to (3.23) and then employing Lemma 3.9(d) for the transversal squares

$$\begin{tikzcd} \mathbb{P}^n \times \mathbb{P}^{n-j} \arrow[r] \arrow[d, "p_{1,n-j}"'] & \mathbb{P}^{n-j} \arrow[d, "p_{n-j}"] \\ \mathbb{P}^n \arrow[r, "p_n"'] & \mathbb{P}^0, \end{tikzcd} \tag{3.24}$$

one gets:

$$0 = p_n^\diamond(p_*^n s_n^\diamond) + \sum_{i,j=1}^{n} a_{ij} p_{n-j}^\diamond(p_*^n s_i^\diamond). \tag{3.25}$$

By 3.9(e), $p_*^n s_i^\diamond = \psi_i$. Hence,

$$0 = p_n^\diamond \psi_n + \sum_{i,j=1}^{n} a_{ij} p_{n-j}^\diamond \psi_i. \tag{3.26}$$

By the PBT, for any $x \in A_*(X)$ we can choose an element $\varphi(x) \in A_*(\mathbb{P}_X^n)$ such that $\psi_n(\varphi(x)) = x$ and $\psi_i(\varphi(x)) = 0$ for $i < n$. Applying operator (3.26) to $\varphi(x)$, we get:

$$0 = p_n^\diamond + \sum_{j=1}^{n} a_{nj} p_{n-j}^\diamond. \tag{3.27}$$

This finishes the proof of case (a). The cohomological relation (b) may be obtained by dualization of these arguments or found in [7], Section 1.10. □

Proposition 3.11. *Let p_n denote, as before, the projection morphism $p_n : \mathbb{P}_X^n \to X$. Then for every element $a \in A_*(X)$, one has*:

$$p_*^n(p_n^!(a)) = p_!^n(1) \frown a.$$

Proof. Rewriting the proposition statement in our notation, we should verify the relation $p_n^\diamond(a) = p_\diamond^n(1) \frown a$. We proceed by induction on n. The case $n = 0$ is trivial. Let the proposition hold for $n < N$. Then for p_N, by Lemma 3.10, we have:

$$p_N^\diamond(a) = -\sum_{j=1}^{N} a_{Nj} p_{N-j}^\diamond(a) \tag{3.28}$$

and

$$p_\diamond^N(1) \frown a = -\sum_{j=1}^{N} a_{Nj} p_\diamond^{N-j}(1) \frown a. \tag{3.29}$$

By the induction hypothesis the expressions on the right-hand side coincide. The induction runs. $\square$

Proposition 3.12. *For every integer $n \geqq 0$ the projection morphism $p = p_n : \mathbb{P}_X^n \to X$ belongs to the class $\mathfrak{V}$.*

Proof. Given $\alpha \in A^*(\mathbb{P}_X^n)$ and $a \in A_*(X)$ one should verify that

$$p_*\big(\alpha \frown p^!(a)\big) = p_!(\alpha) \frown a. \tag{3.30}$$

Clearly, both sides of (3.30) are $A^*(X)$-linear. By the PBT, $A^*(\mathbb{P}_X^n)$ is generated as an $A^*(X)$-module by the elements ζ^j. Thus, it suffices to check the proposition just for these elements. From [7], Lemma 1.9.1, we have a relation $\zeta^j = i_!^j(1)$ in $A^*(\mathbb{P}_X^n)$, where $i^j : \mathbb{P}_X^{n-j} \hookrightarrow \mathbb{P}_X^n$ is the standard embedding map and the element $\zeta^j \in A^*(\mathbb{P}^n)$ is considered here as lying in $A^*(\mathbb{P}_X^n)$ via the pull-back operator for the projection $\mathbb{P}_X^n \to \mathbb{P}^n$. Denote by p_j the projection map $\mathbb{P}_X^{n-j} \to X$. Since $pi^j = p_j$, we have by Corollary 3.7:

$$p_*\big(\zeta^j \frown p^!(a)\big) = p_* i_*^j\big(1 \frown i_j^! p^!(a)\big) = p_*^j p_j^!(a). \tag{3.31}$$

One finishes the proof of Theorem 2.5, using Proposition 3.11:

$$p_*^j p_j^!(a) = p_!^j(1) \frown a = p_! i_!^j(1) \frown a = p_!(\zeta^j) \frown a. \quad \square \tag{3.32}$$

4. Proof of the second projection formula

The strategy of the proof of Theorem 2.7 is very similar to one used in the previous section. It is again convenient to introduce a class $\mathfrak{W}$ consisting of projective morphisms $f : Y \to X$ such that for any $W \in Sm/k$, $\alpha \in A^*(W \times Y)$, and $a \in A_*(X)$ the relation

$$F_!(\alpha)/a = \alpha/f^!(a) \tag{4.1}$$

holds in $A^*(W)$. (Here $F = \mathrm{id} \times f$. Below we use similar notation rules.)

We show that the following classes of morphisms lie in $\mathfrak{W}$:

- Zero-sections of vector bundles: $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathcal{V})$.
- Closed embeddings $i : Y \hookrightarrow X$.
- Projections $p : X \times \mathbb{P}^n \to X$.

Since the class $\mathfrak{W}$ is closed with respect to composition, this will imply our formula for all projective morphisms.

(AutoPDF V7 11/10/07 16:32) WDG Tmath J-1708 CRELLE, PMU: A1(A1) 30/09/2007 pp. 1–21 1708_5499 (p. 15)

Lemma 4.1. *Let $\mathscr{V}$ be a vector bundle over a smooth variety Y and let $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathscr{V})$ be the zero-section of the projection $p : \mathbb{P}(\mathbf{1} \oplus \mathscr{V}) \to Y$. Then the morphism s belongs to the class $\mathfrak{W}$.*

Proof. Let $\alpha \in A^*(W \times Y)$ and $a \in A_*\big(\mathbb{P}(\mathbf{1} \oplus \mathscr{V})\big)$. By functoriality of the slant-product, relation (AR.1), and formulas (2.4), (2.7), one gets:

$$(4.2) \qquad \alpha/s^!(a) = \alpha/p_*\big(s_!(1) \frown a\big) = P^*(\alpha)/\big(s_!(1) \frown a\big)$$
$$\stackrel{(\mathrm{AR.1})}{=} \big(P^*(\alpha) \smile \big(1 \times s_!(1)\big)\big)/a = \big(P^*(\alpha) \smile S_!(1)\big)/a = S_!(\alpha)/a.$$

(Here the relation $1 \times s_!(1) = S_!(1)$ appears from the base-change property applied to the product with W.) $\square$

Proposition 4.2. *Any closed embedding morphism $i : Y \hookrightarrow X$ of smooth varieties belongs to the class $\mathfrak{W}$.*

Proof. Denote by $\mathbb{P}(\mathbf{1} \oplus \mathscr{N})$ the projectivization corresponding to the normal bundle $\mathscr{N} = \mathscr{N}_{X/Y}$. It is endowed with the zero-section morphism $s : Y \hookrightarrow \mathbb{P}(\mathbf{1} \oplus \mathscr{N})$.

As well as in the proof of Theorem 2.5 our arguments are based on the deformation diagram which we obtained from (3.3) by multiplication with a variety $W \in Sm/k$. For convenience, we reproduce this diagram here:

$$(4.3) \qquad \begin{array}{ccccc} & & W \times B - W \times Y \times \mathbb{A}^1 & & \\ & & \downarrow{\scriptstyle K_B} & & \\ W \times \mathbb{P}(\mathbf{1} \oplus \mathscr{N}) & \xhookrightarrow[K_0]{} & W \times B & \xhookleftarrow[K_1]{} & W \times X \\ {\scriptstyle S}\uparrow & & {\scriptstyle I_t}\uparrow & & {\scriptstyle I}\uparrow \\ W \times Y & \xhookrightarrow[J_0]{} & W \times Y \times \mathbb{A}^1 & \xhookleftarrow[J_1]{} & W \times Y. \end{array}$$

First of all, we show that $I_t \in \mathfrak{W}$. Namely, we should prove that for any elements $\alpha \in A^*(W \times Y \times \mathbb{A}^1)$ and $a \in A_*(B)$ the relation

$$(4.4) \qquad \alpha/i_t^!(a) = I_!^t(\alpha)/a.$$

holds in $A^*(W)$.

Exactly as in the proof of Theorem 2.5 one can rewrite a as a sum $k_*^B(a_B) + k_*^0(a_0)$, where $a_0 \in A_*\big(\mathbb{P}(\mathbf{1} \oplus \mathscr{N})\big)$ and $a_B \in A_*(B - Y \times \mathbb{A}^1)$ and obtain the equalities:

$$(4.5) \qquad \alpha/i_t^!(a) = \alpha/i_t^! k_*^0(a_0) = \alpha_0/s^!(a_0),$$

where $\alpha_0 = J_0^*(\alpha)$.

Similarly, one gets the relation:

(AutoPDF V7 11/10/07 16:32) WDG Tmath J-1708 CRELLE, PMU: A1(A1) 30/09/2007 pp. 1–21 1708_5499 (p. 16)

(4.6) $$I_!^t(\alpha)/a = S_! J_0^*(\alpha)/a_0 = S_!(\alpha_0)/a_0.$$

By Lemma 4.1, $\alpha_0/s^!(a_0) = S_!(\alpha_0)/a_0$, which proves (4.4).

Since the map J_1^* is an isomorphism, we can set $\alpha = (J_1^*)^{-1}(\alpha_1) \in A^*(W \times Y \times \mathbb{A}^1)$ and $a = k_*^1(a_1) \in A_*(B)$. Applying Corollary A.3 again, one gets:

(4.7) $$\alpha_1/i^!(a_1) = J_1^*(\alpha)/i^!(a_1) = \alpha/i_t^!(a)$$

and

(4.8) $$I_!(\alpha_1)/a_1 = I_! J_1^*(\alpha)/a_1 = I_!^t(\alpha)/a.$$

Combining these equalities with relation (4.4) proves the proposition. □

Proposition 4.3. *Let $X, W \in Sm/k$, $p : X \times \mathbb{P}^n \to X$ be the projection morphism, and $P = \mathrm{id} \times p : W \times X \times \mathbb{P}^n \to W \times X$. Then for every elements $\alpha \in A^*(W \times X \times \mathbb{P}^n)$ and $a \in A_*(X)$, one has in $A^*(W)$:*

(4.9) $$\alpha/p^!(a) = P_!(\alpha)/a.$$

Proof. Consider the following commutative diagram with transversal square:

(4.10)
$$\begin{tikzcd} X \times \mathbb{P}^n \arrow[dr, "p=p_0"'] & X \times \mathbb{P}^{n-r} \arrow[l, "i"'] \arrow[d, "p_r"] & W \times X \times \mathbb{P}^{n-r} \arrow[l, "\bar{q}"'] \arrow[d, "P_r"] \\ & X & W \times X. \arrow[l, "q"'] \end{tikzcd}$$

Clearly, both sides of (4.9) are $A^*(W)$-linear. So, we may assume that $\alpha = \zeta^r_{W\times X}$. Since $\zeta^r_{W\times X} = I_!(1_{W\times X}) \in A^*(W \times X \times \mathbb{P}^n)$, one has:

(4.11) $$\zeta^r_{W\times X}/p^!(a) = I_!(1_{W\times X})/p^!(a) = 1_{W\times X}/i^! p^!(a)$$
$$= 1/p_r^!(a) = P_r^*(1)/p_r^!(a) = 1/p_*^r p_r^!(a).$$

By Proposition 3.11 and formula (AR.1):

(4.12) $$1/p_*^r p_r^!(a) = 1/\big(p_!^r(1_X) \frown a\big) = q^* p_!^r(1)/a.$$

Applying the base-change property to the square in the diagram above, we get the desired:

(4.13) $$q^* p_!^r(1_X) = P_!^r(1_{W\times X}) = P_!\big(I_!(1)\big) = P_!(\zeta^r_{W\times X}).$$ □

Appendix A. Some properties of a trace structure

In this Appendix we give a brief description of some useful properties of a trace structure, which are utilized in the paper. Although we need to work both with cohomological

(AutoPDF V7 11/10/07 16:32) WDG Tmath J-1708 CRELLE, PMU: A1(A1) 30/09/2007 pp. 1–21 1708_5499 (p. 17)

and homological contexts, the results here are presented only for homology. The cohomological variant is "dual" in the obvious sense and may be found in [7]. All the proofs for the homological case not provided below can be found in [11].

We, first, define a transversal square following A. Merkurjev [5].

Definition A.1. We call a square

$$\begin{array}{ccc} Y' & \xrightarrow{\bar f} & X' \\ {\scriptstyle \bar g}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

in the category *Sm/k transversal* if

(a) it is Cartesian in the category $\mathbf{Sch}/k$ of all schemes over the field k;

(b) the following sequence of tangent bundles over Y' is exact:

$$0 \longrightarrow \mathcal{T}_{Y'} \xrightarrow{d\bar g \oplus d\bar f} \bar g^* \mathcal{T}_Y \oplus \bar f^* \mathcal{T}_{X'} \xrightarrow{dg - df} \bar g^* f^* \mathcal{T}_X \longrightarrow 0.$$

It is not hard to check that this definition is accordant to one given in [7], 1.1.2 or [10], 1.1. Let us check, for example, that for a closed embedding f condition (b) implies the isomorphism: $\bar g^* \mathcal{N}_{X/Y} \simeq \mathcal{N}_{X'/Y'}$. The short exact sequence above may be viewed as a total complex of the bicomplex:

$$\text{(A.1)} \qquad \begin{array}{ccccc} 0 & \longrightarrow & \bar g^* \mathcal{T}_Y & \xrightarrow{-df} & \bar g^* f^* \mathcal{T}_X \\ & & {\scriptstyle d\bar g}\uparrow & & \uparrow{\scriptstyle dg} \\ 0 & \longrightarrow & \mathcal{T}_{Y'} & \xrightarrow[d\bar f]{} & \bar f^* \mathcal{T}_{X'}. \end{array}$$

Since (b) is exact, the bicomplex is acyclic. On the other hand, it is quasiisomorphic to the two-term complex $\bar g^* \mathcal{N}_{X/Y} \leftarrow \mathcal{N}_{X'/Y'}$.

Property A.2 (Base-change for transversal squares). For any transversal square as above with projective morphism f the diagram

$$\begin{array}{ccc} A_*(Y') & \xleftarrow{\bar f^!} & A_*(X') \\ {\scriptstyle \bar g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ A_*(Y) & \xleftarrow{f^!} & A_*(X) \end{array}$$

commutes.

Corollary A.3. *Suppose, we are given a transversal square*

$$\begin{tikzcd} X' \arrow[r, "g"] & X \\ Y' \arrow[u, "\bar f"] \arrow[r, "\bar g"'] & Y \arrow[u, "f"'] \end{tikzcd}$$

with projective morphism f. *Let* $\alpha \in A^*(Y)$ *and* $a \in A_*(X')$. *Then the following relations hold*:

(i) $f_*\big(\alpha \frown f^! g_*(a)\big) = g_* \bar f_*\big(\bar g^*(\alpha) \frown \bar f^!(a)\big)$.

(ii) $f_!(\alpha) \frown g_*(a) = g_*\big(\bar f_! \bar g^*(\alpha) \frown a\big)$.

Moreover, for a variety $W \in Sm/k$ *and* $\beta \in A^*(W \times Y)$, *we have*:

(iii) $\beta/f^! g_*(a) = \bar G^*(\beta)/\bar f^!(a)$.

(iv) $F_!(\beta)/g_*(a) = \bar F_! \bar G^*(\beta)/a$.

Proof. All these relations may be easily obtained using the base-change property. We illustrate it proving the first one:

$$\begin{aligned} f_*\big(\alpha \frown f^! g_*(a)\big) &= f_*\big(\alpha \frown \bar g_* \bar f^!(a)\big) = f_* \bar g_*\big(\bar g^*(\alpha) \frown \bar f^!(a)\big) \\ &= g_* \bar f_*\big(\bar g^*(\alpha) \frown \bar f^!(a)\big). \quad \square \end{aligned} \tag{A.2}$$

Property A.4 (Gysin exact sequence). Let $i : Y \hookrightarrow X$ be a closed embedding and $j : X - Y \hookrightarrow X$ the corresponding open inclusion. Then, the sequence $A_*(X - Y) \xrightarrow{j_*} A_*(X) \xrightarrow{i^!} A_*(Y)$ is exact.

The following lemma is a "dualization" of "Useful Lemma 1.4.2" from [7].

Lemma A.5 (Homological useful lemma). *Consider the following diagram with transversal square*:

$$\begin{tikzcd} & X - Y \arrow[d, "k^1"] \\ V \arrow[r, "k^0"'] \arrow[d, bend left, "p"] & X \\ W \arrow[u, "i"] \arrow[r, "j"'] & Y \arrow[u, hook, "q"'] \end{tikzcd}$$

where the morphism p *is projective*, q *is a closed embedding*, $X - Y$ *is the open complement of* Y *in* X, k_1 *is the corresponding open embedding*, $pi = \mathrm{id}$, *and the morphism* j *induces an isomorphism in homology. Then* $\operatorname{Im} k^0_* + \operatorname{Im} k^1_* = A_*(X)$.

Proof. Let $x \in A_*(X)$. Since the map j_* is an isomorphism and $i^! p^! = \mathrm{id}$, we can, using the transversal base-change property, lift x up to $\bar x = p^!(j_*)^{-1} q^!(x) \in A_*(V)$, such that $q^! k^0_*(\bar x) = q^!(x)$. Then, the Gysin exact sequence implies that $k^0_*(\bar x) - x \in \operatorname{Im} k^1_*$. $\square$

Property A.6 (Projective Bundle Theorem (PBT)). First, we should introduce the notion of an Euler class. For a line bundle $\mathscr{L}$ over X we set $e(\mathscr{L}) := z^* z_!(1)$, where $z : X \to \mathscr{L}$ is the zero-section (see [7], 1.1.4 for details).

For $X \in Sm/k$ and a rank r vector bundle $\mathscr{E} \xrightarrow{p} X$ set $\zeta = e\big(\mathcal{O}_{\mathscr{E}}(1)\big) \in A^*\big(\mathbb{P}(\mathscr{E})\big)$. Then the map

$$\bigoplus_{i=0}^{r-1} \psi_i : A_*\big(\mathbb{P}(\mathscr{E})\big) \xrightarrow{\simeq} \bigoplus_{i=0}^{r-1} A_*(X),$$

where $\psi_i = p_* \circ (\zeta^{\smile i} \frown -)$, is an isomorphism.

Appendix B. Orientation, Chern structure, and homothety involution

Let $\mathscr{A}$ be a symmetric commutative ring T-spectrum. For $\lambda \in k^\times$ consider a map $\lambda : T \to T$ sending x to λx. For any space X it determines an involution (see [15]) on the cohomology groups $A^{*,*}$ as follows:

$$\text{(B.1)} \qquad \varepsilon(\lambda)^* = \Sigma_T^{-1} \lambda^* \Sigma_T : A^{*,*}(X) \to A^{*,*}(X),$$

where $\Sigma_T : A^{*,*}(X) \to A^{*+2,*+1}(X)$ is the T-suspension isomorphism and Σ_T^{-1} is its inverse. Set $\varepsilon = \varepsilon(-1)^*$. The following lemmata show that $\varepsilon = \mathrm{id}$ for orientable T-spectra.

Lemma B.1. *$A^*(\mathbb{P}^1 \times X)$ is a free $A^*(X)$-module with a free basis $\{1, \Sigma_T(1)\}$.*

Proof. For every symmetric commutative ring T-spectrum $\mathscr{A}$ the map $A^*(\mathbb{P}^1) \otimes_{A^*(\mathrm{pt})} A^*(X) \to A^*(\mathbb{P}^1 \times X)$ is an isomorphism. So, it remains to show that $\{1, \Sigma_T(1)\}$ form a free $A^*(\mathrm{pt})$-basis in $A^*(\mathbb{P}^1)$. Note that since the morphism $\mathbb{P}^1 \to \mathrm{pt}$ has a section, one has: $A^*(\mathbb{P}^1) = A^*(\mathrm{pt}) \oplus A^*(\mathbb{P}^1/\{\infty\})$. Using excision and homotopy invariance properties, the latter $A^*(\mathrm{pt})$-bimodule can be rewritten as:

$$\text{(B.2)} \qquad A^*(\mathbb{P}^1/\{\infty\}) = A^*(\mathbb{P}^1/\mathbb{A}^1) = A^*\big(\mathbb{A}^1/(\mathbb{A}^1 - \{0\})\big)$$

$$= A^*(T) \overset{\Sigma_T^{-1}}{\simeq} A^*(\mathrm{pt}). \quad \square$$

Remark B.2. It is worth to mention that for an orientable spectrum $\mathscr{A}$ the set $\{1, \Sigma_T(1)\}$ also form a free basis of the $A^0(\mathrm{pt})$-module $A^0(\mathbb{P}^1)$.

Lemma B.3. *If $\mathscr{A}$ is orientable then $\varepsilon = \mathrm{id}$.*

Proof. We show that for any $\lambda \in k^\times$ one has $\Sigma_T^{-1} \lambda^* \Sigma_T = \mathrm{id}$. By [13], $T \simeq \mathbb{P}^1/\mathrm{pt}$ and the map λ corresponds to the endomorphism of $\mathbb{P}^1$ (preserving the distinguished point 0) sending $[x : y]$ to $[\lambda x : y]$. Let $i : \mathbb{P}^1 \to \mathbb{P}^2$ be a linear embedding. Since $\gamma|_{\mathbb{P}^1} = \Sigma_T(1)$, Lemma B.1 implies that the map $i^* : A^*(\mathbb{P}^2) \to A^*(\mathbb{P}^1)$ is an epimorphism. Now the statement easily follows from [3], Lemma 1.6, Proposition 4.1. $\square$

Finally, we construct a natural orientation of $\mathbb{MGL}$. In [10], Section 6.5, there has been constructed an element $c \in \mathbb{MGL}^{2,1}(\mathbb{P}^\infty)$ such that $c|_{\mathbb{P}^1} = -\Sigma_T(1)$ and $c|_{\mathbb{P}^0} = 0$.

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Clearly, the construction (2.1) being applied to the element c (instead of γ in (2.1)) determines a Chern structure on $\mathbb{MGL}^*$. Set $\gamma = c\big(\mathcal{O}_{\mathbb{P}^\infty}(1)\big) \in \mathbb{MGL}^0(\mathbb{P}^\infty)$.

Proposition B.4. *The element γ is an orientation of the symmetric commutative ring T-spectrum $\mathbb{MGL}$.*

Proof. Obviously, $\gamma|_{\mathbb{P}^0} = 0$. By [9], Lemma 3.6, one has $c\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) = -c\big(\mathcal{O}_{\mathbb{P}^1}(-1)\big)$. The proposition follows as:

$$\gamma|_{\mathbb{P}^1} = c\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) = -c\big(\mathcal{O}_{\mathbb{P}^1}(-1)\big) = -c|_{\mathbb{P}^1} = \Sigma_T(1). \quad \square \tag{B.3}$$

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Eingegangen 4. Oktober 2006, in revidierter Fassung 28. Februar 2007