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Rigidity for Henselian local rings and $\mathbb{A}^1$-representable theories

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Abstract We prove that for a large class of $\mathbb{A}^1$-representable theories including all orientable theories it is possible to construct transfer maps and to prove rigidity theorems similar to those of Gabber for algebraic K-theory. This extends rigidity results of Panin and Yagunov from algebraically closed fields to arbitrary infinite ones.

0 Introduction

The aim of this paper is to establish rigidity results for graded cohomology type functors E on smooth varieties over an infinite base field k. This paper generalizes the results of [14] and [24] where the special case of orientable theories E resp. stably $\mathbb{A}^1$-representable theories on smooth varieties over algebraically closed fields have been studied.

Consider some category of schemes (spaces) $\mathcal{S}$ over a base scheme (space) B together with a cohomology theory $E^*\colon \mathcal{S}^{\mathrm{op}} \to \mathbf{Ab}$. Then we say that E^* satisfies *rigidity* if for every irreducible scheme $X \xrightarrow{\chi} B$, any two sections $\sigma_0, \sigma_1\colon B \to X$ of the structure morphism χ induce the same homomorphism $\sigma_0^* = \sigma_1^*\colon E^*(X) \to E^*(B)$. In classical topology, the rigidity property is an obvious consequence of homotopy invariance of cohomology theories. However, in algebraic geometry $\mathbb{A}^1$-invariance does not always imply rigidity. It only holds under certain restrictions on $\mathcal{S}$ and the cohomology theory E^*. In particular, rigidity fails for K_1 with integral coefficients.

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Rigidity results for finite coefficients have been established for algebraic K-theory by Suslin, Gabber, and others (see [3,4,18]), for hermitian K-theory by [7,9], and for Witt groups by [10,16, p. 208]. The rigidity theorem and one of its corollaries in Gabber's paper [3] are the following:

Theorem 0.1 *Suppose that R is a henselian local ring and $\ell \in \mathbb{Z}$ is invertible in R, $f\colon M \to \operatorname{Spec} R$ is a smooth affine morphism of (pure) relative dimension 1. Let $s_0, s_1\colon \operatorname{Spec} R \to M$ be two sections of f such that $s_0(P) = s_1(P)$, where P is the closed point of $\operatorname{Spec} R$. Then for every homomorphism $R \to F$, where F is any field, the two composed maps $K_*(M, \mathbb{Z}/\ell) \xrightarrow{s_i} K_*(R, \mathbb{Z}/\ell) \to K_*(F, \mathbb{Z}/\ell)$ are equal ($i = 0, 1$).*

The second morphism is known to be injective in many cases at least with integral coefficients if R is regular. In particular, if $F = \operatorname{Frac}(R)$ and R contains a field, this is the Gersten conjecture for algebraic K-theory as proved by Quillen [15] and Panin [13] in this case. See Proposition 2.3 for a proof of the Gersten conjecture with finite coefficients.

Corollary 0.2 *Let M be a smooth scheme over a field k with ℓ invertible in k, $P \in M(k)$ a k-rational point of M, and $R = \mathcal{O}^h_{M,P}$ the henselization of the corresponding local ring. Then the map*

$$K_*(R, \mathbb{Z}/\ell) \xrightarrow{\cong} K_*(k, \mathbb{Z}/\ell)$$

induced by $R \to k$ is an isomorphism.

The proof of Theorem 0.1 relies on the existence of *transfer maps* fulfilling certain properties and on *homotopy invariance* [i.e. $K_*(X) \cong K_*(X \times \mathbb{A}^1)$ if X smooth] whereas the Corollary 0.2 uses moreover that K_* commutes with colimits. Throughout this paper we will always assume that our cohomology functor (also called “cohomology theory”) E commutes with filtered colimits (as all interesting examples do) and extend the domain of E accordingly. In [20, p. 227], Suslin says that the above theorem should hold for other homotopy invariant functors E having transfers for finite flat maps satisfying “the usual properties”. A first axiomatic set of the transfers and their required properties that yield a rigidity statement is published in [21]. (Compare also an unpublished manuscript of Jannsen from 1995 which is now available [6].) A different choice of axioms is proposed in [14]. Panin and Yagunov show that the axioms are satisfied for any *orientable theory* over algebraically closed fields, and deduce a rigidity theorem for orientable theories with finite coefficients. Moreover, they show that base change with respect to an extension of algebraically closed fields is then an isomorphism. In [24], Yagunov shows that these results carry over to all theories that are representable in the stable $\mathbb{A}^1$-homotopy category of [22]. Examples include hermitian K-theory, Balmer Witt groups assuming $\operatorname{char}(k) \neq 2$, and stable cohomotopy groups. Stable $\mathbb{A}^1$-representability allows Yagunov to construct algebraic “Becker–Gottlieb transfers” with respect to a class of morphisms $\mathcal{C}_{\mathrm{triv}}$, which is rather small but still large enough to conclude. We will review these transfers in Sect. 1.

This paper generalizes the above results. Assuming certain additional hypotheses that can be checked in many cases of interest (see Corollary 0.5 below), we can get rid of the condition that our base field is algebraically closed, construct transfers and establish the following generalization of Theorem 0.1 and Corollary 0.2 above:

Theorem 0.3 *Let k be an infinite field and let R be a henselian local ring essentially smooth over k with field of fractions* $\mathrm{Frac}(R) = F$. *Assume that $E = E^{**}$ is a contravariant bigraded functor on the category Sm/k of smooth schemes of finite type over k that is representable in the stable $\mathbb{A}^1$-homotopy category and satisfies $\ell E = 0$ for $\ell \in \mathbb{Z}$ invertible in R. Assume, moreover, that E is normalized with respect to the field F(see Definition* 1.3). *Let $f\colon M \to \mathrm{Spec}\, R$ be a smooth affine morphism of (pure) relative dimension d, and $s_0, s_1\colon \mathrm{Spec}\, R \to M$ two sections of f such that $s_0(P) = s_1(P)$, where P is the closed point of* $\mathrm{Spec}\, R$. *Then the two maps*[1] $E(M) \overset{s_i^*}{\to} E(\mathrm{Spec}\, R)$ *are equal* $(i = 0, 1)$.

Corollary 0.4 *Let E and k be as in Theorem* 0.3, *V a smooth variety over k, $P \in V(k)$ a k-rational point of V, and $R = \mathcal{O}^h_{V,P}$. Then*

$$E(\mathrm{Spec}\, R) \overset{\cong}{\to} E(\mathrm{Spec}\, k)$$

is an isomorphism.

We will see that the proof for a general E is considerably more complicated than in the special case of K-theory.

Given a representable theory E, there is a standard way to construct an associated theory $E(\ ,\ell)$ with $\ell^2 E = 0$, see Sect. 2.

The above hypotheses will hold for orientable theories, but also for Balmer Witt groups W^* in certain degrees, see Sect. 4. For example, we have:

Corollary 0.5 *Let $X \in Sm/k$ with k as above, V a smooth variety over k, $P \in V(k)$, and $F =$* $\mathrm{Frac}(\mathcal{O}^h_{V,P})$. *Let also (as in Theorem* 0.3) *E be a representable cohomology theory such that $\ell E = 0$ for some $\ell \in \mathbb{Z}$ invertible in F. If the map $E(\mathbb{P}^2_{X_L}) \to E(\mathbb{P}^1_{X_L})$ induced by one of the standard inclusions $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is an epimorphism for every finite separable field extension L/F (e.g. $E = MGL$, H_{mot}, or K), then the map*

$$E(X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathcal{O}^h_{V,P}) \to E(X)$$

is an isomorphism. If E is represented by a commutative motivic ring spectrum, then it is sufficient to check the epimorphism condition for $X =$ $\mathrm{Spec}\, k$.

This article is organized as follows: In Sect. 1 we recall the definition of Becker–Gottlieb transfers and some results from [14] and [24]. Cohomology theories with finite coefficients are introduced in Sect. 2. Section 3 contains the proof of Theorem 0.3 and Corollary 0.4. In Sect. 4 we discuss for which theories E the hypotheses of Theorem 0.3 and Corollary 0.4 hold, which will prove, in particular, Corollary 0.5. This includes a short discussion of Witt groups.

Notation remarks Throughout this paper, E will always denote a bigraded cohomology theory which is representable in the stable $\mathbb{A}^1$-homotopy category and thus, in particular, homotopy invariant.

We use the standard "support" notation for cohomology of pairs and denote $E(X, U)$ by $E_Z(X)$, provided that U is an open subscheme of X and $Z = X - U$.

If F is a field we often write $E(F)$ instead of $E(\mathrm{Spec}\, F)$.

[1] Recall that here and below we extend the domain of E and set, for example, $E(\mathrm{Spec}\, R) := \varinjlim E(X_i)$, as $\mathrm{Spec}\, R = \varprojlim X_i$, $X_i \in Sm/k$.

T denotes the Tate object in the stable $\mathbb{A}^1$-homotopy category. Σ_T denotes both T-suspension morphism and the suspension isomorphism induced in cohomology. We omit grading of cohomology groups whenever it is possible. However, to make the T-suspension isomorphism compatible with the usual notation, we write $E^{[d]}$ for cohomology shifted by d. More precisely, if E denotes a cohomology theory $E^{*,*}$ represented by a T-spectrum, we set $E^{[d]} = E^{*+2d,*+d}$.

1 Transfers

Denote by $\mathcal{C}_{\text{triv}}$ a class of equipped morphisms (f, τ, Θ) where f is decomposed as $f\colon X \overset{\tau}{\hookrightarrow} Y \times \mathbb{A}^n \overset{p}{\to} Y$ such that τ is a closed embedding with trivial normal bundle $\mathcal{N}_{Y\times\mathbb{A}^n/X}$, p is the projection morphism, and $\Theta\colon \mathcal{N}_{Y\times\mathbb{A}^n/X} \cong X \times \mathbb{A}^N$ is a trivialization isomorphism. Abusing the notation we often omit τ or Θ if the decomposition or the trivialization is clear from the context.

Following [24], one can construct transfer maps with respect to $\mathcal{C}_{\text{triv}}$. More precisely, for any morphism $(f\colon X \to Y, \tau, \Theta) \in \mathcal{C}_{\text{triv}}$ of codimension d Yagunov defines a Becker–Gottlieb transfer map $(f, \tau, \Theta)_!\colon E(X) \to E^{[d]}(Y)$, sometimes also denoted by $(f, \Theta)_!$ or $f_!^\tau$. In [24] the base field is assumed to be algebraically closed. But this is not needed in the construction and neither in the proof of the following two properties:

Proposition 1.1 (Base change property) *Consider a commutative diagram of Cartesian squares*

$$\begin{tikzcd}
X' \arrow[r, hook, "\tau'"] \arrow[d, "g'"] \arrow[rr, bend left, "f'"] & Y' \times \mathbb{A}^n \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\
X \arrow[r, hook, "\tau"] \arrow[rr, bend right, "f"'] & Y \times \mathbb{A}^n \arrow[r] & Y
\end{tikzcd}$$

where $f \in \mathcal{C}_{\text{triv}}$ is of codimension d, and the morphisms τ, τ' are closed embeddings such that the left-hand-side square is transversal (*see* [14, *Definition* 1.1]). *Assume that Θ' is a base change of Θ in the sense that the square:*

$$\begin{tikzcd}
X' \times \mathbb{A}^{d+n} \arrow[d, "g'\times \mathrm{id}"'] & \mathcal{N}_{Y'\times\mathbb{A}^n/X'} \arrow[l, "\Theta'"'] \arrow[r, equal] \arrow[d, "N(g')"] & \mathcal{N}_{Y\times\mathbb{A}^n/X} \underset{X}{\times} X' \\
X \times \mathbb{A}^{d+n} & \mathcal{N}_{Y\times\mathbb{A}^n/X} \arrow[l, "\Theta"'] &
\end{tikzcd} \tag{1.1}$$

is Cartesian. Then, the diagram:

$$\begin{tikzcd}
E(X') \arrow[r, "(f',\tau',\Theta')_!"] & E^{[d]}(Y') \\
E(X) \arrow[u, "g'^*"] \arrow[r, "(f,\tau,\Theta)_!"] & E^{[d]}(Y).
\arrow[u, from=2-2, to=1-2, "g^*"']
\end{tikzcd}$$

commutes.

Proposition 1.2 (Additivity) *Let $X = X_0 \sqcup X_1 \in Sm/k$ be a disjoint union of subvarieties X_0 and X_1, $e_m\colon X_m \hookrightarrow X$ $(m = 0, 1)$ be the corresponding embedding morphisms, and $(f\colon X \to Y, \tau, \Theta) \in \mathcal{C}_{\mathrm{triv}}$ (codim $f = d$). Setting $f_{m,!} = (f \circ e_m, \tau \circ e_m, \Theta|_{X_m})_!$, we have:*

$$f_{0,!}e_0^* + f_{1,!}e_1^* = f_!.$$

There is a third property which will be necessary for proving our rigidity theorem:

Definition 1.3 (Normalization) *We say that a cohomology functor $E\colon Sm/k \to$ **Ab** satisfies the normalization property for a separable field extension K/k if for any $\lambda \in K^*$ the automorphism $\Sigma_T^{-1}\lambda^*\Sigma_T\colon E(K) \to E(K)$ induced by the λ-homothety of $\mathbb{A}^1_K$ is the identity [here $\Sigma_T\colon E(K) \to E^{[1]}_{\{0\}}(\mathbb{A}_K)$ is the suspension isomorphism]. We call the functor E normalized with respect to the field k if it satisfies the normalization property for every finite separable extension of the field k.*

By [24, Proposition 3.3] the following proposition holds for normalized theories.

Proposition 1.4 *For any decomposition* $\operatorname{Spec} K \overset{\tau}{\hookrightarrow} \mathbb{A}^n_K \to \operatorname{Spec} K$ *of the morphism* (id, Θ)*, the resulting transfer map* $(\mathrm{id}, \tau, \Theta)_!\colon E(K) \to E(K)$ *is the identity.*

Remark 1.5 From the proof of [24, Lemma 3.5], one can easily derive that Proposition 1.4 holds for a graded theory E in a certain degree i if and only if the normalization condition of Definition 1.3 holds for E^i. So, from now on we call both these statements *normalization property*.

The normalization property is fulfilled for algebraically closed fields, see *loc.cit*. In general, we have the following convenient criterion.

Lemma 1.6 *Assume that the map $i^*\colon E(\mathbb{P}^2_K) \to E(\mathbb{P}^1_K)$ induced by one (and thus all) of the standard inclusions $\mathbb{P}^1_K \hookrightarrow \mathbb{P}^2_K$ is surjective. Then E satisfies the normalization property for K.*

Proof As in [24, pp. 38–39], it is sufficient to check that the action of the matrix $d = diag(\lambda, 1)$ induces the identity automorphism on $E(\mathbb{P}^1)$. Consider the embeddings i_0 and i_1 of $\mathbb{P}^1$ into $\mathbb{P}^2$ given by mapping $(x{:}y)$ to $(x{:}0{:}y)$ resp. $(0{:}x{:}y)$. The induced maps i_0^* and i_1^* (both denoted by i^* above) from $E(\mathbb{P}^2)$ to $E(\mathbb{P}^1)$ are equal because E is homotopy invariant and $H\colon \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^2$ given by $H((x{:}y), t) = ((1-t)x{:}tx{:}y)$ is a homotopy between i_0 and i_1. The diagonal matrix $D = \mathrm{diag}(\lambda, 1, 1)$ induces an automorphism D^* of $E(\mathbb{P}^2)$. We have $i_0 d = D i_0$ and $i_1 = D i_1$, hence $i_0^* D^* = d^* i_0^*$ and $i_1^* D^* = i_1^*$. As $i_0^* = i_1^*$ is surjective by hypotheses, the equality $d^* i_0^* = i_1^*$ implies $d^* = \mathrm{id}$. □

Any orientable theory is normalized with respect to any field, since the projective bundle theorem $E(\mathbb{P}^n_K) \cong E(\operatorname{Spec} K)[x]/(x^{n+1})$ holds. On the other hand, in the case of the analytic topology over $\mathbb{R}$, the action induced by -1 [i.e. $\mathrm{diag}(-1, 1)$] on the real projective line $\mathbb{RP}^1 = S^1$ is not the identity on the fundamental group $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$ and the same holds for the cohomology group $H^1(\mathbb{RP}^1, \mathbb{Z})$.

We can rephrase Yagunov's theorem on the existence of transfers as follows:

Theorem 1.7 *Let E be a graded functor on Sm/k which is representable by a T-spectrum in Voevodsky's stable $\mathbb{A}^1$-homotopy category $SH(k)$. Then for every $f\colon X \to Y \in \mathcal{C}_{\mathrm{triv}}$ there is a transfer map $f_!\colon E(X) \to E(Y)$, which satisfies additivity and base change.*

2 Cohomology theories with finite coefficients

We refer the reader to [8,22] for the construction of the motivic stable homotopy category $SH(k)$ and its basic properties. In particular, we have a motivic spectrum $S^{i,j} := S^{i-j} \wedge \mathbf{G}_m^{\wedge j}$ which is the motivic suspension spectrum associated to the corresponding simplicial sheaf. For any positive integer ℓ, the motivic sphere spectrum S has a self map of degree ℓ (take the map of degree ℓ on S^1, for instance). The homotopy cofiber of $\ell\colon S \to S$ is denoted by S/ℓ and is called the *motivic Moore space mod* ℓ. As in topology, we can now define cohomology theories with integral and with finite coefficients.

Definition 2.1 *Let E be a $\mathbb{P}^1$-spectrum in the homotopy category of motivic spectra $SH(k)$. Then for every scheme $X \in Sm/k$, $\ell \in \mathbb{N}$, and $i, m \in \mathbb{Z}$, we set:*

$$E^{i,m}(X) = \mathrm{Hom}_{SH(k)}(\Sigma_T^\infty X_+, S^{i,m} \wedge E)$$
$$E^{i,m}(X, \ell) = \mathrm{Hom}_{SH(k)}(\Sigma_T^\infty X_+, S^{i,m}/\ell \wedge E)$$

In particular, we have $K^{-i,0}(X,\ell) = K_i(X, \mathbb{Z}/\ell)$ for the motivic spectrum $K = \mathbf{BGL}$ introduced in [22, 6.2], and the proof is the same as the one given in *loc. cit.* for integral coefficients. As in topology, one obtains a long exact sequence

$$\dots \to E^{i-1,m}(X,\ell) \to E^{i,m}(X) \stackrel{\times \ell}{\to} E^{i,m}(X) \to E^{i,m}(X,\ell) \to \dots$$

and one deduces:

Lemma 2.2
(a) *There is a natural short exact sequence*

$$0 \to E^{i,m}(X) \otimes \mathbb{Z}/\ell \to E^{i,m}(X,\ell) \to {}_\ell E^{i+1,m}(X) \to 0.$$

(b) *Any element in $E^{i,m}(X,\ell)$ is annihilated by ℓ^2.*

Proof The exactness of (a) is immediate from the long exact sequence. To prove (b), observe that by (a) any ℓ-divisible element in $E^{i,m}(X,\ell)$ maps to zero in ${}_\ell E^{i+1,m}(X)$ and thus lies in the image of $E^{i,m}(X) \otimes \mathbb{Z}/\ell$. □

By the definition above, if E is representable in $SH(k)$, then so is $E(\ ,\ell)$. Hence the following version of the Gersten conjecture also applies to theories with finite coefficients.

Proposition 2.3 *If k is infinite, the Cousin complex* (*see e.g.* [2, *section* 1]) *for E yields a resolution of the Zariski sheaf associated to $X \mapsto E(X)$. In particular, if R is local and essentially smooth over k, then the map*

$$0 \to E(R) \to E(\mathrm{Frac}(R))$$

is a monomorphism.

Proof Setting $E_Z(X) := E(\mathrm{cone}((X - Z) \to X)))$ and $E_x(X) := \mathrm{colim}_{U \ni x} E_{\bar{x} \cap U}(U)$, this is established in [5, Corollary 2.9]. The proof given there for KO is valid for any motivic spectrum E using (as E commutes with colimits by assumption) the isomorphism $E_\eta(R) \cong E(\mathrm{Frac}(R))$ where η is the generic point of $\mathrm{Spec}\, R$. □

Remark 2.4 Note that we can not eliminate the hypothesis that k is infinite as done in [2, Theorem 6.2.5] as we cannot prove the formula in COH6 of *loc. cit.* for our transfer morphisms in general.

3 Proofs

The main purpose of this section is to prove the following theorem which implies Theorem 0.3 thanks to Proposition 2.3. Corollary 0.4 follows from Theorem 0.3 exactly as it does in [3, p. 66].

Theorem 3.1 *Let R be a henselian local ring over k with an infinite field of fractions* $\mathrm{Frac}(R) = F$. *Assume that $E = E^{**}$ is a contravariant bigraded functor on the category Sm/k representable in the stable $\mathbb{A}^1$-homotopy category, $\ell E = 0$, and $\ell \in \mathbb{Z}$ is invertible in R. Assume, moreover, that E is normalized* (*see Definition* 1.3) *with respect to F. Let $f\colon M \to \operatorname{Spec} R$ be a smooth affine morphism of (pure) relative dimension d, $s_0, s_1\colon \operatorname{Spec} R \to M$ two sections of f such that $s_0(P) = s_1(P)$ where P is the closed point of $\operatorname{Spec} R$. Then the two composed maps $s_0^*, s_1^*\colon E(M) \rightrightarrows E(R) \to E(F)$ are equal.*

Proof Parts of the proof follow the one of Gabber [3, pp. 67–69] for algebraic K-theory. As Gabber observes [3, Remark, p. 67], it is enough to consider the case that f is of relative dimension one. We first get rid of local rings and reduce the question to some form of the rigidity theorem for fields. For this, denote by M_F the generic fiber of M, that is the fibered product $M \underset{\operatorname{Spec} R}{\times} \operatorname{Spec} F$ with respect to the canonical morphism $\rho\colon \operatorname{Spec} F \to \operatorname{Spec} R$. Then the maps s_i induce maps $s_i^F\colon \operatorname{Spec} F \to M_F$ via base change, so it suffices to show that $(s_0^F)^* = (s_1^F)^*$.

For $s_i^F\colon \operatorname{Spec} F \to M_F$, we set $P_i\colon = \mathrm{Im}(s_i^F)$. Since the open neighborhoods of the points P_0, P_1 form an inductive system, the value of $(s_0^F)^* - (s_1^F)^*$ is independent of the choice of the containing open neighborhood. So, it is sufficient to establish the equality $(s_0^F)^* = (s_1^F)^*$ for one special affine neighborhood. Setting $\overline{M}$ to be a projective closure of M and C the normalization of the curve $(\overline{M})_F$, we can choose an open neighborhood U of $\{s_0(R), s_1(R)\}$ in M such that $\Omega_{M/R}$ and thus the tangent bundle is trivial when restricted to U. [To find such a U first choose a desired neighborhood for $s_0(R)$ and then deduce that it also contains $s_1(R)$ since both sections coincide on the closed fiber.] We further set $Z := (\overline{M} - U)_{\mathrm{red}}$, $C^\circ := U_F$, and $C_\infty := C - C^\circ$; thus the tangent bundle of C° is also trivial.

Below we identify the invertible sheaf corresponding to the divisor D with its image in the Picard group $\mathrm{Pic}(C, C_\infty)$ and denote it by $\mathcal{O}(D)$ unless any confusion may appear. Recall that the relative Picard group $\mathrm{Pic}(C, C_\infty)$ is by definition the set of isomorphism classes of pairs (L, ψ) where L is a line bundle on C and $\psi\colon L \otimes_{\mathcal{O}_C} \mathcal{O}_{C_\infty} \xrightarrow{\cong} \mathcal{O}_{C_\infty}$ is a trivialization of $L|_{C_\infty}$. The relative Picard group $\mathrm{Pic}(C, C_\infty)$ can be identified with $H^1_{\text{ét}}(C, \mathcal{O}^*_{C,C_\infty})$. This is true for any closed embedding of schemes $C_\infty \subset C$, see [3, p. 67] for more details.

The statement $(s_0^F)^* = (s_1^F)^*$ is implied by the following two theorems:

Theorem 3.2 *For C, C_∞ as above and any integer $\ell \neq 0$ coprime to* $\operatorname{char} F$ *the divisor $\mathcal{O}_C(P_0 - P_1)$ is ℓ-divisible in the relative Picard group* $\mathrm{Pic}(C, C_\infty)$.

Theorem 3.3 *In the above situation, there exists a bilinear pairing*

$$<,>\colon \mathrm{Pic}(C, C_\infty) \times E(C^\circ) \to E(F)$$

such that for any $c \in E(C^\circ)$ and the above F-rational points $P_0, P_1 \in C^\circ$ the equality

$$< \mathcal{O}_C(P_0 - P_1), c >= (s_0^F)^*(c) - (s_1^F)^*(c)$$

holds in $E(F)$.

The first theorem is similar to one proven in [3, Corollary p. 68] (see also [4,21]).

Before proving Theorems 3.2 and 3.3, we show how to derive Theorem 3.1 from them. By Theorem 3.2, there is an element $(\mathcal{L}, \psi)$ in $\mathrm{Pic}(C, C_\infty)$ with $\ell(\mathcal{L}, \psi) = \mathcal{O}_C(P_0 - P_1)$. Then by Theorem 3.3, for any element $c \in E(C^\circ)$ we have: $(s_0^F)^*(c) - (s_1^F)^*(c) =< \mathcal{O}_C(P_0 - P_1), c >= \ell < (L, \psi), c >= 0$ which proves Theorem 3.1. □

Proof of Theorem 3.2 Consider the following short exact sequence of étale sheaves:

$$0 \to j_! \mu_\ell \to \mathcal{O}^*_{C,C_\infty} \xrightarrow{\ell} \mathcal{O}^*_{C,C_\infty} \to 0, \tag{3.1}$$

where $C^\circ \overset{j}{\hookrightarrow} C \overset{i}{\leftarrow} C_\infty$ and $\mathcal{O}^*_{C,C_\infty}$ is the sheaf $\mathrm{Ker}(\mathcal{O}^*_C \to i_* \mathcal{O}^*_{C_\infty})$.

Now we write down a fragment of the cohomology long exact sequence associated to (3.1) in the form:

$$\mathrm{Pic}(C, C_\infty) \xrightarrow{\ell} \mathrm{Pic}(C, C_\infty) \xrightarrow{\delta_F} H^2_{\acute{e}t}(C, j_! \mu_\ell) \tag{3.2}$$

Using the definition of C, C_∞, one sees that the hypotheses of [3, Proposition 4] are satisfied (the slightly different version of [21] requires the existence of a good compactification here). Therefore, in this situation every element of $\mathrm{Pic}(C, C_\infty)$ is of the form $\mathcal{O}_C(D)$ for some $D \in \mathrm{Div}(C, C_\infty)$. Here $\mathrm{Div}(C, C_\infty)$ denotes the group of Cartier divisors on C with support disjoint from C_∞. If $\delta_F(\mathcal{O}_C(D)) = 0$, the exact sequence above yields the equality $D = \ell D' + \mathrm{div}(f)$ in $\mathrm{Div}(C, C_\infty)$ for a suitable divisor D' and f a meromorphic function on C which is regular around C_∞ and satisfies $f\ |_{C_\infty} = 1$. To establish that $\delta_F(\mathcal{O}_C(P_0 - P_1)) = 0$ it suffices to show that the class of $s_0 - s_1 := s_0(\mathrm{Spec}\, R) - s_1(\mathrm{Spec}\, R)$ in the relative Picard group of $\overline{M}$ lies in $\mathrm{Ker}(\mathrm{Pic}(\overline{M}, Z) \xrightarrow{\delta} H^2_{\acute{e}t}(\overline{M}, J_! \mu_\ell))$, where $J \colon \overline{M} - Z \to \overline{M}$. In fact, the class of $P_0 - P_1$ is the pull-back of $s_0 - s_1$ with respect to the restriction to the generic fiber. Finally, we show, following [3,21], that $\delta(s_0 - s_1) = 0$. Let $\overline{M}_\omega = \overline{M} \times_{\mathrm{Spec}\, R} \omega$ be the special fiber of $\overline{M} \to \mathrm{Spec}\, R$. One has the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\overline{M}, Z) & \xrightarrow{\delta} & H^2_{\acute{e}t}(\overline{M}, J_! \mu_\ell) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle r} \\ \mathrm{Pic}(\overline{M}_\omega, Z_\omega) & \xrightarrow{\delta_\omega} & H^2_{\acute{e}t}(\overline{M}_\omega, J_{\omega !} \mu_\ell), \end{array} \tag{3.3}$$

where $J_\omega \colon \overline{M}_\omega - Z_\omega \to \overline{M}_\omega$. By the proper base change theorem [17, Corollaire XII. 5.5], [12] the map r is an isomorphism. Since the sections s_0 and s_1 coincide at the closed point ω, the map α is zero on the class of $s_0 - s_1$. The statement $\delta(s_0 - s_1) = 0$ follows, which completes the proof of the theorem. □

Remark 3.4 One can see that the claim of the proper base change theorem

$$H^2_{\acute{e}t}(\overline{M}, J_! \mu_\ell) \simeq H^2_{\acute{e}t}(\overline{M}_\omega, J_{\omega !} \mu_\ell) \tag{3.4}$$

is in fact a particular case of the rigidity theorem for étale cohomology.

In order to prove Theorem 3.3, we need the following auxiliary definitions.

Definition 3.5 *Given a finite separable field extension L/F and a closed embedding $\mathcal{F} \colon \mathrm{Spec}\, L \hookrightarrow \mathbb{A}^n = \mathbb{A}^n_F$, we define a map $\mathrm{tr}^{\mathcal{F}}_{L/F} \colon E(L) \to E(F)$ in the following way.*

Choose a trivialization of the normal bundle $\Phi\colon \mathcal{N}_{\mathbb{A}^n_F/\operatorname{Spec} L} \simeq \mathbb{A}^n_L$. *Then we define* $\operatorname{tr}^{\mathcal{F}}_{L/F} = (f, \mathcal{F}, \Phi)_!$ *to be the Becker–Gottlieb transfer map corresponding to the morphism* $f\colon \operatorname{Spec} L \overset{\mathcal{F}}{\hookrightarrow} \mathbb{A}^n_F \to \operatorname{Spec} F$.

Remark 3.6 The definition of this map does not depend on the choice of the isomorphism Φ. To see this, recall [24] that the transfer map is defined by the composition

$$E(L) = E_L(L) \to E^{[n]}_{L\times\{0\}}(\mathbb{A}^n_L) \overset{\Phi}{\to} E^{[n]}_L(\mathcal{N}_{\mathbb{A}^n_F/L}) \to E^{[n]}_L(\mathbb{A}^n_F) \to E(F).$$

Now two different trivializations differ by an automorphism of $\mathbb{A}^n_L$ which induces the identity map on $E^{[n]}_{L\times\{0\}}(\mathbb{A}^n_L)$ by the normalization property (compare [24, Lemma 3.6]). We do not claim that tr is independent of the factorization of f.

First, we construct the desired pairing for a group $\widetilde{\operatorname{Pic}}(C, C_\infty)$, which we shall define now.

Definition 3.7 *Let C be a regular projective curve over a field F. A divisor on C is called separable if it can be written as $\sum a_i D_i$ such that the structure morphisms $D_i\colon \operatorname{Spec} L_i \to \operatorname{Spec} F$ are given by finite separable field extensions L_i/F. A separable divisor having all multiplicities equal to ± 1 is called unramified. If f is a function such that* $\operatorname{div}(f)$ *is unramified, we also say that f is unramified.*

For a regular projective curve C with dense open subscheme C°, and $C_\infty := C - C^\circ$, let us denote by $\operatorname{Div}_S(C, C_\infty) \subset \operatorname{Div}(C, C_\infty)$ the subgroup of all separable divisors on C whose support does not meet C_∞. We also denote by $\mathcal{M}$ the (multiplicative) group of all meromorphic functions taking the value 1 on C_∞, and we use the same notation for the corresponding subgroup of $\operatorname{Div}(C, C_\infty)$. Finally, we set

$$\widetilde{\operatorname{Pic}}(C, C_\infty) = \operatorname{Div}_S(C, C_\infty)/(\operatorname{Div}_S(C, C_\infty) \cap \mathcal{M}). \tag{3.5}$$

We now define the crucial pairing.

Definition 3.8 *For C, C°, and $C_\infty := C - C^\circ$ as above and a regular closed embedding $\mathcal{F}\colon C^\circ \hookrightarrow \mathbb{A}^n_F$, we define a bilinear pairing $<,>^{\mathcal{F}}\colon \operatorname{Div}_S(C, C_\infty) \otimes E^*(C^\circ) \to E^*(F)$ as follows. For a divisor $D = \sum_i a_i(\operatorname{Spec} L_i \overset{x_i}{\to} C^\circ)$ and $c \in E^*(C^\circ)$ we set:*

$$< D, c >^{\mathcal{F}} = \sum_i a_i \operatorname{tr}^{\mathcal{F}\circ x_i}_{L_i/F} x_i^*(c),$$

where L_i/F are the corresponding finite field extensions.

The pairing is well-defined by Remark 3.6, and the condition $< \mathcal{O}_C(P_0 - P_1), c >^{\mathcal{F}} = (s_0^F)^*(c) - (s_1^F)^*(c)$ of Theorem 3.3 is satisfied by Proposition 1.4.

Proposition 3.9 *For C, C°, and C_∞ as in* 3.8*, choose a regular closed embedding $\mathcal{F}\colon C^\circ \hookrightarrow \mathbb{A}^n_F$. Let $f\colon C \to \mathbb{P}^1_F$ be a meromorphic function such that* $\operatorname{div}(f) \in \operatorname{Div}_S(C, C_\infty) \cap \mathcal{M}$. *Then the map* $< \operatorname{div}(f), - >^{\mathcal{F}}$ *is trivial.*

Proof First, we assume that f is unramified. Then the proof is similar to the one of [24, Theorem 1.6.]. Denote by $\tilde{C}^\circ$ the open locus $f \neq 1$ on C°. Since the morphism $(\mathcal{F}, f)\colon C^\circ \to \mathbb{A}^n \times \mathbb{P}^1$ is a closed embedding, one has (after the natural identification $\mathbb{A}^1 = \mathbb{P}^1 - \{1\}$) the closed embedding $\mathcal{G} = (\mathcal{F}, f)_{\tilde{C}^\circ}\colon \tilde{C}^\circ \to \mathbb{A}^n \times \mathbb{A}^1$. Recall that the curve U and consequently $\tilde{C}^\circ$ were chosen in such a way that by Remark 3.10 below the embedding $\mathcal{G}\colon \tilde{C}^\circ \to \mathbb{A}^n \times \mathbb{A}^1$ has a trivial normal bundle. Thus, the morphism $\tilde{C}^\circ \xrightarrow{\mathcal{G}} \mathbb{A}^n \times \mathbb{A}^1 \xrightarrow{\text{pr}} \mathbb{A}^1$ belongs to $\mathcal{C}_{\text{triv}}$. Setting $D = \operatorname{div}_0(f)$ to be the locus of $f = 0$ and $D' = \operatorname{div}_\infty(f)$ the locus of $f = \infty$, we now show that $< D, - >^{\mathcal{F}} = < D', - >^{\mathcal{F}}$. Consider the square corresponding to the restriction of f to its fiber over the point $\{0\}$ which is commutative by the base change and additivity properties:

$$\begin{tikzcd}
E(\tilde{C}^\circ) \arrow[r, "\oplus x_i^*"] \arrow[d] \arrow[dd, bend right=60, "f_!^{\mathcal{G}}"'] & \bigoplus\limits_{x_i\in f^{-1}(0)} E(\operatorname{Spec} L_i) \arrow[d] \arrow[ddr, "\oplus(f|_{x_i})_!^{\mathcal{G}}"] & \\
E(\mathbb{A}^n_F\times\mathbb{A}^1_F) \arrow[d] & \oplus E(\mathbb{A}^n_F\times\{0\}) \arrow[d] & \\
E(\mathbb{A}^1_F) \arrow[r, "\simeq", "i_0^*"'] & E(\{0\}) \arrow[r, equal] & E(\operatorname{Spec} F),
\end{tikzcd} \tag{3.6}$$

where the field extensions L_i/F correspond to the points x_i lying over $\{0\}$.

For a point $x \in f^{-1}(0)$, one has $\mathcal{F}|_x \times \{0\} = \mathcal{G}|_x$. Therefore, the transfer map on the right hand side becomes the sum of $\operatorname{tr}^{\mathcal{F}|_{x_i}}_{L_i/F}$, so it does not depend on f.

The diagram shows that $i_0^* f_!^{\mathcal{G}} = \sum_{x_i} \operatorname{tr}^{\mathcal{F}|_{x_i}}_{L_i/F} x_i^* =< D, - >^{\mathcal{F}}$. Since the same can be done over the point $\{\infty\}$ and the maps i_0^*, i_∞^* are equal by homotopy invariance, the claim is proven. □

Remark 3.10 Any closed embedding of the curve $\tilde{C}^\circ$ in $\mathbb{A}^{n+1}_F$ has trivial normal bundle for the following reason: since the tangent bundles of both our curve and $\mathbb{A}^{n+1}$ are trivial, the normal bundle is stably trivial. A stably trivial vector bundle on an affine curve is already trivial: a rank n bundle $\mathcal{E}$ on a smooth affine curve contains a rank $n-1$ trivial summand, so $\mathcal{E} = \mathcal{O}^{n-1} \oplus \mathcal{L}$. Now let q be such an integer that the vector bundle $\mathcal{E} \oplus \mathcal{O}^q$ is trivial. Since $\det(\mathcal{E} \oplus \mathcal{O}^q) = \mathcal{L}$, $\mathcal{E}$ is stably trivial if and only if $\mathcal{L} = \mathcal{O}$.

We now show that any divisor Q of $\operatorname{Div}_S(C, C_\infty) \cap \mathcal{M}$ can be written as a sum of unramified principal divisors admitting representatives taking value 1 on C_∞. This follows applying the lemma below to a principal divisor $Q = \operatorname{div}(f)$. This completes the proof of Proposition 3.9.

Lemma 3.11 *Every divisor* $Q \in \operatorname{Div}_S(C, C_\infty)$ *can be written in the form* $Q = \sum_i \operatorname{div}(g_i) + Q'$*, where* $g_i \in \mathcal{M}$*, all the divisors on the right-hand-side belong to* $\operatorname{Div}_S(C, C_\infty)$ *and are unramified.*

Proof This as an easy modification of the proof of [23, Lemma 3.16]. Let $Q = \sum_i \pm P_i$, where P_i are (not necessary different) closed points on C°. Applying Voevodsky's proof to each of these points, one gets equivalences $P_i \sim Q_i'$. All the equivalences can be obtained by unramified functions $g_i \in \mathcal{M}$ such that the divisors $Q_i' \in \operatorname{Div}_S(C, C_\infty)$ are unramified. Since there are infinitely many closed points of C° with separable residue fields, the procedure used enables us to chose the equivalences in such a way that the supports of Q_i' are pairwise disjoint. Finally, one just sets $Q' = \sum_i \pm Q_i'$. □

We thus have constructed the desired pairing for the group $\widetilde{\operatorname{Pic}}(C, C_\infty)$. Using Proposition 3.12 below, one obtains the pairing required in Theorem 3.3.

Proposition 3.12 *The natural map* $\widetilde{\mathrm{Pic}}(C, C_\infty) \to \mathrm{Pic}(C, C_\infty)$ *is an isomorphism.*

Proof The injectivity easily follows from the definitions. The surjectivity follows by [23, Lemma 3.16]. Observe that the proof in *loc. cit.* is written for curves having a smooth compactification, whereas over fields of finite characteristic one only has regular compactification in general. Fortunately, this condition is not essential, since the Riemann–Roch theorem implicitly used there also holds without this assumption, see e.g. [11, Section 7.3.2]. □

Remark 3.13 Observe that we cannot replace our proof of Theorem 3.3 by the argument of [4, Lemma 2.2]. As divisors may have multiplicities, this would require (unique) transfers for finite morphisms $\mathrm{Spec}\, L \to \mathrm{Spec}\, F$ even when L is not a field and hence the normalization property for such L. Moreover, the squares for which we would need base change are no longer transversal in this case.

4 Examples

We now prove Corollary 0.5. The first claim follows from Corollary 0.4, applied to the functor $\tilde{E}(Y) := E(Y \times X)$. Observe that since E is representable, so is $\tilde{E}$ again by using an adjoint of $\wedge X_+$ (compare [8, p. 459]). Alternatively, one may check this directly, compare also [14, Proposition 2.17]. The assumptions made on E trivially imply that $\tilde{E}$ satisfies the normalization property with respect to F.

We note that any orientable theory satisfies the projective bundle theorem and thus this hypothesis. In particular, this applies to algebraic K-theory, motivic cohomology H, and algebraic cobordism MGL (which is orientable by [14, Section 6.5]).

We now prove the last claim of Corollary 0.5. Assume that E is represented by a commutative ring spectrum $\mathcal{E}$. That means by definition that we have a multiplication map $\mathcal{E} \wedge \mathcal{E} \overset{\mu}{\to} \mathcal{E}$ and a unit map $S \overset{\iota}{\to} \mathcal{E}$ such that the standard diagrams in $SH(k)$ commute. For $X, Y \in SH(k)$, consider the cohomology external product $\wedge\colon E(X) \otimes E(Y) \to E(X \wedge Y)$ sending the cohomology classes $\alpha\colon X \to \mathcal{E}$, $\beta\colon Y \to \mathcal{E}$ to

$$\alpha \wedge \beta\colon X \wedge Y \overset{\alpha\wedge\beta}{\to} \mathcal{E} \wedge \mathcal{E} \overset{\mu}{\to} \mathcal{E}.$$

The coefficient group $E(S^0) = E$ becomes a graded commutative bigraded ring with unit (the class of the unit morphism $\iota\colon S^0 \to \mathcal{E}$). After the natural identifications $S^0 \wedge X = X = X \wedge S^0$, all groups $E(X)$ become left and right E-modules via the above external product, and the map $E(X) \otimes_E E(Y) \overset{\wedge}{\to} E(X \wedge Y)$ becomes a map of E-modules. The $\wedge$-product is functorial in the following sense. Let $f\colon U \to W$. Then for every V, the following diagram commutes:

$$\begin{tikzcd} E(W) \otimes_E E(V) \arrow[r, "\wedge"] \arrow[d, "f^* \otimes \mathrm{id}"] & E(W \wedge V) \arrow[d, "(f \wedge \mathrm{id})^*"] \\ E(U) \otimes_E E(V) \arrow[r, "\wedge"] & E(U \wedge V). \end{tikzcd}$$

From now on, X and $\mathbb{P}^1$ are unpointed varieties, and $(\ )_+$ denotes an added base-point to an unpointed variety. Recall that $\mathrm{Spec}\, k_+ = S^0$. We drop Σ_T^∞ from our notation, thus identifying a pointed variety with its motivic suspension spectrum. The claim above follows immediately from the result below as the epimorphism condition implies that the (induced by a homothety) map d^* in the proof of Lemma 1.6 acts as the identity.

Proposition 4.1 *Let E be a cohomology theory represented by a commutative motivic ring spectrum. Let $\lambda\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism which induces the identity endomorphism λ^* on the cohomology group $E(\mathbb{P}^1_+)$. Then for every X, the endomorphism $\Lambda^* = (\lambda \times \mathrm{id})^*\colon E((\mathbb{P}^1 \times X)_+) \to E((\mathbb{P}^1 \times X)_+)$ is the identity, as well.*

Proof Recall that $\mathbb{P}^1_+ \wedge X_+ = (\mathbb{P}^1 \times X)_+$ and set $U = W = \mathbb{P}^1_+$, $V = X_+$, and $f = \lambda$ in the diagram above.

Let us now show that the product map $E(\mathbb{P}^1_+) \otimes_E E(X_+) \to E(\mathbb{P}^1_+ \wedge X_+)$ is an isomorphism of E-modules. Consider the splitting cofiber sequence of pointed spaces $S^0 \rightleftarrows \mathbb{P}^1_+ \longrightarrow T$, where T is the Tate object naturally isomorphic to the T-suspension of S^0. Passing to cohomology, taking the tensor product with $E(X_+)$, and the $\wedge$-product with X on the other hand, one obtains the following commutative diagram of splitting short exact sequences of E-modules:

$$\begin{tikzcd}
E(T) \otimes_E E(X_+) \arrow[r] \arrow[d, "\wedge"] & E(\mathbb{P}^1_+) \otimes_E E(X_+) \arrow[r] \arrow[d, "\wedge"] & E(S^0) \otimes_E E(X_+) \arrow[d, equal, "\wedge"] \\
E(T \wedge X_+) \arrow[r] & E(\mathbb{P}^1_+ \wedge X_+) \arrow[r] & E(S^0 \wedge X_+)
\end{tikzcd}$$

One can check that the map τ sending $\alpha \in E(T \wedge X_+)$ to $\tau(\alpha) = \Sigma_T(1) \otimes \Sigma_T^{-1}(\alpha) \in E(T) \otimes_E E(X_+)$ is an inverse to $\wedge$. So that, the product map $E(T) \otimes_E E(X_+) \xrightarrow{\wedge} E(T \wedge X_+)$ is an isomorphism and the Proposition easily follows as $\Lambda^* = \wedge \circ (\lambda^* \otimes \mathrm{id}) \circ (\wedge^{-1}) = \mathrm{id}$. □

Graded Witt groups W^* are representable by a motivic spectrum KT (see [5, Theorem 5.8]). Moreover, they are independent of the weight, that is $W^{i,m} = W^{i-m,0} = W^{i-m}$ (see [5, Corollary 5.7]). Therefore, the statement about Witt groups is related to the following result (where the part concerning W^0 with integral coefficients is classical, due to Arason) which is a consequence of [5].

Proposition 4.2 *For any field K of characteristic $\neq 2$ the inclusion $\mathbb{P}^1_K \hookrightarrow \mathbb{P}^2_K$ induces epimorphisms*

$$W^i(\mathbb{P}^2_K, \ell) \to W^i(\mathbb{P}^1_K, \ell) \quad \text{for } \ i = 2, 3.$$

Proof By Sect. 2 and the fact that $W^i(K) = 0$ for $i = 1, 2, 3$ [1, Theorem 5.6], we have a short exact sequence $W^3(K, \ell) \to W^0(K) \to W^0(K) \to W^0(K, \ell)$ and $W^1(K, \ell) = 0 = W^2(K, \ell)$. Combining this with the long exact sequence associated to the homotopy fibration $\Omega^2 KT(K) \to KT(\mathbb{P}^2_K) \to KT(\mathbb{P}^1_K)$ of [5, Proposition 6.2] (in particular KT is the motivic spectrum representing Witt groups) with finite coefficients, the claim follows. □

From this, Remark 1.5, and Lemma 1.6, the rigidity theorem follows for $W^i(-, \ell)$ if $i = 1, 2$ and $X = \operatorname{Spec} k$. But in this case the groups involved are known to be zero by [1, Theorem 5.6]. If some information on $W^*(X)$ was available concerning the ℓ-torsion part, then these methods would give more general rigidity results for Witt groups. For instance, if one knows that $W^*(X, \ell) = 0$, then applying [5, Proposition 6.2] the above methods show that $W^*(X \times_k \mathcal{O}^h_{M,P}, \ell)$ is still 0.

We finish this paragraph by pointing out that even if the above rigidity theorems were available for KO and W in all bidegrees, Gabber's strategy to deduce from this a statement similar to [3, Corollary 1] would not carry over immediately. The main problem when trying to follow Gabber's strategy with the orthogonal group instead of GL seems to establish the good analogues of [19, Proposition 1.3 and Corollary 1.6].

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