Rigidity II: Non-Orientable Case

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Abstract. The following paper is devoted to the construction of transfer maps (Becker-Gottlieb transfers) for non-orientable cohomology theories on the category of smooth algebraic varieties. Since non-orientability makes obstruction to the existence of transfer structure, we define transfers for a specially constructed class of morphisms. Being rather small, this class is yet big enough for application purposes. As an application of the developed transfer technique we get the proof of rigidity theorem for all cohomology theories represented by T-spectra.

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Introduction

The purpose of the current paper is to generalize the results obtained in [PY] to non-orientable cohomology theories on the category of smooth varieties over an algebraically closed field. This generalization seems to be especially interesting after a recent paper of Hornbostel [Ho] who proved T-representability of higher Witt groups and Hermitian K-theory. These results give us two good examples of non-orientable theories, which have important algebraic and arithmetic meaning.

In our proof we mostly follow the strategy described in [PY]. This, roughly speaking, includes constructing of transfer maps for a given theory, checking such fundamental properties as commutativity of base-change diagrams for transversal squares, finite additivity, and normalization. Finally, we use these properties to establish the main result (Rigidity theorem). Employing further the technique of Suslin [Su1], one can obtain the results similar to ones obtained in Suslin's paper for the extension of algebraically closed fields. Slightly

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adapting methods of Gabber [Ga] we may generalize the rigidity property for Hensel local rings to an arbitrary cohomology theory.

Certainly, this program would fail already on the first step, because of the result of Panin [Pa] showing that there exists an one-to-one correspondence between orientations and transfer structures. However, shrinking the class of morphisms for which we define transfers to some smaller class $\mathcal{C}_{\text{triv}}$ we may construct a satisfactory transfer structure. On the other hand, the class $\mathcal{C}_{\text{triv}}$ is still big enough to be used in the proof of the rigidity theorem.

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Notation remarks. We use the standard 'support' notation for cohomology of pairs and denote $A(X, U)$ by $A_Z(X)$, provided that U is an open subscheme of X and $Z = X - U$. Moreover, in this case we often denote the pair (X, U) by $(X)_Z$.

We omit grading of cohomology groups whenever it is possible. However, to make the T-suspension isomorphism compatible to the usual notation, we write $A^{[d]}$ for cohomology shifted by d. For example, if A denotes a cohomology theory $A^{*,*}$ represented by a T-spectrum, we set $A^{[d]} = A^{*+2d,*+d}$.

For a closed smooth subscheme $Z \subset X \in Sm/k$ we denote by $B(X, Z)$ the deformation to the normal cone of Z in X. Namely, we set $B(X, Z)$ to be the blow-up of $X \times \mathbb{A}^1$ with center at $Z \times \{0\}$. More details for this well-known construction may be found in [Fu, Chapter 5],[MV, Theorem 3.2.23], or [Pa].

The notation pt is reserved for the final object $\operatorname{Spec} k$ in Sm/k.

1. Rigidity Theorem

Denote by $\mathcal{C}_{\text{triv}}$ the class of equipped morphisms (f, Θ) where f is decomposed as $f\colon X \overset{\tau}{\hookrightarrow} Y \times \mathbb{A}^n \xrightarrow{p} Y$ such that τ is a closed embedding with trivial normal bundle $\mathcal{N}_{Y\times\mathbb{A}^n/X}$, p is a projection morphism, and $\Theta\colon \mathcal{N}_{Y\times\mathbb{A}^n/X} \cong X \times \mathbb{A}^N$ is a trivialization isomorphism. Abusing the notation we often omit Θ if the trivialization is clear from the context. The main purpose of this paper is to show that the class $\mathcal{C}_{\text{triv}}$ may be endowed with a transfer structure, which makes given cohomology theory A a functor with weak transfers (see [PY, Definition 1.8]) with respect to this class. We also show that $\mathcal{C}_{\text{triv}}$ is still big enough to fit all the requirements of constructions used in [PY] to prove the Rigidity Theorem. This, finally, yields Theorem 1.10, which may be applied to concrete examples of theories.

Let Sm/k be a category of smooth varieties over an algebraically closed field k. Denote by Sm^2/k a category whose objects are pairs (X, Y), where $X, Y \in Sm/k$, the scheme Y is a locally closed subscheme in X and morphisms are defined in a usual way as morphisms of pairs. A functor $\mathcal{E}\colon X \mapsto (X, \emptyset)$ identifies Sm/k with a full subcategory of Sm^2/k.

Definition 1.1. *We say that a functor* $F\colon Sm/k \to G$ *admits extension to pairs by a functor* $\mathcal{F}\colon Sm^2/k \to G$ *if* $F = \mathcal{F} \circ \mathcal{E}$.

Definition 1.2. *We call a contravariant functor* $\mathcal{A}\colon Sm^2/k \to Gr\text{-}Ab$ *to the category of graded abelian groups a cohomology theory if it satisfies the following four properties:*

(1) Suspension Isomorphism. *For a scheme* $X \in Sm/k$ *and its open subscheme* U *set* $W = X - U$. *Then, we are given a functorial isomorphism*

$$\mathcal{A}_W(X) \overset{\Sigma}{\cong} \mathcal{A}^{[1]}_{W\times\{0\}}(X \times \mathbb{A}^1)$$

induced by the T*-suspension morphism.*

(2) Zariski Excision. *Let* $X \overset{i}{\supseteq} X_0 \supseteq Z$ *be objects of* Sm/k *such that* X_0 *is open in* X *and* Z *is closed in* X. *Then, the induced map* $i^*\colon \mathcal{A}_Z(X) \xrightarrow{\cong} \mathcal{A}_Z(X_0)$ *is an isomorphism.*

(3) Homotopy Invariance. *For every* $(X, Y) \in Sm^2/k$ *the map* $p^*\colon \mathcal{A}(X, Y) \to \mathcal{A}(X \times \mathbb{A}^1, Y \times \mathbb{A}^1)$ *induced by the projection is an isomorphism.*

(4) Homotopy purity. *Let* $Z \subset Y \subset X \in Sm/k$ *be closed embeddings of smooth varieties. Let* $\mathcal{N}$ *be the corresponding normal bundle over* Y, $i_0\colon \mathcal{N} \hookrightarrow B(X, Y)$ *and* $i_1\colon X \hookrightarrow B(X, Y)$ *be canonical embeddings over* 0 *and* 1, *respectively. Then, the induced maps:*

$$\mathcal{A}_Z(\mathcal{N}) \underset{\cong}{\overset{i_0^*}{\leftarrow}} \mathcal{A}_{Z\times\mathbb{A}^1}(B(X, Y)) \underset{\cong}{\overset{i_1^*}{\rightarrow}} \mathcal{A}_Z(X)$$

are isomorphisms.

Definition 1.3. *We call a contravariant functor* $A\colon Sm/k \to Gr\text{-}Ab$ *a cohomology theory if it admits an extension to pairs by the functor* $\mathcal{A}$ *which is a cohomology theory.*

In what follows we often use the same notation for functors and their extensions to pairs. We also usually identify objects X and $(X, \emptyset)$.
Most important examples of cohomology theories may be obtained in the following way.

Example 1.4. *Every functor represented by a* T*-spectrum in the sense of Voevodsky (see* [Vo]*) is a cohomology theory.*
Since the category of spaces, introduced by Voevodsky [Vo, p.583], *has fibred coproducts, we can extend any functor* $A\colon Sm/k \to G$ *to* Sm^2/k *setting* $\mathcal{A}(X, Y) = A(X/Y)$. *All functors represented by* T*-spectra satisfy conditions (1)–(3) of Definition 1.2 (see* [MV, PY, Pa]*). Condition (4) is actually Theorem 2.2.8 from* [Pa].

THEOREM 1.5. *Every cohomology theory A given on the category Sm/k of smooth varieties over an algebraically closed field k may be endowed with the structure of a functor with weak transfers for the class $\mathcal{C}_{\mathrm{triv}}$, i.e. for every $f\colon X \to Y \in \mathcal{C}_{\mathrm{triv}}$ we assign the transfer map $f_!\colon A(X) \to A(Y)$, which satisfy properties 3.1–3.3 below.*

We postpone the proof of this theorem till the last section and show, first, that the class $\mathcal{C}_{\mathrm{triv}}$ is big enough to make the proof of the Rigidity Theorem given at [PY] running. We reproduce here some constructions and arguments from [PY]. From now on we consider the case of algebraically closed base field k. Let $A\colon (Sm/k) \to \text{Gr-Ab}$ be a cohomology theory and X be a smooth curve over k. We can construct a map $\Phi\colon \mathrm{Div}(X) \to \mathrm{Hom}(A(X), A(k))$ defined on canonical generators as: $[x] \mapsto x^*$, where $x^*\colon A(X) \to A(k)$ is the pull-back map, corresponding to the point $x \in X(k)$.

THEOREM 1.6. *Let X be a smooth affine curve with trivial tangent bundle, $\bar{X}$ be its projective completion, and $X_\infty = \bar{X} - X$. Let also A be a homotopy invariant contravariant functor with weak transfers for the class $\mathcal{C}_{\mathrm{triv}}$. Then, the map Φ can be decomposed in the following way:*

$$\begin{tikzcd}
\mathrm{Div}(X) \arrow[rr, "\Phi"] \arrow[dr, "\Omega"'] & & \mathrm{Hom}(A(X), A(k)) \\
& \mathrm{Pic}(\bar{X}, X_\infty), \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

where $\mathrm{Pic}(\bar{X}, X_\infty)$ *is the relative Picard group (see* [SV]*) and the map Ω is the canonical homomorphism.*

Proof. Let us recall that a divisor $\mathcal{D}$ lies in the kernel of Ω if and only if there exists a function $f \in k(\bar{X})$ such that $f|_{X_\infty} = 1$ and $\mathcal{D} = [f]$. We denote zero and pole locuses of f by $\mathrm{div}_0(f) = D$, and $\mathrm{div}_\infty(f) = D'$, respectively. It is now sufficient to check that $\Phi(D) = \Phi(D')$.
Denote by X^0 the open locus $f \neq 1$ on $\bar{X}$. By the choice of the function f, we have: $i\colon X^0 \subset X$ and the morphism $f\colon X^0 \to \mathbb{P}^1 - \{1\} = \mathbb{A}^1$ is finite. This shows the existence of a decomposition $f\colon X^0 \xrightarrow{\tau} \mathbb{A}^n \xrightarrow{p} \mathbb{A}^1$, with closed embedding τ and projection p. For the normal bundle $\mathcal{N}_{\mathbb{A}^n/X^0}$ we have a short exact sequence:

$$0 \longrightarrow T_{X^0} \longrightarrow \tau^*(T_{\mathbb{A}^n}) \longrightarrow \mathcal{N}_{\mathbb{A}^n/X^0} \longrightarrow 0. \tag{1.1}$$

Since the tangent bundles T_{X^0} and $T_{\mathbb{A}^n}$ are trivial, the normal bundle $\mathcal{N}_{\mathbb{A}^n/X^0}$ is stably trivial. Finally, because every stably trivial bundle over a curve is trivial, we have: $f \in \mathcal{C}_{\mathrm{triv}}$.
Moreover, since we may assume (as well as in [PY, Proof of Theorem 1.11]) that the corresponding divisors are unramified, the map f is étale over the

points $\{0\}$ and $\{\infty\}$. Consider now the diagram:

$$\begin{tikzcd}
 & & A(X) \arrow[d, "i^*"] & & \\
\bigoplus\limits_{x/\infty} A(x) \arrow[dr, "\sum\limits_{x/\infty} \tilde{f}_{x,!}"'] & A(D') \arrow[l] \arrow[d, "\tilde{f}_{\infty,!}"] & A(X^0) \arrow[l] \arrow[r] \arrow[d, "\tilde{f}_!"] & A(D) \arrow[r] \arrow[d, "\tilde{f}_{0,!}"] & \bigoplus\limits_{x/0} A(x) \arrow[dl, "\sum\limits_{x/0} \tilde{f}_{x,!}"] \\
 & A(pt) & A(\mathbb{A}^1) \arrow[l, "i^*_\infty"'] \arrow[r, "i^*_0"] & A(pt) &
\end{tikzcd} \tag{1.2}$$

where $\tilde{f}_{0,!}$ ($\tilde{f}_{\infty,!}$) denotes the transfer map corresponding to the morphism $\tilde{f}_0$ (resp. $\tilde{f}_\infty$), which is a restriction of the morphism $\tilde{f}$ to the divisor D (D', respectively). Due to the discussion above, all vertical arrows are well defined, since all corresponding morphisms belong to $\mathcal{C}_{\mathrm{triv}}$. (For morphisms $\tilde{f}_0$ and $\tilde{f}_\infty$ we choose trivialization maps as restrictions of Θ.)

Using the standard properties of the functor with weak transfers (see 3.1–3.3), one can see that the diagram above is commutative.

On the other hand, it is easy to check that going from $A(X)$ to two different copies of $A(pt)$ one obtains the maps $\Phi(D)$ and $\Phi(D')$, respectively.

Finally, using the homotopy invariance of the functor A, one has:

$$\Phi(D) = i_0^* \tilde{f}_! i^* = i_\infty^* \tilde{f}_! i^* = \Phi(D'). \tag{1.3}$$

□

Since the group $\mathrm{Pic}(\bar{X}, X_\infty)^\circ$ of relative divisors of degree 0 is n-divisible over an algebraically closed field of characteristic relatively prime to n, we get the following corollary:

COROLLARY 1.7. *Let us assume, in addition to the hypothesis of Theorem 1.6, that there exists an integer n coprime to the exponential characteristic* $\mathrm{Char}(k)$ *such that $nA(Y) = 0$ for any $Y \in Sm/k$. Then, the map Ψ can be passed through the degree map* $\mathrm{Pic}(\bar{X}, X_\infty) \overset{deg}{\to} \mathbb{Z}$. *Namely, if D, D' are two divisors of the same degree, one has:* $\Phi(D) = \Phi(D') \colon A(X) \to A(\operatorname{Spec} k)$.

Now we want to get rid of the normal bundle triviality assumption. For this end, we need the following simple geometric observation.

LEMMA 1.8. *For a smooth curve X and a divisor D on X one can choose such an open neighborhood X^0 of* $\operatorname{Supp} D$ *that the tangent bundle T_{X^0} is trivial.*

Proof. Let Υ be an invertible sheaf on X corresponding to the tangent bundle T_X. Denote by $\mathcal{O}_{X,D}$ a localization of $\mathcal{O}_X$ at the support of the divisor D. The scheme $\operatorname{Spec} \mathcal{O}_{X,D}$ is a spectrum of a regular semi-local ring endowed with a natural morphism $j \colon \operatorname{Spec} \mathcal{O}_{X,D} \to X$. Therefore, the sheaf $j^*\Upsilon$ is free. This means there exists an open neighborhood $X^0 \supset \operatorname{Supp} D$ such that the restriction $\Upsilon|_{X^0}$ is free as well. □

The proofs of following two theorems are same, word by word, to ones of Theorems 1.13 and 2.17 from [PY].

THEOREM 1.9 (THE RIGIDITY THEOREM). *Let $A\colon Sm/k \to Gr\text{-}Ab$ be a homotopy invariant functor with weak transfers for the class $\mathcal{C}_{\mathrm{triv}}$. Assume that the field k is algebraically closed and $nA = 0$ for some integer n, coprime to* $\operatorname{Char} k$. *Then, for every smooth affine variety V and any two k-rational points $t_1, t_2 \in V(k)$ the induced maps $t_1^*, t_2^*\colon A(V) \to A(\operatorname{Spec} k)$ coincide.*

THEOREM 1.10. *Let $k \subset K$ be an extension of algebraically closed fields. Let also A be a cohomology theory vanishing after multiplication by n, coprime to the exponential field characteristic. Then, for any $X \in Sm/k$, we have:*

$$A(X) \xrightarrow{\cong} A(X_K).$$

Besides Theorem 1.10 we would like to mention briefly the following nice application of the developed technique [2]

THEOREM 1.11. *Let A and k be as above, $M \in Sm/k$, and R be the henselization of M at some closed point. Then, the map*

$$A(R) \xrightarrow{\cong} A(\operatorname{Spec} k)$$

is an isomorphism.

The proof of the theorem may be achieved as a direct compilation of Gabber [Ga], Suslin-Voevodsky [Su2, SV] (see also an approach of [GT]) results and Theorem 1.6. By general strategy, one reduces the above statement to a form of the Rigidity Theorem. Namely, it is possible to construct such a curve M over the field k with some special divisor D that the statement of Theorem 1.11 would follow from the fact that $\Phi(D) = 0$. The divisor D, by its construction, can be written in the form $D = n \cdot \tilde{D} + [f]$, for some divisor $\tilde{D}$ and rational function f on M (as follows from the proper base change theorem [Mi, SGA4]). Finally, it is sufficient to apply the statement of Theorem 1.6 to complete the proof.

2. BECKER–GOTTLIEB TRANSFERS

In this section we construct transfer maps required in Theorem 1.5. First of all, we build transfers with support for closed embeddings. Let $W \hookrightarrow X \overset{f}{\hookrightarrow} Y$ be closed embeddings such that $W, X, Y \in Sm/k$ and $(f, \Theta) \in \mathcal{C}_{\mathrm{triv}}$ of codimension n. We now define a map $(f, \Theta)_!\colon A_W(X) \to A_W^{[n]}(Y)$. Consider, first, following isomorphisms:

$$\varphi_W(\Theta)\colon A_W(X) \xrightarrow[\cong]{\Sigma^n} A^{[n]}_{W\times\{0\}}(X \times \mathbb{A}^n) \xrightarrow[\cong]{\Theta^*} A^{[n]}_W(\mathcal{N}_{Y/X}). \tag{2.1}$$

The next step involves Homotopy Purity property. Consider the map:

$$\chi_W\colon A_W(\mathcal{N}_{Y/X}) \underset{\cong}{\overset{(i_0^*)^{-1}}{\to}} A_{W\times\mathbb{A}^1}(B(Y,X)) \underset{\cong}{\overset{i_1^*}{\to}} A_W(Y), \tag{2.2}$$

[2] This observation was obtained jointly with Jens Hornbostel.

Definition 2.1. *The composite map:* $(f,\Theta)_!^W = \chi_W \circ \varphi_W(\Theta)\colon A_W(X) \to A_W^{[n]}(Y)$ *is called Becker–Gottlieb transfer for the closed embedding f with support W.*

In case $W = X$ we often omit any mentioning of the support. One can easily verify that the defined transfer map commutes with support extension homomorphisms. Namely, the following lemma holds.

Lemma 2.2. *Suppose we have a chain of closed embeddings:* $W_2 \hookrightarrow W_1 \hookrightarrow X \overset{f}{\hookrightarrow} Y$, *with* $f \in \mathcal{C}_{\mathrm{triv}}$. *Then, the diagram*

$$\begin{tikzcd} A_{W_2}(X) \arrow[r, "f_!^{W_2}"] \arrow[d, "ext."'] & A_{W_2}(Y) \arrow[d, "ext."] \\ A_{W_1}(X) \arrow[r, "f_!^{W_1}"] & A_{W_1}(Y) \end{tikzcd}$$

commutes. (Here ext. denotes the support extension homomorphism.)

Construction-Definition 2.3. *Let now* $(f\colon X \to Y, \Theta) \in \mathcal{C}_{\mathrm{triv}}$ *be a morphism of relative dimension d endowed with a decomposition* $X \overset{\tau}{\hookrightarrow} Y \times \mathbb{A}^n \overset{p}{\to} Y$, *with closed embedding τ and projection p. We define Becker–Gottlieb*[3] *transfer map* $(f,\Theta)_!$ *in the following way. Consider the standard open embedding* $\mathbb{A}^n \overset{j}{\hookrightarrow} \mathbb{P}^n$ *and denote the complement of* $\mathbb{A}^n$ *by* $\mathbb{P}_\infty$. *The following morphisms of pairs are induced by standard embeddings:*

(2.3)

$$(Y \times \mathbb{A}^n)_X \overset{j_X}{\hookrightarrow} (Y \times \mathbb{P}^n)_X \overset{\alpha}{\leftarrow} (Y \times \mathbb{P}^n, Y \times \mathbb{P}_\infty) \overset{\beta}{\to} (Y \times \mathbb{P}^n)_{Y\times\{0\}} \overset{j_Y}{\hookleftarrow} (Y \times \mathbb{A}^n)_{Y\times\{0\}}.$$

Since the morphism β identifies $\mathbb{P}_\infty$ *with zero-section of the line bundle* $\mathbb{P}^n - \{0\}$ *over* $\mathbb{P}^{n-1}$, *it induces an isomorphism of cohomology groups* $\beta^*\colon A_Y(Y \times \mathbb{P}^n) \overset{\cong}{\to} A(Y \times \mathbb{P}^n, Y \times \mathbb{P}_\infty)$. *The morphism j_X gives us an excision isomorphism* $j_X^*\colon A_X(Y \times \mathbb{P}^n) \overset{\cong}{\to} A_X(Y \times \mathbb{A}^n)$.
We define $(f,\Theta)_!$ *as a the following composite map:*

(2.4)

$$A(X) \xrightarrow{(\tau,\Theta)_!} A_X^{[d+n]}(Y \times \mathbb{A}^n) \xrightarrow{j_Y^*\circ(\beta^*)^{-1}\circ\alpha^*\circ(j_X^*)^{-1}} A_{Y\times\{0\}}^{[d+n]}(Y \times \mathbb{A}^n) \xrightarrow[\cong]{\Sigma^{-n}} A^{[d]}(Y),$$

where Σ^{-n} *denotes the n-fold T-desuspension. We usually denote the map* $\Sigma^{-n} \circ j_Y^* \circ (\beta^*)^{-1} \circ \alpha^* \circ (j_X^*)^{-1}$ *by* $p_!$.

3. Proof of Theorem 1.5

We now prove Theorem 1.5 checking consequently all necessary properties of a functor with weak transfers.

[3]We call the transfer maps under construction Becker–Gottlieb transfers, since we generally follow the philosophy of their paper [BG]. However, our algebraic construction is much more restrictive than the original topological one.

Proposition 3.1 (Base change property). *Given a diagram with Cartesian squares*

$$\begin{tikzcd} X' \arrow[r, hook, "\tau'"] \arrow[d, "g'"] \arrow[rr, bend left, "f'"] & Y' \times \mathbb{A}^n \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, hook, "\tau"] \arrow[rr, bend right, "f"'] & Y \times \mathbb{A}^n \arrow[r] & Y \end{tikzcd}$$

where $f \in \mathcal{C}_{\mathrm{triv}}$ of codimension d, and morphisms τ, τ' are closed embeddings such that the left-hand-side square is transversal. We also require Θ' to be a base-change of Θ in the sense that the square:

$$\begin{tikzcd} X' \times \mathbb{A}^{d+n} \arrow[d, "g' \times \mathrm{id}"'] & \mathcal{N}_{Y' \times \mathbb{A}^n / X'} \arrow[l, "\Theta'"'] \arrow[d, "N(g')"] \arrow[r, equal] & \mathcal{N}_{Y \times \mathbb{A}^n / X} \underset{X}{\times} X' \\ X \times \mathbb{A}^{d+n} & \mathcal{N}_{Y \times \mathbb{A}^n / X} \arrow[l, "\Theta"'] & \end{tikzcd} \tag{3.1}$$

is Cartesian. Then, the diagram:

$$\begin{array}{ccc} A(X') & \xrightarrow{(f',\Theta')_!} & A^{[d]}(Y') \\ {\scriptstyle g'^*}\big\uparrow & & {\scriptstyle g^*}\big\uparrow \\ A(X) & \xrightarrow{(f,\Theta)_!} & A^{[d]}(Y). \end{array}$$

commutes.

Proof. Let us look at the diagram appearing on the first step of the computation of $f_!$.

$$\begin{tikzcd} A(X') \arrow[r, "\Sigma^{d+n}"] & A^{[d+n]}_{X' \times \{0\}}(X' \times \mathbb{A}^{d+n}) \arrow[r, "\Theta'^*"] & A^{[d+n]}_{X'}(\mathcal{N}_{Y' \times \mathbb{A}^n / X'}) \\ A(X) \arrow[u, "g'^*"] \arrow[r, "\Sigma^{d+n}"] & A^{[d+n]}_{X \times \{0\}}(X \times \mathbb{A}^{d+n}) \arrow[u, "\Sigma^n(g')^*"'] \arrow[r, "\Theta^*"] & A^{[d+n]}_{X}(\mathcal{N}_{Y \times \mathbb{A}^n / X}) \arrow[u, "N(g')^*"'] \end{tikzcd} \tag{3.2}$$

The left-hand-side square commutes because of the suspension functoriality. Commutativity of the right-hand-side one follows from Diagram 3.1. Going further along the construction of $f_!$, we may see that all other squares whose commutativity has to be checked are either commute already in the category Sm^2/k or include (de-)suspension isomorphisms like the very left one. This shows the required base-change diagram commutes. □

Proposition 3.2 (Additivity). *Let $X = X_0 \sqcup X_1 \in Sm/k$ be a disjoint union of subvarieties X_0 and X_1, $e_m\colon X_m \hookrightarrow X$ $(m = 0, 1)$ be embedding maps, and*

$(f\colon X \to Y, \Theta) \in \mathcal{C}_{\mathrm{triv}}$ $(\operatorname{codim} f = d)$. Setting $f_m = f \circ e_m$, we have:

$$f_{0,!}e_0^* + f_{1,!}e_1^* = f_!.$$

(All necessary decompositions and trivializations for morphisms f_0, f_1 are assumed to be corresponding restrictions of ones for f.)

Proof. As it follows from the proof of [PY, Proposition 4.3], in order to show the additivity property it is sufficient to check the commutativity of the following pentagonal diagram:

(3.3)
$$\begin{tikzcd}
A(X_0) \arrow[d, "\tau_{0,!}"'] & A_{X_0}(X) \arrow[l, "\psi^*"', "\cong"] \arrow[r, "\varphi^*"] \arrow[dl, dotted] & A(X) \arrow[d, "\tau_!"] \\
A^{[d+n]}_{X_0}(Y \times \mathbb{A}^n) \arrow[rr, "\chi^*"] \arrow[dr, "p_{0,!}"'] & & A^{[d+n]}_{X}(Y \times \mathbb{A}^n) \arrow[dl, "p_!"] \\
& A^{[d]}(Y), &
\end{tikzcd}$$

where ψ^* is the excision homomorphism, φ^* and χ^* are extension of support maps, and the composites $p_{0,!} \circ \tau_{0,!}$ ($p_! \circ \tau_!$, resp.) form the transfer maps $f_{0,!}$ ($f_!$, respectively). We prove, first, the commutativity of the bottom triangle. Both oblique arrows may be factored through the group $A(Y \times \mathbb{P}^n, Y \times \mathbb{P}_\infty)$. Since the diagram:

(3.4)
$$\begin{tikzcd}
(Y \times \mathbb{A}^n)_{X_0} \arrow[d, "j_{X_0}"'] & & (Y \times \mathbb{A}^n)_X \arrow[ll, hook', "\chi"'] \arrow[d, "j_X"] \\
(Y \times \mathbb{P}^n)_{X_0} & & (Y \times \mathbb{P}^n)_X \arrow[ll, hook'] \\
& (Y \times \mathbb{P}^n, Y \times \mathbb{P}_\infty) \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

commutes already in Sm^2/k, the required triangle commutes as well. We now show the commutativity of the rectangular part of Diagram 3.3. Define the dotted map as a transfer with the support X_0 corresponding to the embedding $X \hookrightarrow Y \times \mathbb{A}^n = \mathcal{Y}$. This (due to Lemma 2.2) makes the right trapezium part of the diagram commutative. Commutativity of the upper-left triangle is equivalent, by the definition, to a claim that the following diagram commutes.
(3.5)
$$\begin{tikzcd}
A_{X_0}(X) \arrow[r, "\Sigma^{d+n}"] \arrow[d, "\psi^*"', "\cong"] \arrow[dr, phantom, "(1)"] & A^{[d+n]}_{X_0}(X \times \mathbb{A}^{d+n}) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "(2)"] & A^{[d+n]}_{X_0}(\mathcal{N}_{\mathcal{Y}/X}) \arrow[d, "\cong"] \arrow[dr, phantom, "(3)"] & A^{[d+n]}_{X_0 \times \mathbb{A}^1}(B(\mathcal{Y}, X)) \arrow[l, "\cong"'] \arrow[r] \arrow[dr, phantom, "(4)"] & A^{[d+n]}_{X_0}(\mathcal{Y}) \arrow[d, equal] \\
A(X_0) \arrow[r, "\Sigma^{d+n}"] & A^{[d+n]}_{X_0}(X_0 \times \mathbb{A}^{d+n}) \arrow[r] & A^{[d+n]}_{X_0}(\mathcal{N}_{\mathcal{Y}/X_0}) & A^{[d+n]}_{X_0 \times \mathbb{A}^1}(B(\mathcal{Y}, X_0)) \arrow[l, "\cong"'] \arrow[r] \arrow[u, "\sigma^*"] & A^{[d+n]}_{X_0}(\mathcal{Y})
\end{tikzcd}$$

(Here the map σ^* is induced by a morphism blowing down a component X_1. See [PY, Section 4] for more details.) Simple arguments utilized at the end of the proof of Proposition 3.1 may be used here as well. Namely, the suspension isomorphism functoriality implies commutativity of square (1). Square (2) commutes, because its bottom horizontal arrow is the restriction of the top one. Squares (3) and (4) are induced by commutative diagrams of varieties. The proposition follows. □

PROPOSITION 3.3 (NORMALIZATION). *For the morphism* $\mathrm{id}\colon pt \to pt = \mathrm{Spec}(k)$ *endowed with an arbitrary decomposition* $pt \overset{\tau}{\hookrightarrow} \mathbb{A}^n \to pt$ *the map* $\mathrm{id}_!\colon A(pt) \to A(pt)$ *is identical.*

In order to prove this statement we show, first, that in case of the identity map $pt \to pt$ the constructed transfer map does not depend on the choice of normal bundle trivialization.

LEMMA 3.4. *Let* $X \in Sm^2/k$ *be a smooth pair endowed with a linear action of* $SL_n(k)$*. Then, for any matrix* $\alpha \in SL_n(k)$ *the induced isomorphism* $A(X) \xrightarrow{\alpha^*} A(X)$ *is the identity map.*

Proof. Every matrix of $SL_n(k)$ may be written as a product of elementary matrices. Every elementary matrix acts trivially on X, because of existence of a canonical contracting homotopy $H(e_{ij}(a), t) = e_{ij}(at)$. □

LEMMA 3.5. *The standard homothety action of* $\lambda \in k^*$ *on the affine line* $\mathbb{A}^1$ *induces the identity isomorphism* $A_{\{0\}}(\mathbb{A}^1) \xrightarrow{\lambda^*} A_{\{0\}}(\mathbb{A}^1)$.

Proof. Consider a diagram:

(3.6)
$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, "\lambda"] \arrow[d, hook] & \mathbb{A}^1 \arrow[d, hook] \\ \mathbb{P}^1 \arrow[r, "\Lambda"] & \mathbb{P}^1, \end{tikzcd}$$

where $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ and the vertical arrows are standard open embeddings given by: $a \mapsto (a\colon 1)$. Due to the excision axiom, this diagram yields the following commutative diagram of cohomology groups:

(3.7)
$$\begin{tikzcd} A_{\{0\}}(\mathbb{A}^1) & A_{\{0\}}(\mathbb{A}^1) \arrow[l, "\lambda^*"'] \\ A_{\{0\}}(\mathbb{P}^1) \arrow[u, "\cong"] & A_{\{0\}}(\mathbb{P}^1). \arrow[l, "\Lambda^*"'] \arrow[u, "\cong"] \end{tikzcd}$$

Let us get rid of the support. Since the natural map $A(\mathbb{P}^1) \to A(\mathbb{A}^1) = A(pt)$ is split by the projection $\mathbb{P}^1 \to pt$, the cohomology long exact sequence shows that the support extension map $A_{\{0\}}(\mathbb{P}^1) \xrightarrow{ext} A(\mathbb{P}^1)$ is a monomorphism. The action of diagonal matrices clearly commutes with the map ext. Therefore, it is sufficient to show that the matrix Λ acts trivially on $A(\mathbb{P}^1)$. This matrix is SL_2-equivalent to the matrix $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$. (Let us recall that the field k is algebraically closed.) Due to Lemma 3.4 two SL_2-equivalent matrices induce the same action in cohomology and the latter matrix obviously acts trivially on $\mathbb{P}^1$. $\square$

Lemma 3.6. *Any matrix of $GL_n(k)$ acting on $\mathbb{A}^n$ by left multiplication induces trivial action on cohomology groups $A_{\{0\}}(\mathbb{A}^n)$.*

Proof. Changing, if necessary, the acting matrix by its SL_n-equivalent, we may assume that the action is given by the diagonal matrix $\Lambda = diag(\lambda, 1, \ldots, 1)$. Let us also mention that the pair $(\mathbb{A}^n, \mathbb{A}^n - \{0\})$ is the n-fold T-suspension of $T = (\mathbb{A}^1, \mathbb{A}^1 - \{0\})$. Since we have chosen the matrix Λ in a special way (acting only on the first factor), the suspension isomorphism and Lemma 3.5 complete the proof. $\square$

Proof of Proposition 3.3 Let us consider the chain of maps giving the transfer map: $\mathrm{id}_! \colon A(pt) \to A(pt)$. We take into account that the normal bundle to pt in $\mathbb{A}^n$ is canonically isomorphic to $\mathbb{A}^n$.

(3.8)

$$\begin{tikzcd}
A(pt) \arrow[d, "\Sigma^n"] & & & & \\
A_{\{0\}}(\mathbb{A}^n) \arrow[r, "(i_0^*)^{-1}"] \arrow[rr, dotted, bend right, "\delta"'] & A_{\mathbb{A}^1}(B(\mathbb{A}^n, pt)) \arrow[r, "i_1^*"] & A_{\{0\}}(\mathbb{A}^n) \arrow[r, "(j^*)^{-1}"] \arrow[dr, dotted, "\gamma"] & A_{\{0\}}(\mathbb{P}^n) \arrow[r, "\alpha^*"] & A(\mathbb{P}^n, \mathbb{P}_\infty) \arrow[d, "(\beta^*)^{-1}"'] \\
 & & & A_{\{0\}}(\mathbb{A}^n) \arrow[d, "\Sigma^{-n}"] & A_{\{0\}}(\mathbb{P}^n) \arrow[l, "j^*"'] \\
 & & & A(pt) &
\end{tikzcd}$$

In this diagram j^* denotes the excision isomorphism and maps γ and δ are just set to be composites of the fitting arrows. As it was shown in [PY, Lemma 5.8], the map δ is identical. Since in the considered case both maps α^* and β^* are induced by the same embedding $(\mathbb{P}^n, \mathbb{P}_\infty) \hookrightarrow (\mathbb{P}^n, \mathbb{P}^n - \{0\})$, the map γ is also identical. This finishes the proof of Normalization property. $\square$

The latter three propositions actually check all the conditions required by the definition of a functor with weak transfers. This completes the proof of Theorem 1.5 as well.

References

[BG] Becker, J. C.; Gottlieb, D. H. *The transfer map and fiber bundles.* Topology 14 (1975), pp. 1–12.

[Fu] Fulton, W. *Intersection theory.* Springer-Verlag, Berlin, 1998, xiv+470 pp.

[Ga] Gabber, O. *K-theory of Henselian Local Rings and Henselian Pairs.* Contemp. Math. 126 (1992), pp. 59-70.

[GT] H. Gillet and R. Thomason: *The K-theory of strict hensel local rings and a theorem of Suslin.* JPAA 34 (1984), pp. 241-254

[Ho] Hornbostel, J. *$\mathbb{A}^1$-representability of hermitian K-theory and Witt groups.* (preprint) `www.math.uiuc.edu/K-theory/0578/` 2002.

[Mi] Milne, J. *Étale cohomology.* Princeton University Press 1980, xiii+323 pp.

[MV] Morel, F.; Voevodsky, V. *A^1-homotopy theory of schemes.* Inst. Hautes Études Sci. Publ. Math. No. 90, pp. 45–143 (2001).

[Pa] Panin, I. *Push-forwards in Oriented Cohomology Theories of Algebraic Varieties: II* (preprint) `www.math.uiuc.edu/K-theory/0619/` 2003.

[PY] Panin, I.; Yagunov S. *Rigidity for Orientable Functors.* Journal of Pure and Applied Algebra, vol.172, (2002) pp. 49–77.

[SGA4] *Théorie des topos et cohomologie étale des schémas*(SGA 4) dirigé par M.Artin, A.Grothendieck et J.-L.Verdier, Tome III, Lecture Notes in Mathematics 305, Springer-Verlag, 1973.

[Su1] Suslin, A. *On the K-theory of algebraically closed fields.* Invent. Math. 73 (1983), no. 2, pp. 241–245.

[Su2] Suslin, A. *Algebraic K-Theory of Fields.* Proceedings of the ICM 1986 (Berkeley), Vol.1, pp. 222-244.

[SV] Suslin, A.; Voevodsky, V. *Singular homology of abstract algebraic varieties.* Invent. Math. 123 (1996), no. 1, pp. 61–94.

[Vo] Voevodsky,V. *$\mathbb{A}^1$-homotopy theory.* Documenta Mathematica, Extra volume ICM 1998, pp. 579-604.

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