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On the Homology of GL_n and Higher Pre-Bloch Groups

Serge Yagunov

Abstract. For every integer $n > 1$ and infinite field F we construct a spectral sequence converging to the homology of $\mathrm{GL}_n(F)$ relative to the group of monomial matrices $\mathrm{GM}_n(F)$. Some entries in E^2-terms of these spectral sequences may be interpreted as a natural generalization of the Bloch group to higher dimensions. These groups may be characterized as homology of GL_n relatively to GL_{n-1} and GM_n. We apply the machinery developed to the investigation of stabilization maps in homology of General Linear Groups.

Introduction

The purpose of the present work is to extend an approach used for the investigation of the homology of GL_n by Suslin, Sah, and others. Studying of groups $H_*(\mathrm{GL}_n)$ seems rather important, in particular, because of their close relation to algebraic K-theory. Unfortunately, these groups are much too big and complicated to be computed explicitly [12]. Therefore, all results allowing to compare groups $H_*(\mathrm{GL}_n)$ for different values of n become quite important.

Almost all known methods of getting information about the homology of GL_n (see, for example, [4], [7], [8], [10], [11], [13], [14], [15], [17]) are based on the following approach. We choose some GL_n-resolution of the coefficient ring such that the corresponding hyperhomology spectral sequence has a desired form. This spectral sequence converges to $H_*(\mathrm{GL}_n)$ and we usually want to build the GL_n-resolution which, after passing to coinvariants, has first nontrivial homology in high enough dimension. This allows us to analyze the relative homology group in corresponding dimensions. In [10], [14] the authors use a complex constructed from points in general position of the affine space. This proves the coincidence of groups $H_n\big(\mathrm{GL}_n(F), \mathrm{GL}_{n-1}(F)\big)$ and $K_n^M(F)$.

In [11], [15] we see a complex cooked up from distinct points of the projective line $\mathbb{P}^1(F)$. After applying to this complex the functor of GL_n-coinvariants, we get a complex, whose first non-trivial homology group lies in dimension 3. This group $\wp(F)$ (we call it pre-Bloch group) coincides with $H_3\big(\mathrm{GL}_2(F), \mathrm{GM}_2(F)\big)$.

The extension of this method to the n-dimensional projective space $\mathbb{P}^n$ would be quite powerful tool for the investigation of $H_{n+1}(\mathrm{GL}_n, \mathrm{GM}_n)$. Unfortunately, differentials in corresponding spectral sequences are very hard to compute.

One possible approach to eliminate computational difficulties, at least in low dimensions, is presented in this paper. For any natural n we construct a complex of GL_n-modules such that the corresponding hyperhomology spectral sequence converges to $H_*(\mathrm{GL}_n, \mathrm{GM}_n)$. From these spectral sequences we obtain the sequence of groups $\wp^n(F)$

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which serves as some generalization of the pre-Bloch group to higher dimensions. We expect the following conjectures to be true:

Conjecture 0.1 Assume that the field F is infinite, $n > 1$ is an integer, and $n!$ is invertible in the coefficient ring $\mathbb{A}$. Then we have the following natural isomorphism:

$$\wp^n(F) = \bigoplus_{i \geq 0} H_{n+1-2i}(\mathrm{GL}_{n-2i}, \{\mathrm{GM}_{n-2i}, \mathrm{GL}_{n-2i-1}\})_{\mathrm{ind}},$$

where "$_{\mathrm{ind}}$" means the indecomposable part of the homology.

Conjecture 0.2 Let us assume in addition that the field F is algebraically closed. Then the groups $\wp^n(F)$ are divisible.

In the present paper we prove the first conjecture for small dimensions ($n < 5$). For $n = 2$ Conjecture 0.2 is proven in [11], [15]. Nothing seems to be known about this conjecture for bigger n.

In order to clarify the intuitive meaning of these conjectures, let us make one step down in dimension. In [10], [14] it was shown that there exists a sequence of groups $S_n(F)$ explicitly given by generators and relations such that

$$S_n(F) = \bigoplus_{i \geq 0} H_{n-2i}(\mathrm{GL}_{n-2i}, \mathrm{GL}_{n-2i-1}). \tag{0.1}$$

These groups may be decomposed as a direct sum of certain Milnor K-groups. Therefore, Milnor K-theory becomes an obstruction for the injectivity of the map $H_n(\mathrm{GL}_n) \to H_n(\mathrm{GL}_{n+1})$. Over an algebraically closed field the Milnor K-groups are, obviously, divisible. This consideration closely relates to the well known Friedlander-Milnor Isomorphism Conjecture. (See, for example, [11], [14], [18] for more details. The case of number fields is exposed in [2].) Here I just allow myself to retype one relevant paragraph from [11].

> **4.13. Problem (Suslin).** Let F be an infinite field. Is it true that $H_i\big(\mathrm{GL}(n, F)\big) \to H_i\big(\mathrm{GL}(n+1, F)\big)$ is injective modulo torsion for all i and n?
> By using the general stability theorem of Suslin (...), we can assume $i > n > 1$ in Problem 4.13. (...) provides an affirmative answer for the first case of $i = 3$ and $n = 2$. Dupont's work suggests than an affirmative answer is available for $i = 4$ and $n = 2$.

It seems that now, a decade after the cited paper appeared, the things are still almost in the same position. Proving of Conjectures 0.1, 0.2 would imply a significant breakthrough in the case $i = n + 1$. And, at least we know that the case $i = 4$ and $n = 3$ follows from Conjecture 0.2 for $\wp^4(F)$.

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Remarks on Notation Everywhere in this paper F denotes an infinite field. All appearing vectors, matrices, *etc.*, are assumed to be over F. We denote by e_i the i-th standard basis n-column $(0, 0, \dots, 1, \dots, 0)^T$. Symbol E denotes the n-column $(1, 1, \dots, 1)^T$. We use Σ_n for the group of n-permutations or a subgroup of GL_n naturally isomorphic to the group of n-permutations. GM_n denotes a group of non-degenerated monomial matrices of size $n \times n$, *i.e.*, $\mathrm{GM}_n(F) = (F^*)^n \rtimes \Sigma_n$.

Unless the opposite mentioned all the homology groups are assumed to have coefficients in some ring $\mathbb{A}$. This ring should satisfy the condition of invertibility of $n!$, where n is defined in each concrete case.

All the induced modules are implicitly over the ring $\mathbb{A}$. Namely, if H is a subgroup of a group G, then

$$\mathrm{Ind}_H^G M \overset{\text{def}}{=} \mathbb{A}[G] \otimes_{\mathbb{A}[H]} M.$$

If M_* is a complex of H-modules and N_* is a G-complex, we denote by $\mathrm{Cone}(M_* \to N_*)$ the *Cone* of morphisms of complexes $\mathrm{Ind}_H^G M_* \to N_*$ over G.

We also use the nonstandard notation $F_{q/q-1}$ for the factor-filtration F_q/F_{q-1}.

1 Complexes Related to Groups GM_n

We are beginning to construct the desired spectral sequence. Let n be a fixed positive integer. We call an n-column over F "monomial" if it has exactly one nonzero entry and "affine" if all its entries are nonzero.

Definition 1.1 We say that an $(n \times k)$-matrix M has columns in general position (or satisfies GP-condition) if either $\operatorname{rank} M = k$ if $k \le n$, or any n-minor of M is nonzero (if $k > n$, respectively).

Definition 1.2 We say that a certain matrix satisfies the strong general position condition (SGP-condition) if any minor of any size of this matrix is non-zero.

Denote by $\tilde{D}^n_m$ the set of all $n \times (m + 1)$ matrices of the form

$$(v_0, v_1, \dots, v_j, w_{j+1}, \dots, w_m), \tag{1.1}$$

where v_i are monomial columns with nonzero entries in pairwise different positions, w_j are affine columns, and such that the submatrix $(w_{j+1}, \dots, w_m)$ satisfies the SGP-condition. (Both monomial and affine parts of this matrix are allowed to be empty.) The positions of nonzero entries in the monomial part of a matrix from $\tilde{D}^n_*$ uniquely define an ordered subset of $\{1,\ldots,n\}$. Two matrices of $\tilde{D}^n_m$ are said to be D-equivalent if they have the same affine parts and the same corresponding ordered subsets. Denote by D^n_m a free $\mathbb{A}$-module generated by all classes of equivalence formed by matrices of $\tilde{D}^n_m$. Let us define a differential operator $d\colon D^n_m \to D^n_{m-1}$ as an alternating sum of face operators d_i, which throw away the i-th column of matrix. One can easily see that the complex D^n_* is acyclic in positive dimensions. The natural action of the group GM_n on D^n_* supplies the complex D^n_* with the structure of GM_n-module. Let us now calculate the hyperhomology of GM_n with coefficients in D^n_*. First we compute the homology of the coinvariant complex $(D^n_*)_{\mathrm{GM}_n}$.

Proposition 1.3

$$H_i\big((D^n_*)_{\mathrm{GM}_n}\big) = \begin{cases} \mathbb{A} & \text{for } i=0 \\ 0 & \text{for } 0<i<n+1 \end{cases}$$

and there exists a natural epimorphism $\mathbb{A}\otimes(F^*)^{[n/2]} \twoheadrightarrow H_{n+1}\big((D^n_*)_{\mathrm{GM}_n}\big)$.

Proof Let us consider a filtration:

$$0 = F_{-1} \subset F_0 \subset \cdots \subset F_n = D^n_*, \tag{1.2}$$

where $F_kD^n_*$ is the subcomplex of D^n_* generated by matrices having at most k monomial columns. One can easily check that this filtration is compatible with differential. Since the action of GM_n doesn't change the number of monomial columns, the introduced filtration induces a filtration on the complex of coinvariants $(D^n_*)_{\mathrm{GM}_n}$ for which we also use F. Let $E^*_{*,*}$ be the spectral sequence associated to the latter filtration. Its first term has the form $E^1_{p,q} = H_{p+q}\Big(F_{q/q-1}\big((D^n_*)_{\mathrm{GM}_n}\big)\Big)$.

Let us note that for $p\geq 0$ any generator of $F_{q/q-1}\big((D^n_{p+q})_{\mathrm{GM}_n}\big)$ can be written uniquely as the following class of matrices:

$$\begin{pmatrix} & & & & & w_{11} & w_{12} & \cdots & w_{1,p} \\ e_1 & e_2 & \cdots & e_q & E & w_{21} & w_{22} & \cdots & w_{2,p} \\ & & & & & \vdots & \vdots & & \vdots \\ & & & & & w_{n1} & w_{n2} & \cdots & w_{n,p} \end{pmatrix}_{\Sigma_{n-q}} \tag{1.3}$$

where the group Σ_{n-q} acts on a matrix permuting its last $n-q$ rows. For $p=-1$ (*i.e.*, in the lowest nontrivial dimension $q-1$) the corresponding term of this complex equals to $\mathbb{A}$. One can easily see that the differential in the factor-complex $F_{q/q-1}$ acts only in the affine part throwing away the columns successively. Therefore, we can forget about the monomial part of matrix.

Lemma 1.4 *Suppose that $n!$ is invertible in the coefficient ring $\mathbb{A}$. Then*

$$E^\infty_{p,q} = E^2_{p,q} = \begin{cases} \mathrm{Ker}(H_{p+n}F_{n/n-1} \xrightarrow{d_1} H_{p+n-1}F_{n-1/n-2}); & \text{if } q=n \\ 0 & \text{otherwise.} \end{cases}$$

Sketch of the proof We are going to show that the spectral sequence corresponding to the considered filtration degenerates at term E^2. For this purpose we construct the following homotopy operator. Let us denote for a moment the matrix

$$\begin{pmatrix} & & & & a_{11} & \cdots & a_{1m} \\ & & & & \vdots & \ddots & \vdots \\ e_1 & \cdots & e_k & E & a_{k1} & \cdots & a_{km} \\ & & & & \vdots & \ddots & \vdots \\ & & & & a_{n1} & \cdots & a_{nm} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} A_1 \\ \vdots \\ A_k \\ \text{---} \\ \vdots \end{pmatrix}_k .$$

We will also need a function $\psi_k \colon \Sigma_k \to \mathbb{N}$ defined as follows. Every permutation $\sigma \in \Sigma_k$ can be considered as the permutation $\{i_1, i_2, \ldots, i_k\}$ of the standard ordinal $\{1, 2, \ldots, k\}$. We put $\psi_k(\sigma) = \min\{k, \{j : i_j > i_{j+1}\}\}$. Now we are ready to write down the necessary homotopy operator.

$$h_0 \begin{pmatrix} \text{---} \\ A_1 \\ \vdots \\ A_n \end{pmatrix}_0 = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} A_i \\ \text{---} \\ \vdots \\ \hat{A}_i \\ \vdots \end{pmatrix}_1 , \tag{1.4}$$

and for $k > 0$, we set:

$$h_k \begin{pmatrix} A_1 \\ \vdots \\ A_k \\ \text{---} \\ \vdots \\ A_n \end{pmatrix}_k = \frac{(1-n)(n-k-1)!}{n!} \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma) P^n_{\psi_k(\sigma)} \sum_{i=k+1}^{n} \begin{pmatrix} A_{\sigma^{-1}(1)} \\ \vdots \\ A_{\sigma^{-1}(k)} \\ A_i \\ \text{---} \\ \vdots \\ \hat{A}_i \\ \vdots \end{pmatrix}_{k+1} \tag{1.5}$$

where $\operatorname{sgn}(\sigma)$ is just the usual sign of a permutation and P^n_k is given by the following relations:

$$P^n_1 = 1, \tag{1.6}$$

$$P^n_k = (k-n-1)P^n_{k-1} - \sum_{i=1}^{k-2} P^n_i .$$

One can easily verify that these operators give us a contracting homotopy which commutes with the differential in the factor-filtration and, therefore, define a contracting homotopy on E^1-term of the spectral sequence associated to the introduced filtration. Lemma now follows. Formulas above are the result of tedious combinatorial computations which we prefer to omit here. The interested reader can follow [18] and [19] for more details. ■

Let us now finish the computation of the E^2-term.

Consider a complex of $\mathbb{A}$-modules

$$\mathbb{A} \longleftarrow \mathbb{A}[\mathcal{F}_0] \longleftarrow \mathbb{A}[\mathcal{F}_1] \longleftarrow \cdots \tag{1.7}$$

where $\mathbb{A}[\mathcal{F}_i]$ are free $\mathbb{A}$-modules generated by sets $\mathcal{F}_i$ of all $n \times (i+1)$-matrices satisfying the SGP-condition. The differential operator is defined in the standard way. This complex gives us a free $(F^*)^n$-resolution of $\mathbb{A}$. One can easily verify that

$$F_{n-1/n-2}\big((D_*^n)_{\mathrm{GM}_n}\big)[n-1] \simeq F_{n/n-1}\big((D_*^n)_{\mathrm{GM}_n}\big)[n] \simeq \mathbb{A}[\mathcal{F}_*]_{(F^*)^n}, \tag{1.8}$$

where the last term is the complex of $(F^*)^n$-coinvariants of $\mathbb{A}[\mathcal{F}_*]$. These isomorphisms imply that

$$H_{p+n}F_{n/n-1} \simeq H_{p+n-1}F_{n-1/n-2} \simeq H_1(\mathbb{A}[\mathcal{F}_*]_{(F^*)^n}) \simeq \mathbb{A} \otimes (F^*)^n. \tag{1.9}$$

The map $d_1 : \mathbb{A} \otimes (F^*)^n \to \mathbb{A} \otimes (F^*)^n$ looks now as follows:

$$d_1\big(1 \otimes (a_1, a_2, \ldots, a_n)\big) = 1 \otimes \Big(\frac{a_2}{a_1}, 1, \frac{a_4}{a_3}, 1, \ldots, \frac{a_n}{a_{n-1}}, 1\Big) \tag{1.10}$$

provided that n is even and

$$d_1\big(1 \otimes (a_1, a_2, \ldots, a_n)\big) = 1 \otimes \Big(a_2, a_2, a_4, a_4, \ldots, a_{\frac{n-1}{2}}, a_{\frac{n-1}{2}}, \prod_{i=1}^{n} a_i^{(-1)^{i+1}}\Big) \tag{1.11}$$

if n is odd.

Therefore, the desired kernel is generated by elements of the form

$$(x_1, x_1, x_2, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}}) \tag{1.12}$$

for even n and

$$\Big(x_1, 1, x_2, 1, \ldots, x_{\frac{n-1}{2}}, 1, \prod_{i=1}^{(n-1)/2} x_i^{-1}\Big) \tag{1.13}$$

for odd one.

In both cases the group $(F^*)^{[n/2]}$ covers the kernel that completely proves Proposition 1.3. ■

Let us now find the homology of GM_n with coefficients in D_m^n.

Proposition 1.5 *Let $p > 0$ and let $n!$ be invertible in the coefficient ring. Then*

$$H_p(\mathrm{GM}_n, D^n_q) = \begin{cases} H_p\big((F^*)^{q+1} \times \mathrm{GM}_{n-q+1}\big) & 0 \le q \le n-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof The Shapiro lemma reduces the problem to the computation of the homology of stabilizers for each orbit of GM_n-action onto D^n_q.

The considered action splits the module D^n_q into three types of orbits. The first of them consists of an orbit defined by matrices without affine part. Such an orbit appears only if $q < n$ and its stabilizer is $(F^*)^{q+1} \times \mathrm{GM}_{n-q+1}$.

The second type consists of matrices with only one affine column. There is only one orbit of this type, presented by the matrix $(e_1, \dots, e_q, E)$. The stabilizer of this matrix is Σ_{n-q}. This orbit is nonempty iff $q \le n$. Since the order of Σ_{n-q} is invertible in the coefficient ring, we have $H_p(\Sigma_{n-q}) = 0$ for $p > 0$. (See [3, Corollary 10.2].)

The third type includes all the orbits consisting of matrices having more than one affine column. One can easily see that the stabilizer of such a matrix is trivial.

Therefore, only the summand corresponding to the orbit of the first type really appears in homology. ■

2 Complexes Related to Groups $H_*(\mathrm{GL}_n, \mathrm{GM}_n)$

In this section we are going to introduce a GL_n-complex C^n_* constructed using a mixture of points in general position of projective and affine spaces. For the induced complex $\mathrm{Ind}^{\mathrm{GL}_n}_{\mathrm{GM}_n} D^n_*$ of GL_n-modules we construct a natural morphism of complexes $\mathrm{Ind}^{\mathrm{GL}_n}_{\mathrm{GM}_n} D^n_* \to C^n_*$, which happens to be an epimorphism.

The kernel of this map K^n_* will be an object of our main interest. Hyperhomology spectral sequence of GL_n with coefficients in K^n_* converges to the relative homology of the pair $(\mathrm{GL}_n, \mathrm{GM}_n)$ and is exactly the spectral sequence mentioned in the introduction.

Let us start with the following construction. Let $\tilde{C}^n_0 \leftarrow \tilde{C}^n_1 \leftarrow \cdots$ be a complex which has the free $\mathbb{A}$-module generated by $n \times (m+1)$ matrices with columns in general position in dimension m. The differential operator d is given by the formula

$$d(a_0, a_1, \dots, a_q) = \sum_{i=0}^{q} (-1)^i (a_0, a_1, \dots, \hat{a}_i, \dots, a_q). \tag{2.1}$$

Denote by $C^{n;p}_*$ a subcomplex of $\tilde{C}^n_*$ generated by elements

$$(x_0, x_1, \dots, x_p, x_{p+1}, \dots, x_m) - (\alpha_0 x_0, \alpha_1 x_1, \dots, \alpha_p x_p, x_{p+1}, \dots, x_m) \tag{2.2}$$

in dimension m if $m > p$ and by

$$(x_0, x_1, \dots, x_m) - (\alpha_0 x_0, \alpha_1 x_1, \dots, \alpha_m x_m) \tag{2.3}$$

otherwise. The coefficients α_i above are arbitrary elements of F^*.

Now, we set $C^n_* = \tilde{C}^n_* / C^{n;n-1}_*$. One can easily verify that the complex C^n_* has no nontrivial homology in positive dimensions.

Consider the hyperhomology spectral sequence of GL_n with coefficients in C^n_*. The group GL_n acts on C^n_m transitively for $m < n$ and freely for $m \geq n$. The canonical generator in dimension m has the form $(e_1, e_2, \ldots, e_{m+1})$ if $m < n$ and $(e_1, e_2, \ldots, e_n, E)$ in dimension $m = n$. Its stabilizer is the affine group

$$\mathrm{Aff}_{n-m-1,m+1} = \begin{pmatrix} F^* & & 0 & \\ & \ddots & & * \\ 0 & & F^* & \\ 0 & \cdots & 0 & \mathrm{GL}_{n-m-1} \end{pmatrix} \tag{2.4}$$

if $m < n$ and trivial if $m \geq n$.

Now we can compute the E^1-term of the spectral sequence under consideration. The Shapiro lemma together with an easy modification of [14, Theorem 1.8] gives us the following

Lemma 2.1

$$E^1_{p,q} = H_p(\mathrm{GL}_n, C^n_q) = \begin{cases} H_p\big((F^*)^{q+1} \times \mathrm{GL}_{n-q-1}\big) & \textit{for } q < n \\ 0 & \textit{otherwise.} \end{cases}$$

(We set GL_0 *to be the trivial group.)* ■

Let us compute also some entries in the 0-th row of the E^2-term. We have $E^2_{0,q} = H_q\big((C^n_*)_{\mathrm{GL}_n}\big)$. Since GL_n acts transitively in low dimensions, the corresponding coinvariant groups are $\mathbb{A}$ and differentials are alternatively trivial and identical. Therefore, we can see that

$$E^2_{0,q} = \begin{cases} \mathbb{A} & \text{for } q = 0 \\ 0 & \text{for } 0 < q \leq n. \end{cases} \tag{2.5}$$

Consider a GL_n-complex obtained from D^n_* by the extension of scalars.

Lemma 2.2 *The map* $\varphi\colon \mathrm{Ind}^{\mathrm{GL}_n}_{\mathrm{GM}_n} D^n_* \to C^n_*$*, sending* $a \otimes g$ *to* ag *is an epimorphism (of complexes).*

Proof For any $(n \times q)$-matrix B satisfying GP-condition we can find such a matrix $G \in \mathrm{GL}_n$ that $GB = (e_1, e_2, \ldots, e_n, b_1, b_2, \ldots, b_{q-n})$. (If $q < n$ the result just needs to be cut at the point e_q.) Let us note now that the submatrix $(b_1, b_2, \ldots, b_{q-n})$ should satisfy SGP-condition, *i.e.*, the matrix GB belongs to $\tilde{D}^n_*$. Therefore, $B = \varphi(G^{-1} \otimes GB)$. ■

Denote the kernel of the map φ by K^n_*. Now we want to compute the hyperhomology of GL_n with coefficients in K^n_*.

Proposition 2.3 $H_p(\mathrm{GL}_n, K^n_*) = H_{p+1}(\mathrm{GL}_n, \mathrm{GM}_n)$.

Proof Consider the epimorphism of GL_n-modules $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} \mathbb{A} \to \mathbb{A}$, where GL_n acts trivially on $\mathbb{A}$. Denote its kernel by $\tilde{K}^n$. Since complexes $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D_*^n$ and C_*^n are GL_n-resolutions of modules $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} \mathbb{A}$ and $\mathbb{A}$, respectively, the complex K_*^n is a resolution of the module $\tilde{K}^n$ as well. Therefore, $H_*(\mathrm{GL}_n, K_*^n) = H_*(\mathrm{GL}_n, \tilde{K}^n)$. Standard homological algebra shows that $H_p(\mathrm{GL}_n, \tilde{K}^n) = H_{p+1}(\mathrm{GL}_n, \mathrm{GM}_n)$. ■

Now we can find out the E^1-term of the hyperhomology spectral sequence of GL_n with coefficients in K_*^n.

Notation 2.4 From now on we often denote the pair $(\mathrm{GL}_k \times X, \mathrm{GM}_k \times X)$ by $(\mathrm{GL}_k, \mathrm{GM}_k) \times X$.

Proposition 2.5 *If $p > 0$, then*

$$E^1_{p,q} = H_p(\mathrm{GL}_n, K_q^n) = H_{p+1}\big((\mathrm{GL}_{n-q-1}, \mathrm{GM}_{n-q-1}) \times (F^*)^{q+1}\big).$$

Proof The short exact sequence of coefficients

$$0 \longrightarrow K_q^n \longrightarrow \mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D_q^n \longrightarrow C_q^n \longrightarrow 0 \tag{2.6}$$

and calculations of the homology of GL_n with coefficients in $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D_q^n$ and C_q^n makes the proof similar to one of Proposition 2.3. ■

Let us look at short exact sequence (2.6). It gives us a long exact sequence of GL_n-homology groups. Consider its final terms. Since the map $H_1\big(\mathrm{GM}_{n-q} \times (F^*)^q\big) \to H_1\big(\mathrm{GL}_{n-q} \times (F^*)^q\big)$ is an epimorphism for any q, we have the following short exact sequence:

$$0 \longrightarrow H_0(\mathrm{GL}_n, K_q^n) \longrightarrow (D_q^n)_{\mathrm{GM}_n} \longrightarrow (C_q^n)_{\mathrm{GL}_n} \longrightarrow 0. \tag{2.7}$$

Assembling together these sequences for all q we obtain the short exact sequence of complexes.

$$0 \longrightarrow H_0(\mathrm{GL}_n, K_*^n) \longrightarrow (D_*^n)_{\mathrm{GM}_n} \longrightarrow (C_*^n)_{\mathrm{GL}_n} \longrightarrow 0. \tag{2.8}$$

The homology long exact sequence corresponding to (2.8) together with (2.5) and Proposition 1.4 gives us the following

Lemma 2.6

$$E^2_{0,q} = H_0(\mathrm{GL}_n, K_q^n) = \begin{cases} 0 & q < n \\ \mathrm{Coker}\big(H_{n+1}(D_*^n)_{\mathrm{GM}_n} \to H_{n+1}(C_*^n)_{\mathrm{GL}_n}\big) & q = n. \end{cases}$$ ■

Definition 2.7 We define an $\mathbb{A}$-module $\wp^n(F)$ (or $\wp^n$ for shortness) as

$$\mathrm{Coker}\big(H_{n+1}(D^n_*)_{\mathrm{GM}_n} \longrightarrow H_{n+1}(C^n_*)_{\mathrm{GL}_n}\big).$$

Let us mention that in even dimensions the considered module is generated by symbols $(a_1, \ldots, a_n)$, where $a_i \in F^*$ and $a_i \neq a_j$ for $i \neq j$. In odd dimensions it is generated by linear combinations of these symbols, lying in the kernel of augmentation map.

We are now ready to prove the Theorem mentioned in the introduction.

Theorem 2.8 *Assume that the base field F is infinite and $n!$ is invertible in the coefficient ring $\mathbb{A}$. Then there exists a naturally defined first quadrant spectral sequence converging to $E^\infty_p = H_{p+1}(\mathrm{GL}_n, \mathrm{GM}_n)$ and with the E^1-term, which has the following form for $p > 0$:*

$$E^1_{p,q} = \begin{cases} H_{p+1}\big(\mathrm{GL}_{n-q-1} \times (F^*)^{q+1}, \mathrm{GM}_{n-q-1} \times (F^*)^{q+1}\big) & \text{if } q < n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$E^2_{0,q} = \begin{cases} 0 & \text{for } q < n \\ \wp^n(F) & \text{for } q = n. \end{cases}$$

Proof One can easily see that almost everything was already proven above. We just need to check vanishing of the E^1-term of the considered spectral sequence in columns $n-1$ and n. But in these columns the E^1-term should have the homology of pairs $(\mathrm{GL}_1, \mathrm{GM}_1) \times (F^*)^{n-1}$ and $(\mathrm{GL}_0, \mathrm{GM}_0) \times (F^*)^n$, which are obviously trivial. ■

Remark 2.9 From now on we will call the constructed above spectral sequence “The Main Spectral Sequence”.

3 Structure of $\wp^n(F)$

The main purpose of this section is to measure the deviation between $\mathbb{A}$-modules $\wp^n(F)$ defined above and $\mathbb{A} \otimes \wp^n(F)_{\mathrm{cl}}$, where $\wp^n(F)_{\mathrm{cl}}$ denotes the “classical” pre-Bloch group. We will abuse the notation and define these modules by $\wp^n(F)_{\mathrm{cl}}$ as well.

Really, it seems that only the group $\wp^2(F)_{\mathrm{cl}}$ appeared in publications (see [1], [11], [15]) so, we need to clarify what we do mean under “classical” for $n > 2$.

Let us fix some integer $n \geq 2$ and an infinite field F. Consider a complex

$$P^n_*(F)\colon P^n_0(F) \xleftarrow{d} P^n_1(F) \xleftarrow{d} \cdots \tag{3.1}$$

where $P^n_k(F)$ is the free abelian group spanned by $(k+1)$-tuples of rational points of the projective space $\mathbb{P}^{n-1}(F)$, which are in general position (*i.e.*, the matrix built from these points satisfying GP-condition). The differential operator d is defined in the standard way.

The complex $P^n_*(F)$ has a canonical structure of GL_n-module and clearly acyclic since the field F is infinite. One can easily check that the lowest nontrivial homology group (in positive dimension) of the complex $\big(P^n_*(F)\big)_{\mathrm{GL}_n}$ appears in dimension $n+1$.

Definition 3.1 We set $\wp^n(F)_{\mathrm{cl}} = H_{n+1}\big(P^n_*(F)_{\mathrm{GL}_n}\big)$.

Remark 3.2 The definition above is the most natural generalization of the definition which can be found, for example in papers of Suslin [15] and Sah [11] for dimension 2. We hope it gives us enough motivation to call the groups of this family "classical".

We shall also introduce one auxiliary object.

Definition 3.3 Let us define $\widehat{\wp^n}(F)$ as $H_{n+1}\Big(\big(C^n_*(F)\big)_{\mathrm{GL}_n}\Big)$, where $C^n_*(F)$ is the complex defined in the previous section.

Let $\mathbf{a} = (a_1, \dots, a_n)^T$ be an n-column such that $a_1, \dots, a_n \in F^*$ and $a_i \neq a_j$ for $i \neq j$. (From now on and up to the end of this section we consider only n-tuples satisfying this condition.) We reserve a notation $[\mathbf{a}]$ for the element of $\big(C^n_{n+1}(F)\big)_{\mathrm{GL}_n}$ given by the matrix $(e_1, e_2, \dots, E, \mathbf{a})$. The module $\widehat{\wp^n}(F)$ is generated by elements of the form $[\mathbf{a}]$ in even dimensions and by $[\mathbf{a}] - [\mathbf{b}]$ in odd ones. (Abusing the notation we use the symbol $[\mathbf{a}]$ to denote the corresponding element of $\widehat{\wp^n}(F)$ as well.)

We start from the following commutative diagram:

(3.2)
$$\begin{tikzcd}
& H_{n+1}(D^n_*)_{\mathrm{GM}_n} \arrow[d, "\beta"] & & \\
F^* \arrow[r, "\chi"] & \widehat{\wp^n}(F) \arrow[r, "\varphi_2"'] \arrow[d, "\varphi_1"] & \wp^n_{\mathrm{cl}} \arrow[l, bend right, "\psi"'] \arrow[dl, "\nu"] \arrow[r] & 0\,. \\
& \wp^n \arrow[d] & & \\
& 0 & &
\end{tikzcd}$$

Let us define all the maps above. The map φ_1 was introduced in the definition of $\wp^n(F)$, φ_2 is the "projectivization" mapping induced by the projection $\mathbb{A}^n(F) \setminus \{0\} \to \mathbb{P}^{n-1}(F)$ with center at the coordinate origin. To define the map ψ we just set: $\psi([\mathbf{a}]) = [a_1^{-1}\mathbf{a}]$. We also set $\nu = \varphi_1\psi$. In order to construct χ we need to prove one auxiliary result.

Lemma 3.4 *For any $n > 1$, $k \in F^*$, and an n-columns $\mathbf{a}, \mathbf{b}$ the following relation holds in $\widehat{\wp^n}(F)$:*

$$[\mathbf{a}] - [k\mathbf{a}] = [\mathbf{b}] - [k\mathbf{b}]$$

Proof Without loosing of generality we can assume that the triple $(E, \mathbf{a}, \mathbf{b})$ satisfies SGP-condition. Let

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & ka_1 & kb_1 \\ 0 & 1 & & 0 & 1 & ka_2 & kb_2 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & ka_n & kb_n \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & a_1 & b_1 \\ 0 & 1 & & 0 & 1 & a_2 & b_2 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & a_n & b_n \end{pmatrix}. \tag{3.3}$$

Consider $d(X) = \sum_{i=0}^{n+2}(-1)^i d_i(X)$, where d_i is the face operator throwing away the i-th column.

It can be easily checked that for $0 \le i < n+1$

$$d_i\big((e_1, e_2, \ldots, e_n, E, \mathbf{a}, \mathbf{b})\big) = [\mathbf{f_i}(\mathbf{a}, \mathbf{b})], \tag{3.4}$$

where $\mathbf{f_i}\colon {F^*}^n \times {F^*}^n \to {F^*}^n$ are homogeneous rational functions of $\mathbf{a}, \mathbf{b}$ such that $\deg_{\mathbf{a}} \mathbf{f_i} = -1$ and $\deg_{\mathbf{b}} \mathbf{f_i} = 1$. This implies that all these terms annihilate in $d(X)$. Thus, we have $d(X) = (-1)^{n+1}\big(d_{n+1}(X) - d_{n+2}(X)\big)$. This shows that the element

$$([k\mathbf{a}] - [\mathbf{a}]) - ([k\mathbf{b}] - [\mathbf{b}]) \in C_{n+1}^n(F)_{\mathrm{GL}_n} \tag{3.5}$$

belongs to the image of the differential d and therefore, vanishes in the homology group $\widehat{\wp^n}(F)$. ■

Definition 3.5 We set $\chi(k) = [\mathbf{a}] - [k\mathbf{a}]$.

Now one can easily verify that the row in Diagram 3.2 is exact and the map φ_2 split by ψ. Thus, we have the following fact.

Lemma 3.6 $\widehat{\wp^n}(F) = \wp^n(F)_{\mathrm{cl}} \oplus \operatorname{Im}\chi$.

In case of odd n we can prove even a stronger result.

Lemma 3.7 *If n is odd, then $\chi = 0$.*

Proof Let $k, l \in F^*$. Consider

$$d\big((e_1, e_2, \ldots, e_n, E, l\mathbf{a}, k\mathbf{b}) - (e_1, e_2, \ldots, e_n, E, \mathbf{a}, \mathbf{b})\big). \tag{3.6}$$

The proof of the previous lemma tells us that this expression equals to

$$\underbrace{\chi(k/l) - \chi(k/l) + \cdots - \chi(k/l)}_{n+1 \text{ times}} + \chi(k) - \chi(l). \tag{3.7}$$

Therefore, in $\widehat{\wp^n}(F)$ we have $\chi(k) = \chi(l)$. Since χ is a homomorphism, this completes the proof. ■

Corollary 3.8 *If $n > 1$ is odd, then the map $\widehat{\wp^n}(F) \xrightarrow{\varphi_2} \wp^n(F)_{\mathrm{cl}}$ is an isomorphism.*

Let us now consider the case of even n.

Lemma 3.9 *Let $n > 0$ be an even integer. Then for any $k \in F^*$ the element $2^{\frac{n}{2}}\varphi_1\big(\chi(k)\big)$ vanishes in $\wp^n(F)$.*

Proof Let us consider the group ring $\mathbb{Z}[\Sigma_n]$. The following construction supplies C^n_{n+1} with a structure of $\mathbb{Z}[\Sigma_n]$-module. Let $[\mathbf{x}] = (e_1, \dots, e_n, E, \mathbf{x}) \in C^n_{n+1}$ and $\Xi = \sum_i c_i\xi_i \in \mathbb{Z}[\Sigma_n]$. ($\mathbf{x}$ here is an n-column and ξ_i are n-permutations acting on $\mathbf{x}$.)

We set

$$\Xi[\mathbf{x}] = \sum_i c_i(e_1, e_2, \dots, e_n, E, \xi_i\mathbf{x}) \in C^n_{n+1}. \tag{3.8}$$

Now, we construct a product map $\mathbb{Z}[\Sigma_k] \otimes \mathbb{Z}[\Sigma_l] \xrightarrow{\times} \mathbb{Z}[\Sigma_{k+l}]$ in the following way. Let $\sigma \in \Sigma_k$ and $\tau \in \Sigma_l$ be permutations acting on sets $\{1, \dots, k\}$ and $\{1, \dots, l\}$, respectively. Then $\sigma \times \tau \in \Sigma_{k+l}$ acts on the set $\{1, \dots, k+l\}$ by the formula

$$\sigma \times \tau(i) = \begin{cases} \sigma(i) & \text{for } i \le k \\ k + \tau(i-k) & \text{for } i > k. \end{cases} \tag{3.9}$$

We extend this definition to the group rings by linearity. Let now γ be only nontrivial permutation in Σ_2. Set $\Gamma = 1 + \gamma \in \mathbb{Z}[\Sigma_2]$ and consider $\Gamma^m = \Gamma \times \Gamma \times \cdots \times \Gamma \in \mathbb{Z}[\Sigma_{2m}]$. The following lemma can be easily checked by direct calculation.

Lemma 3.10 *Let n be a positive even. Then for any n-tuple $\mathbf{x}$ an element $\Gamma^{\frac{n}{2}}[\mathbf{x}]$ belongs to the image of the canonical map $D^n_{n+1} \to C^n_{n+1}$ and its $\mathbb{Z}$-augmentation is $2^{\frac{n}{2}}$.*

Consider now the element $\Gamma^{\frac{n}{2}}[k\mathbf{x}] - \Gamma^{\frac{n}{2}}[\mathbf{x}] \in \widehat{\wp^n}(F)$. Since it comes from D^n_{n+1}, its image in $\wp^n(F)$ is 0. On the other hand it should be equal to $2^{\frac{n}{2}}\chi(k)$.

Proposition 3.11 *Assume that n is a positive integer. If n is even we should also assume that 2 is invertible in the coefficient ring. In this case the map $\nu\colon \wp^n(F)_{\mathrm{cl}} \to \wp^n(F)$ is an epimorphism.*

Proof It is just an obvious corollary of Lemmas 3.8 and 3.9. ■

Let us, finally, consider the case $n = 2$.

Proposition 3.12 *Assume that 2 is invertible in the coefficient ring. Then $\wp^2(F) = \wp^2(F)_{\mathrm{cl}}$.*

Proof Mention, first, that in the case under consideration $\varphi_2\beta = 0$. Really, the image of $\varphi_2\beta$ is generated by classes $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] + \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$. Using the notation of [15] it equals to $[a/b] + [b/a] = \langle a/b\rangle$ and from the proof of [15, Lemma 1.2] we have $2\langle a/b\rangle = 0$.

This, together with Proposition 3.11, shows that the map φ_2 can be pushed down to the factor-group $\wp^n$ and the resulting map is inverse to ν. ■

Remark 3.13 Using the proofs given above, one can easily check that the map ν: $\wp^2(F)_{\mathrm{cl}} \to \wp^2(F)$ is an isomorphism for quadratically closed field F even with $\mathbb{Z}$-coefficients.

4 Application to Low Dimensional Cases

Now we have introduced all the players and are ready to get some applications of the developed machinery. The main objectives of this section is to give a homological interpretation of groups $\wp^n(F)$ and to show some of their possible applications to the investigation of groups $H_*(\mathrm{GL}_n)$. The most important properties of groups $\wp^n(F)$ are summarized in Conjectures 0.1, 0.2 above.

Remark 4.1 In case of algebraically closed field F the assertion of Conjecture 0.2 easily follows from the divisibility of groups $\wp^n(F)_{\mathrm{cl}}$ which look more geometrically motivated.

In this section we shall prove Conjecture 0.1 for $n < 5$. (For $n = 4$ we will have to assume, in addition, that the field F is algebraically closed.) Conjecture 0.2 was proven for $n = 2$ in [11], [15]. Unfortunately, we do not know how to prove 0.2 for $n > 2$. Rather, we are going to show the importance of this conjecture for the investigation of the homology of linear groups.

Before proceeding, let us make some remarks concerning the definition of a product on homology and the "indecomposable part of the homology" which appears in Conjecture 0.1.

For integers $i, j > 0$ we can define a homology product

$$H_i(\mathrm{GL}_n, \mathrm{GM}_n) \otimes H_j(F^*) \xrightarrow{\smile} H_{i+j}(\mathrm{GL}_{n+1}, \mathrm{GM}_{n+1}) \tag{4.1}$$

as a composition of the external homological product and the natural map induced by the inclusion

$$F^* \times \mathrm{GL}_n \mapsto \begin{pmatrix} F^* & 0 \\ 0 & \mathrm{GL}_n \end{pmatrix} \subset \mathrm{GL}_{n+1}\,. \tag{4.2}$$

One can easily check that this product is well-defined. The same construction defines a product on birelative homology groups as well.

We call the factor-group

$$H_k(\mathrm{GL}_n, \{\mathrm{GM}_n, \mathrm{GL}_{n-1}\}) \Big/ \Big(\bigoplus_{i+j=k} H_i(\mathrm{GL}_{n-1}, \{\mathrm{GM}_{n-1}, \mathrm{GL}_{n-2}\}) \smile H_j(F^*) \Big)$$

the indecomposable part of homology $H_k(\mathrm{GL}_n, \{\mathrm{GM}_n, \mathrm{GL}_{n-1}\})_{\mathrm{ind}}$.

4.1 The Case $n = 2$

Theorem 2.8 implies the following fact as an immediate corollary.

Theorem 4.2

$$H_i(\mathrm{GL}_2, \mathrm{GM}_2) = \begin{cases} 0 & i < 3 \\ \wp^2(F) & i = 3. \end{cases}$$

Proof Actually, if $n = 2$, then the E^1-term of The Main Spectral Sequence has no nonzero entries except for the 0-th row. ■

4.2 The Case $n = 3$

In this case the E^1-term of The Main Spectral Sequence is transgressive. Theorem 4.2 and the relative version of the Künneth formula [16] allow us to write down its E^2-term in terms of relative homology groups.

$$\begin{array}{ccccc} H_4(\mathrm{GL}_2, \mathrm{GM}_2) \oplus \wp^2 \otimes F^* & & & & \\ \wp^2 & & \mathrm{ZERO} & & \\ 0 & 0 & & & \\ 0 & 0 & 0 & \wp^3 & \cdots . \end{array} \tag{4.3}$$

This implies the following long exact sequence:

$$\cdots \longrightarrow E^2_{0,q} \xrightarrow{d^{(q)}} H_q\big((\mathrm{GL}_2, \mathrm{GM}_2) \times (F^*)\big) \longrightarrow H_q(\mathrm{GL}_3, \mathrm{GM}_3) \longrightarrow \cdots . \tag{4.4}$$

The statement of Conjecture 0.1 for $n = 3$ read as follows.

Theorem 4.3 $\wp^3(F) = H_4(\mathrm{GL}_3, \{\mathrm{GM}_3, \mathrm{GL}_2\})_{\mathrm{ind}}$.

Proof We want to construct the following commutative diagram with exact columns.

$$\begin{tikzcd} H_4(\mathrm{GL}_2, \mathrm{GM}_2) \arrow[r] \arrow[d] & H_4(\mathrm{GL}_2, \mathrm{GM}_2) \oplus (\wp^2 \otimes F^*) \arrow[d] \\ H_4(\mathrm{GL}_3, \mathrm{GM}_3) \arrow[r, equal] \arrow[d] & H_4(\mathrm{GL}_3, \mathrm{GM}_3) \arrow[d] \\ H_4(\mathrm{GL}_3, \{\mathrm{GM}_3, \mathrm{GL}_2\}) \arrow[r] \arrow[d] & \wp^3(F) \arrow[d] \\ H_3(\mathrm{GL}_2, \mathrm{GM}_2) \arrow[r, "\sim"] \arrow[d] & \wp^2(F) \arrow[d] \\ H_3(\mathrm{GL}_3, \mathrm{GM}_3) \arrow[r, equal] \arrow[d] & H_3(\mathrm{GL}_3, \mathrm{GM}_3) \arrow[d] \\ 0 & 0. \end{tikzcd} \tag{4.5}$$

Let us remind that we denoted by $\tilde{K}^n$ the kernel of map $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} \mathbb{A} \to \mathbb{A}$ and by K^n_k the kernel of map $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D^n_k \to C^n_k$. Let us also denote by $i\colon\ \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ the standard embedding:

$A \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)$. Consider the diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_3} \tilde{K}^2 & & K_0^3 \\
\downarrow & & \downarrow \\
\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_3} \mathrm{Ind}_{\mathrm{GM}_2}^{\mathrm{GL}_2} \mathbb{A} & \xrightarrow{\varphi} & \mathrm{Ind}_{\mathrm{GM}_3}^{\mathrm{GL}_3} D_0^3 \\
\downarrow & & \downarrow \\
\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_3} \mathbb{A} & \xrightarrow{\psi} & C_0^3
\end{array}
\tag{4.6}
$$

where $\varphi(g_1 \otimes g_2 \otimes 1) = g_1 i(g_2) \otimes e_1$ and $\psi(g \otimes 1) = g \otimes e_1$. Since the group GL_2 doesn't act on the vector e_1, one can easily check that these maps are well-defined, the diagram above commutes and has exact columns. Maps φ and ψ induce the map on kernels $\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_3} \tilde{K}^2 \xrightarrow{\xi} K_0^3$.

This map ξ (together with the identity map in the lowest degree) induces the morphism of two-term complexes:

$$
\begin{array}{ccc}
(\tilde{K}^3 & \longleftarrow & \mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_3} \tilde{K}_2) \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \xi_*}\downarrow \\
(\tilde{K}^3 & \longleftarrow & K_0^3).
\end{array}
\tag{4.7}
$$

Consider the hyperhomology $H_*\big(\mathrm{GL}_3, (\tilde{K}^3 \leftarrow \mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_3} \tilde{K}_2)\big)$. Using the standard machinery of homological algebra one can check that the corresponding spectral sequence converges to $H_*(\mathrm{GL}_3, \{\mathrm{GM}_3, \mathrm{GL}_2\})$.

The E^1-term of one of associated spectral sequences has only two nontrivial rows. It has groups $H_*(\mathrm{GL}_3, \mathrm{GM}_3)$ in the 0-th row and groups $H_*(\mathrm{GL}_2, \mathrm{GM}_2)$ in the first one.

Let us now look at the hyperhomology spectral sequence of GL_3 with coefficients in the complex $(\tilde{K}^3 \leftarrow K_0^3)$. The map $K_0^3 \to \tilde{K}^3$ is an epimorphism. Consider the following GL_3-resolution of $\mathrm{Ker}(K_0^3 \to \tilde{K}^3)$: $K_1^3 \leftarrow K_2^3 \leftarrow \cdots$. Since $H_i(\mathrm{GL}_3, K_q^3) = 0$ for $i > 0, q > 0$, we have:

$$
H_q\big(\mathrm{GL}_3, (\tilde{K}^3 \longleftarrow K_0^3)\big) \simeq H_0(\mathrm{GL}_3, K_{q-1}^3).
\tag{4.8}
$$

The E^1-term of the other spectral sequence associated to the latter two-term complex has two nonzero rows

$$
\begin{array}{ccc}
\cdots & H_*\big((\mathrm{GL}_2, \mathrm{GM}_2) \times F^*\big) & \cdots \\
\cdots & H_*\big((\mathrm{GL}_3, \mathrm{GM}_3)\big) & \cdots .
\end{array}
\tag{4.9}
$$

Evidently, the map of corresponding spectral sequences induced by ξ is a canonical isomorphism in the 0-th rows and is induced by the inclusion $(\mathrm{GL}_2, \mathrm{GM}_2) \hookrightarrow (\mathrm{GL}_2, \mathrm{GM}_2) \times F^*$ in the first rows. We have constructed all the maps in Diagram (4.5). Desired properties now follow from functoriality.

Standard diagram chasing in (4.5) gives us the exact sequence

$$\wp^2 \otimes F^* \xrightarrow{\gamma} H_4(\mathrm{GL}_3, \{\mathrm{GM}_3, \mathrm{GL}_2\}) \longrightarrow \wp^3(F) \longrightarrow 0. \tag{4.10}$$

Now we want to check that γ is a product map. The homomorphism γ is induced by the augmentation map $K_0^3 \to \tilde{K}^3$ sending any column to 1. One can easily verify that this map defines the morphism of stabilizers

$$\mathrm{Stab}_{\mathrm{GL}_3} e_1 = F^* \times \mathrm{GL}_2 \longrightarrow \mathrm{Stab}_{\mathrm{GL}_3} 1 = \mathrm{GL}_3, \tag{4.11}$$

given as follows:

$$F^* \times \mathrm{GL}_2 \mapsto \begin{pmatrix} F^* & 0 \\ 0 & \mathrm{GL}_2 \end{pmatrix}. \tag{4.12}$$

The latter morphism induces the desired product on homology. This fact, together with Diagram (4.5), finishes the proof of Theorem 4.3. ■

Before proceeding with the case $n = 4$ we shall prove the following theorem.

Theorem 4.4 *The natural embedding* $(\mathrm{GL}_2, \mathrm{GM}_2) \hookrightarrow (\mathrm{GL}_3, \mathrm{GM}_3)$ *induces the isomorphism* $\wp^2 = H_3(\mathrm{GL}_2, \mathrm{GM}_2) \simeq H_3(\mathrm{GL}_3, \mathrm{GM}_3)$, *provided that* 3! *is invertible in the coefficient ring.*

Proof Let us look at the end terms of long exact sequence (4.4) associated to the case $n = 3$:

$$\wp^3 \xrightarrow{d} \wp^2 \longrightarrow H_3(\mathrm{GL}_3, \mathrm{GM}_3) \longrightarrow 0. \tag{4.13}$$

This proves that the map under consideration is an epimorphism. We shall now prove that the map d is zero, which requires more work. First of all, we introduce a new complex $C_*^{n|k}$ defined as follows. We set $C_q^{n|k} = C_q^n$ and

$$d^{n|k}(a_0, \ldots, a_{k-1}|a_k, \ldots, a_q) = \sum_{i=1}^{q} (-1)^i (a_0, \ldots, a_{k-1}|a_k, \ldots, \hat{a}_i, \ldots, a_q). \tag{4.14}$$

(Since the differential doesn't affect first k columns we will separate them by "|".) We also define a complex $\bar{D}_*^{n|k}$ as the following modification of D_*^n. We let $\bar{D}_l^{n|k}$ be the free $\mathbb{A}$-module generated by all classes of D-equivalence of matrices having the form:

$$(\underbrace{*, \ldots, *}_{k \text{ times}} | \underbrace{m, \ldots, m}_{p \text{ times}}, \underbrace{a, \ldots, a}_{q \text{ times}}).$$

Here p and q are nonnegative integers, $p + q = l$; the symbol "m" means a monomial column, the symbol "a" means an affine column, and the stars denote columns of any of these two types. In addition, we assume that the maximal affine submatrix $(a, \ldots, a)$ satisfies SGP-condition.

We introduce also a subcomplex of $\bar{D}_*^{n|k}$ denoted by $D_*^{n|k}$. For any q, we set $D_q^{n|k} = D_q^n/F_{k-1}D_q^n$. Less formally, generators of $D_*^{n|k}$ are the generators of $\bar{D}_*^{n|k}$, having the form

$$(m, \ldots, m|m, \ldots, m, a, \ldots, a).$$

Differentials in all of the introduced complexes are given by formula (4.14).

Proposition 4.5 *The following list of properties holds for the complexes* $C_*^{n|k}$, $\bar{D}_*^{n|k}$, *and* $D_*^{n|k}$:

a) *Complexes* $C_*^{n|k}$, $\bar{D}_*^{n|k}$, *and* $D_*^{n|k}$ *are acyclic.*
b) *They have canonical structure of* GL_n- *(resp.* GM_n-*) modules.*
c) *The associated hyperhomology spectral sequence converges to* 0 *and has* E^1-*term coinciding with the* E^1-*term of the spectral sequence associated to the complex* C_*^n *(resp.* D_*^n*).*
d) *There is the natural epimorphism* $\varphi\colon \mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D_*^{n|k} \twoheadrightarrow C_*^{n|k}$.
e) *The map* ψ, *given by the formula*

$$\psi(a_0, a_1, \ldots, a_q) = \sum_{i=0}^{q} (-1)^i (a_i|a_0, a_1, \ldots, \hat{a}_i, \ldots, a_q) \tag{4.15}$$

induces morphisms of complexes $C_*^n \to C_*^{n|1}$ *and* $D_*^n \to \bar{D}_*^{n|1}$.
f) *The diagram*

$$\begin{array}{ccccc} \mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D_*^n & \xrightarrow{\varphi} & C_*^n & \longrightarrow & 0 \\ {\scriptstyle 1\otimes\psi}\downarrow & & {\scriptstyle\psi}\downarrow & & \\ \mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} \bar{D}_*^{n|1} & \xrightarrow{\varphi} & C_*^{n|1} & \longrightarrow & 0 \end{array} \tag{4.16}$$

commutes. This condition shows that ψ *defines the morphism of cones*

$$Cone(D_*^n \to C_*^n) \xrightarrow{\psi} Cone(\bar{D}_*^{n|1} \to C_*^{n|1}). \tag{4.17}$$

All the statements above are easy corollaries of the results of Sections 1 and 2. ∎

(Let us remind that we denote by $\mathrm{Cone}(D_*^n \to C_*^n)$ the *Cone* of morphism $\mathrm{Ind}_{\mathrm{GM}_n}^{\mathrm{GL}_n} D_*^n \to C_*^n$.) We can also see that from the "homological point of view" there is no difference between complexes $(D_*^{n|1})_{\mathrm{GM}_n}$ and $(\bar{D}_*^{n|1})_{\mathrm{GM}_n}$.

Lemma 4.6 *If* $n!$ *is invertible in the coefficient ring, then the natural embedding* $(D_*^{n|1})_{\mathrm{GM}_n} \hookrightarrow (\bar{D}_*^{n|1})_{\mathrm{GM}_n}$ *induces an isomorphism*

$$(\bar{D}_*^{n|1})_{\mathrm{GM}_n} \simeq (D_*^{n|1})_{\mathrm{GM}_n} \oplus Q_*,$$

where the complex Q_* *is contractible.*

Proof Decomposition is obvious. The contracting homotopy for Q_* can be cooked up by standard methods. (See, for example, [10]). ■

Corollary 4.7 *Under the same conditions as before, we have:*

$$H_*\big((\bar{D}_*^{n|1})_{\mathrm{GM}_n}\big) \simeq H_*\big((D_*^{n|1})_{\mathrm{GM}_n}\big).$$

Consider the spectral sequence associated to $\mathrm{Cone}(\bar{D}^{3|1} \to C^{3|1})$. The E^2-term of this spectral sequence looks as follows.

$$\begin{array}{ccccc} H_3(\mathrm{GL}_2, \mathrm{GM}_2) & 0 & \cdots & & \\ 0 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & \wp^{3|1} & \cdots . \end{array} \tag{4.18}$$

(We denote by $\wp^{n|1}$ an analog of the group $\wp^n$ which appears in the case of $n|1$-complexes.)

Since the spectral sequence above converges to 0, we obtain an isomorphism $d_3^{3|1}$: $\wp^{3|1} \overset{\simeq}{\to} H_3(\mathrm{GL}_2, \mathrm{GM}_2)$. The morphism ψ gives us the morphism of E^2-terms of corresponding hyperhomology spectral sequences which is identical on the first columns. So, we have the following commutative diagram:

$$\begin{tikzcd} \wp^3 \arrow[r, "d"] \arrow[d, "\psi_*"'] & H_3(\mathrm{GL}_2, \mathrm{GM}_2) \arrow[r] & H_3(\mathrm{GL}_3, \mathrm{GM}_3) \arrow[r] & 0 . \\ \wp^{3|1} \arrow[ur, "d_3^{3|1}", "\simeq"'] & & & \end{tikzcd} \tag{4.19}$$

We are going to extend this diagram up to the following one:

$$\begin{tikzcd} \wp^3 \arrow[r, "d"] \arrow[d, "\psi_*"'] & H_3(\mathrm{GL}_2, \mathrm{GM}_2) \arrow[r] & H_3(\mathrm{GL}_3, \mathrm{GM}_3) \arrow[r] & 0 \\ \wp^{3|1} \arrow[ur, "d_3^{3|1}", "\simeq"'] \arrow[d, "\pi_*"'] & & & \\ \wp^2 \arrow[uur, "\simeq"'] & & & \end{tikzcd} \tag{4.20}$$

and show that the vertical path $\wp^3 \to \wp^{3|1} \to \wp^2$ is the zero map. Let us first define a map π_*. This map is induced by the projection map $\mathbb{P}^2(F) \xrightarrow{\pi} \mathbb{P}^1(F)$ with the center at the point e_3. More precisely, in any class of GL_3-equivalence we can always choose a matrix A of the form

$$A = \left(\begin{array}{c|ccc} 0 & w_{11} & \cdots & w_{1k} \\ 0 & w_{21} & \cdots & w_{2k} \\ 1 & w_{31} & \cdots & w_{3k} \end{array}\right). \tag{4.21}$$

We set

$$\pi(A) = \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ w_{21} & \cdots & w_{2k} \end{pmatrix}. \tag{4.22}$$

The same construction works for terms of $(D_*^{3|1})_{\mathrm{GM}_3}$ as well. One can easily check that the constructed map is well-defined. The map just constructed gives us the morphism of bicomplexes:

$$\begin{tikzcd}
D_1^{3|1} \arrow[dr, "\pi_*"] \arrow[dd] & & D_2^{3|1} \arrow[ll, "d^{3|1}"'] \arrow[dr, "\pi_*"] \arrow[dd] & & D_3^{3|1} \arrow[ll, "d^{3|1}"'] \arrow[dr, "\pi_*"] \arrow[dd] & & \cdots \arrow[ll] \\
& D_0^2 \arrow[dd] & & D_1^2 \arrow[ll, "d"'] \arrow[dd] & & D_2^2 \arrow[ll, "d"'] \arrow[dd] & \cdots \arrow[l] \\
C_1^{3|1} \arrow[dr, "\pi_*"] & & C_2^{3|1} \arrow[ll, "d^{3|1}"'] \arrow[dr, "\pi_*"] & & C_3^{3|1} \arrow[ll, "d^{3|1}"'] \arrow[dr, "\pi_*"] & & \cdots \arrow[ll] \\
& C_0^2 & & C_1^2 \arrow[ll, "d"'] & & C_2^2 \arrow[ll, "d"'] & \cdots \arrow[l]
\end{tikzcd} \tag{4.23}$$

Comparing the E^2-terms of corresponding hyperhomology spectral sequences one can easily verify that the bottom triangle of Diagram (4.20) commutes.

Lemma 4.8 *The map* $\wp^3 \xrightarrow{\psi} \wp^{3|1} \xrightarrow{\pi_*} \wp^2$ *is zero.*

Proof We can write down this map explicitly. Let $\mathbf{c}$ denote a 2-column. Denoting the generator $(e_1, e_2, E, \mathbf{c})$ of $\wp^2(F)$ by $(\mathbf{c})^*$, we have:

(4.24)

$$\pi_*\psi \begin{pmatrix} 1 & 0 & 0 & 1 & a_1 \\ 0 & 1 & 0 & 1 & a_2 \\ 0 & 0 & 1 & 1 & a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}^* - \begin{pmatrix} a_1 \\ a_3 \end{pmatrix}^* + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^* - \begin{pmatrix} a_3 - a_1 \\ a_3 - a_2 \end{pmatrix}^* + \begin{pmatrix} a_3^{-1} - a_1^{-1} \\ a_3^{-1} - a_2^{-1} \end{pmatrix}^*.$$

On the other hand, we have

$$\begin{aligned} \partial \begin{pmatrix} 1 & 0 & 1 & a_3^{-1} - a_1^{-1} & a_3 - a_1 \\ 0 & 1 & 1 & a_3^{-1} - a_2^{-1} & a_3 - a_2 \end{pmatrix} &= \begin{pmatrix} -a_1a_2 \\ -a_1a_3 \end{pmatrix}^* - \begin{pmatrix} -a_1a_2 \\ -a_2a_3 \end{pmatrix}^* + \begin{pmatrix} -a_1a_3 \\ -a_2a_3 \end{pmatrix}^* \\ &\quad - \begin{pmatrix} a_3 - a_1 \\ a_3 - a_2 \end{pmatrix}^* + \begin{pmatrix} a_3^{-1} - a_1^{-1} \\ a_3^{-1} - a_2^{-1} \end{pmatrix}^*. \end{aligned} \tag{4.25}$$

From the previous section we already know that in $\wp^2(F)$ there exists a relation

$$\begin{pmatrix} x \\ y \end{pmatrix}^* = \begin{pmatrix} ax \\ ay \end{pmatrix}^*. \tag{4.26}$$

Therefore, the image of $\pi_*\psi$ is a boundary. This observation finishes the proof of the lemma and the proof of Theorem 4.4 as well. ■

We obtain the following fact, proven in [15] with $\mathbb{Z}$-coefficients in a different way, as an easy corollary of Theorem 4.4.

Corollary 4.9 *The kernel of the homomorphism $H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$ lies in $H_3(\mathrm{GM}_2)$.*

Proof It is clear from the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H_3(\mathrm{GM}_2) & \longrightarrow & H_3(\mathrm{GL}_2) & \longrightarrow & H_3(\mathrm{GL}_2, \mathrm{GM}_2) \\ \big\downarrow & & \big\downarrow & & \simeq\big\downarrow \\ H_3(\mathrm{GM}_3) & \longrightarrow & H_3(\mathrm{GL}_3) & \longrightarrow & H_3(\mathrm{GL}_3, \mathrm{GM}_3). \end{array} \tag{4.27}$$

■

The last case when we could get through with direct calculations is the next one.

4.3 The Case $n = 4$

Throughout this section we shall assume, in addition, that the base field F is algebraically closed. Using Theorem 2.8 and the relative Künneth formula [16] we can write down terms of interest in the E^1-term of Spectral Sequence (2.8), in the case $n = 4$.

(For the shortness we will often denote $H_m(\mathrm{GL}_n, \mathrm{GM}_n)$ by $H_m(n)$.)

$$\begin{array}{ccc} \bigoplus_{i+j=5}\big(H_i(3) \otimes H_j(F^*)\big) & \bigoplus_{i+j+k=5}\big(H_i(3) \otimes H_j(F^*) \otimes H_k(F^*)\big) & \\ H_4(3) \oplus F^* \otimes H_3(3) & H_4(2) \oplus (F^* \times F^*) \otimes H_3(2) & \\ H_3(3) & H_3(2) & \mathrm{ZERO} \\ 0 & 0 & \\ * & * & * \end{array} \tag{4.28}$$

(we have $E^1_{p,q} = 0$ for $p > 0$ and $q > 1$).

Remark 4.10 All the Tor-summands in the considered dimensions happen to be zero. This is implied by vanishing groups $H_m(\mathrm{GL}_n, \mathrm{GM}_n)$ for $m < 3$ and unique divisibility of the group

$$H_3(\mathrm{GL}_3, \mathrm{GM}_3) \simeq H_3(\mathrm{GL}_2, \mathrm{GM}_2) = \wp^2(F)$$

for algebraically closed field F. (See [15] and also [5], [6].)

Theorem 4.4 easily implies vanishing of the differential $d_1 : E^1_{21} \to E^1_{20}$ $\big(H_3(2) \to H_3(3)\big)$. In order to investigate other differentials we use complexes $C^{4|2}_*$ and $D^{4|2}_*$. Consider homomorphisms $C^{4|2}_* \xrightarrow{\varphi} C^4_*$ and $D^{4|2}_* \xrightarrow{\varphi} D^4_*$ given by the formula

$$\varphi\colon (a_0, a_1|a_2, \dots, a_q) = (a_0, a_1, a_2, \dots, a_q) + (a_1, a_0, a_2, \dots, a_q). \tag{4.29}$$

They induce the morphism of cones

$$\mathrm{Cone}(D_*^{4|2} \to C_*^{4|2}) \xrightarrow{\varphi} \mathrm{Cone}(D_*^4 \to C_*^4). \tag{4.30}$$

The E^1-term of hyperhomology spectral sequence of GL_4 with coefficients in $\mathrm{Cone}(D_*^{4|2} \to C_*^{4|2})$ looks as follows:

$$E^1_{p,q} = \begin{cases} \tilde{E}^1_{p,q+1} & q > 0 \\ 0 & q = 0. \end{cases} \tag{4.31}$$

Here $\tilde{E}^1$ denotes the E^1-term of the spectral sequence associated to the case $n = 4$. (See (4.28).)

One can easily check that the latter spectral sequence converges to zero and has no nontrivial differentials, but coming out of the 0-th row. Thus, all differentials in (4.28) vanish on the image of φ_*. We can also conclude that every element lying in the image of φ_* belongs to the image of corresponding differential (from the 0-th row) and, therefore, goes to zero in $E^\infty_{*,*}$. The homomorphism φ_* acts on groups

$$\bigoplus_{i+j+k=N} \big(H_i(\mathrm{GL}_3, \mathrm{GM}_3) \otimes H_j(F^*) \otimes H_k(F^*)\big) \tag{4.32}$$

as an endomorphism $\mathrm{id} + s$, where s swaps two copies of F^*.

Now we apply this discussion to the calculation of differentials in spectral sequence (4.28).

Lemma 4.11 *The group* $\wp^2 = H_3(\mathrm{GL}_2, \mathrm{GM}_2)$ *is a direct summand of* $\wp^4$*, provided that* 6 *is invertible in the coefficient ring.*

Proof Since $\wp^2 = H_3(\mathrm{GL}_2, \mathrm{GM}_2) = E^2_{2,1} \subset \mathrm{Im}\,\varphi_*$, there is no nontrivial differentials in spectral sequence (4.31) going out of the term of dimension $(2, 1)$ and the map

$$\wp^4 = E^3_{0,4} \xrightarrow{d_3} E^3_{2,1} = E^1_{2,1} = H_3(\mathrm{GL}_2, \mathrm{GM}_2) = \wp^2 \tag{4.33}$$

is an epimorphism. One can easily check that the map

$$H_3(\mathrm{GL}_2, \mathrm{GM}_2) = \wp^2 \xrightarrow{\varphi_*} \wp^2 = H_3(\mathrm{GL}_2, \mathrm{GM}_2) \tag{4.34}$$

is multiplication by 2. (The group on the left-hand side is an entry of the spectral sequence, corresponding to the case $\{4|2\}$. The right-hand side one is the corresponding entry of (4.28)).

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \wp^{4|2} & \xrightarrow{\simeq} & H_3(\mathrm{GL}_2, \mathrm{GM}_2) \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\times 2} \\ \wp^4 & \xrightarrow{d_3} & H_3(\mathrm{GL}_2, \mathrm{GM}_2), \end{array} \tag{4.35}$$

which gives us the section of differential d_3. ■

Lemma 4.12 *The map*

$$H_4(\mathrm{GL}_2, \mathrm{GM}_2) \oplus (F^* \times F^*) \otimes H_3(\mathrm{GL}_2, \mathrm{GM}_2) \xrightarrow{d_1} H_4(\mathrm{GL}_3, \mathrm{GM}_3) \oplus F^* \otimes H_3(\mathrm{GL}_3, \mathrm{GM}_3)$$

is given by the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \psi \end{smallmatrix}\right)$, *such that the following sequence*

$$0 \longrightarrow \operatorname{Im} \varphi_* \longrightarrow (F^* \times F^*) \otimes H_3(\mathrm{GL}_2, \mathrm{GM}_2) \xrightarrow{\psi} F^* \otimes H_3(\mathrm{GL}_3, \mathrm{GM}_3)$$

is exact.

Proof One can easily see that the map

$$d_1 : \bigoplus_{i+j+k=N} H_i(F^*) \otimes H_j(F^*) \otimes H_k(2) \longrightarrow \bigoplus_{l+m=N} H_l(F^*) \otimes H_m(3) \tag{4.36}$$

looks as follows

$$d_1(f \otimes g \otimes h) = f \otimes (g \smile h) - g \otimes (f \smile h). \tag{4.37}$$

This shows immediately that $d_1\big(H_4(\mathrm{GL}_2, \mathrm{GM}_2)\big) = 0$.

Since $i_* : H_3(\mathrm{GL}_2, \mathrm{GM}_2) \to H_3(\mathrm{GL}_3, \mathrm{GM}_3)$ is an isomorphism (see Theorem 4.4), each of the terms $F^* \otimes H_3(\mathrm{GL}_2, \mathrm{GM}_2)$ maps isomorphically onto $F^* \otimes H_3(3)$. (These maps are different by their signs.) Therefore, the cokernel of d_1 is $H_4(\mathrm{GL}_3, \mathrm{GM}_3)$. The map

$$(F^* \times F^*) \otimes H_3(\mathrm{GL}_2, \mathrm{GM}_2) / \operatorname{Im} \varphi \longrightarrow F^* \otimes H_3(\mathrm{GL}_3, \mathrm{GM}_3) \tag{4.38}$$

is an isomorphism. This implies the coincidence of $\operatorname{Ker} d_1$ and $\operatorname{Im} \varphi_*$. ∎

Corollary 4.13 *Since* $\operatorname{Ker}(E^1_{1,3} \xrightarrow{d_1} E^1_{0,3}) \subset \operatorname{Im} \varphi_*$, *the natural projection* $E^2_{1,3} \to E^\infty_{1,3}$ *is zero.*

Now we are able to draw the E^4-term of the spectral sequence associated to the case $n = 4$.

$$\begin{array}{cccccc}
\bigoplus_{i+j=5} H_i(3) \otimes H_j(F^*)/\operatorname{Im} d_1 & * & & & & \\
H_4(3) & 0 & & \mathrm{ZERO} & & \\
H_3(3) & 0 & \vdots & & & \\
0 & 0 & 0 & 0 & \cdots & \\
0 & 0 & 0 & 0 & \wp^4/\wp^2 & \cdots
\end{array} \tag{4.39}$$

This immediately implies two corollaries.

Corollary 4.14 *We obtained the following stabilization result:*

$$H_3(\mathrm{GL}_2, \mathrm{GM}_2) \simeq H_3(\mathrm{GL}_3, \mathrm{GM}_3) \simeq \cdots \simeq H_3(\mathrm{GL}, \mathrm{GM}).$$

Corollary 4.15 *The following sequence*

$$\bigoplus_{i+j=5} H_i(3)\otimes H_j(F^*) \longrightarrow H_5(4) \longrightarrow \wp^4/\wp^2 \longrightarrow H_4(3) \longrightarrow H_4(4) \longrightarrow 0$$

is exact.

We can, finally, prove the main theorem of this subsection.

Theorem 4.16 *Assume that* 3! *is invertible in the coefficient ring. Then the group* $\wp^4(F)$ *is a direct sum of the group* $H_3(\mathrm{GL}_2,\mathrm{GM}_2)$ *and the indecomposable component* $H_5(\mathrm{GL}_4,\{\mathrm{GL}_3,\mathrm{GM}_4\})_{\mathrm{ind}}$.

Proof We have already established that the group $H_3(\mathrm{GL}_2,\mathrm{GM}_2)\simeq\wp^2(F)$ is a direct summand of $\wp^4(F)$. Let us now consider the following diagram.

$$\begin{array}{ccc} H_5(\mathrm{GL}_3,\mathrm{GM}_3) & \longrightarrow & \bigoplus_{i+j=5} H_i(\mathrm{GL}_3,\mathrm{GM}_3)\otimes H_j(F^*) \\ \downarrow & & \downarrow{\scriptstyle\mu} \\ H_5(\mathrm{GL}_4,\mathrm{GM}_4) & = & H_5(\mathrm{GL}_4,\mathrm{GM}_4) \\ \downarrow & & \downarrow \\ H_5(\mathrm{GL}_4,\{\mathrm{GL}_3,\mathrm{GM}_4\}) & \longrightarrow & \wp^4(F)/\wp^2(F) \\ \downarrow & & \downarrow \\ H_4(\mathrm{GL}_3,\mathrm{GM}_3) & = & H_4(\mathrm{GL}_3,\mathrm{GM}_3) \\ \downarrow & & \downarrow \\ H_4(\mathrm{GL}_4,\mathrm{GM}_4) & = & H_4(\mathrm{GL}_4,\mathrm{GM}_4) \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array} \tag{4.40}$$

Using the technique developed for the case $n=3$, we can show that this diagram is commutative, has exact columns, and the map μ coincides with the $\smile$-product on one-relative homology groups. The standard diagram chase gives us the following exact sequence

$$\bigoplus_{\substack{i+j=5\\ j>0}} H_i(\mathrm{GL}_3,\mathrm{GM}_3)\otimes H_j(F^*) \xrightarrow{\eta} H_5(\mathrm{GL}_4,\{\mathrm{GL}_3,\mathrm{GM}_4\}) \longrightarrow \wp^4/\wp^2 \longrightarrow 0. \tag{4.41}$$

The map η here is obtained as a composition of μ and the projection map from the homology long exact sequence. We should now check that the image of the map η coincides with the image of the $\smile$-product on bi-relative homology groups. Consider the following commutative diagram, whose left vertical path is exact:

$$
\begin{tikzcd}
H_4(\mathrm{GL}_2,\mathrm{GM}_2)\otimes H_1(F^*) \arrow[d] & \\
H_4(\mathrm{GL}_3,\mathrm{GM}_3)\otimes H_1(F^*) \arrow[r, "\mu"] \arrow[d] & H_5(\mathrm{GL}_4,\mathrm{GM}_4) \arrow[d] \\
H_4(\mathrm{GL}_3,\{\mathrm{GM}_3,\mathrm{GL}_2\})\otimes H_1(F^*) \arrow[r, "\smile"] \arrow[d] & H_5(\mathrm{GL}_4,\{\mathrm{GM}_4,\mathrm{GL}_3\}) \\
0. &
\end{tikzcd}
\tag{4.42}
$$

Since the left vertical arrow of the commutative square above is an epimorphism and all elements coming from $H_4(\mathrm{GL}_2,\mathrm{GM}_2)\otimes H_1(F^*)$ go to zero in the group $H_5(\mathrm{GL}_4,\{\mathrm{GM}_4,\mathrm{GL}_3\})$, we obtain the desired result. ■

Proposition 4.17 *Assume that the claim of Conjecture 0.2 holds for $n = 2, 4$. (It is known only for $n = 2$. See [5], [15].) Let F be an algebraically closed field and $p > n$ be a prime. Then the natural mapping $H_4\big(\mathrm{GL}_3(F),\mathbb{Z}/p\big) \to H_4\big(\mathrm{GL}_4(F),\mathbb{Z}/p\big)$ is a monomorphism.*

Proof One can easily show that Conjecture 0.2 implies vanishing of the group $H_5(\mathrm{GL}_4,\{\mathrm{GM}_4,\mathrm{GL}_3\},\mathbb{Z}/p)$. Let us now look at the following commutative diagram with exact rows:

$$
\begin{tikzcd}
H_5(\mathrm{GM}_4,\mathrm{GM}_3,\mathbb{Z}/p) \arrow[r] \arrow[d] & H_4(\mathrm{GM}_3,\mathbb{Z}/p) \arrow[r, "i_*"] \arrow[d] & H_4(\mathrm{GM}_4,\mathbb{Z}/p) \arrow[d] \\
H_5(\mathrm{GL}_4,\mathrm{GL}_3,\mathbb{Z}/p) \arrow[r] & H_4(\mathrm{GL}_3,\mathbb{Z}/p) \arrow[r, "i_*"] & H_4(\mathrm{GL}_4,\mathbb{Z}/p).
\end{tikzcd}
\tag{4.43}
$$

Any class x in $H_4(\mathrm{GL}_3,\mathbb{Z}/p)$ belonging to the kernel of i_* lifts up to some class x' in $H_5(\mathrm{GM}_4,\mathrm{GM}_3,\mathbb{Z}/p)$. It can be easily checked that the map i_* in the upper row is a monomorphism. Therefore, the image of x' in $H_4(\mathrm{GM}_3,\mathbb{Z}/p)$ and, moreover, $H_4(\mathrm{GL}_3,\mathbb{Z}/p)$ are zero. ■

The latter proposition shows the significant importance of Conjecture 0.2 for the investigation of the homology of linear groups. Questions connected to the estimation of kernels of maps induced by the natural embedding seem to be the most difficult in this field.

5 Construction of Homomorphism $\xi\colon H_n\big(\mathrm{GL}_n(F)\big) \to \wp^{n-1}(F)$

In this section we generally follow the strategy proposed by Suslin in [15]. Some changes of the construction make our approach more convenient.

Remark 5.1 In order to make our computations fully compatible with ones in [15] we've chosen here the different embedding $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$, given by the formula $\mathrm{GL}_{n-1} \mapsto \left(\begin{smallmatrix} \mathrm{GL}_{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_n$. This, of course, doesn't change the maps induced on homology.

Consider the complex $P_*^n(F)$ defined in Section 3. Let us define another differential d' such that

$$d'(x_0, \ldots, x_p) = \sum_{i=0}^{p-1} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_p). \tag{5.1}$$

One can easily prove

Lemma 5.2 *The following complexes are acyclic:*

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ \epsilon\ } P_0^n \xleftarrow{\ d\ } P_1^n \xleftarrow{\ d\ } \cdots$$

$$0 \longleftarrow P_0^n \xleftarrow{\ d'\ } P_1^n \xleftarrow{\ d'\ } \cdots .$$

We define the homomorphism $\nu \colon P_m^n \to P_m^n$ by the formula:

$$\nu(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_p, x_i). \tag{5.2}$$

Direct computation shows that $d'\nu = \nu d$. Consider now the following morphism of complexes:

$$\begin{array}{ccccccc} P_1^n & \xleftarrow{\ d\ } & P_2^n & \xleftarrow{\ d\ } & P_3^n & \xleftarrow{\ d\ } & \cdots \\ \nu \big\downarrow & & \nu \big\downarrow & & \nu \big\downarrow & & \\ P_1^n & \xleftarrow{\ d'\ } & P_2^n & \xleftarrow{\ d'\ } & P_3^n & \xleftarrow{\ d'\ } & \cdots \end{array} \tag{5.3}$$

and denote its cone by $\mathcal{P}_*$. Let also define the "natural" augmentation τ as the composition $\tau \colon \mathcal{P}_0 = P_1^n \xrightarrow{d'} P_0^n \xrightarrow{\epsilon} \mathbb{Z}$. One can easily check that the complex $0 \leftarrow \mathbb{Z} \xleftarrow{\tau} \mathcal{P}_*$ is acyclic. This complex has a natural GL_n-module structure that determines the canonical homomorphism $H_n\big(\mathrm{GL}_n(F)\big) \to H_n\big((\mathcal{P}_*)_{\mathrm{GL}_n(F)}\big)$. The groups $(P_m^n)_{\mathrm{GL}_n(F)} = \bar{P}_m^n$ can be easily computed. In particular, $\bar{P}_m^n = \mathbb{Z}$ for $m \le n$, $\bar{P}_{n+1}^n = \coprod \mathbb{Z}[a_1, \ldots, a_{n-1}]$, where $[a_1, \ldots, a_{n-1}]$ is the orbit of

$$\begin{pmatrix} & & & & & 1 & \\ e_1 & e_2 & \cdots & e_{n-1} & E & a_1 & e_n \\ & & & & & a_2 & \\ & & & & & \vdots & \\ & & & & & a_{n-1} & \end{pmatrix} \tag{5.4}$$

and summation is over all vectors $[a_1, \dots, a_{n-1}]$ for which the matrix above satisfies GP-condition. In the same way, the group $\bar{P}^n_{n+1}$ can be written as $\coprod \mathbb{Z} \left[\begin{smallmatrix} a_1, & \dots, & a_{n-1} \\ b_1, & \dots, & b_{n-1} \end{smallmatrix}\right]$ under the same conditions. Let us define the following homomorphisms:

$$\omega_1 \colon (\mathcal{P}^n_n)_{\mathrm{GL}_n} = \mathbb{Z} \oplus \coprod \mathbb{Z}[a_1, \dots, a_{n-1}] \to (\bar{P}^{n-1}_n) = \coprod \mathbb{Z}[x_1, \dots, x_{n-2}] \tag{5.5}$$

and

$$\begin{aligned} \omega_2 \colon (\mathcal{P}^n_{n+1})_{\mathrm{GL}_n} &= \coprod \mathbb{Z}[a_1, \dots, a_{n-1}] \oplus \coprod \mathbb{Z} \begin{bmatrix} a_1, & \dots, & a_{n-1} \\ b_1, & \dots, & b_{n-1} \end{bmatrix} \\ &\to (\bar{P}^{n-1}_{n+1}) = \coprod \mathbb{Z} \begin{bmatrix} x_1, & \dots, & a_{x-2} \\ y_1, & \dots, & y_{n-2} \end{bmatrix} \end{aligned} \tag{5.6}$$

setting

$$\omega_1(1) = 0 \tag{5.7}$$

$$\omega_1([a_1, \dots, a_{n-1}]) = \langle a_1, \dots, a_{n-2} \rangle \tag{5.8}$$

$$\omega_2\left(\begin{bmatrix} a_1, & \dots, & a_{n-1} \\ b_1, & \dots, & b_{n-1} \end{bmatrix} \right) = \left\langle \begin{matrix} a_1, & \dots, & a_{n-2} \\ b_1, & \dots, & b_{n-2} \end{matrix} \right\rangle \tag{5.9}$$

$$\omega_2([a_1, \dots, a_{n-1}]) = \left\langle \begin{matrix} a_1, & \dots, & a_{n-2} \\ \frac{a_1(1-a_{n-1})}{a_1 - a_{n-1}}, & \dots, & \frac{a_{n-2}(1-a_{n-1})}{a_{n-2} - a_{n-1}} \end{matrix} \right\rangle. \tag{5.10}$$

The above notation $\left\langle \begin{smallmatrix} a_1, & \dots, & a_{n-2} \\ b_1, & \dots, & b_{n-2} \end{smallmatrix} \right\rangle$ means the class of the matrix

$$\begin{pmatrix} & & & & & 1 & 1 \\ e_1 & e_2 & \cdots & e_n & E & a_1 & b_1 \\ & & & & & \vdots & \vdots \\ & & & & & a_{n-2} & b_{n-2} \end{pmatrix} \tag{5.11}$$

in $(P^{n-1}_{n+1})_{\mathrm{GL}_{n-1}}$.

Direct computation shows that the following diagram commutes.

$$\begin{tikzcd} \bar{P}^n_{n+1} \arrow[d, "\nu"] \arrow[drr, bend left, "\omega_2"] & & \\ \bar{P}^n_{n+1} \arrow[d, "\omega_1"] & & \bar{P}^n_{n+2} \arrow[ll, "d'"'] \arrow[d, "\omega_2"] \\ \bar{P}^{n-1}_n & & \bar{P}^{n-1}_{n+1}. \arrow[ll, "d"'] \end{tikzcd} \tag{5.12}$$

This gives us the well-defined homomorphism $\omega\colon H_n\big((\mathcal{P}^n_*)_{\mathrm{GL}_n}\big) \to \wp^{n-1}_{\mathrm{cl}}$.

Combining two homomorphisms obtained in this section with the homomorphism α constructed in Section 3 we get the following map:

$$\xi\colon H_n(\mathrm{GL}_n) \to H_n\big((\mathcal{P}^n_*)_{\mathrm{GL}_n}\big) \xrightarrow{\omega} \wp^{n-1}_{\mathrm{cl}} \xrightarrow{\alpha} \wp^{n-1}. \tag{5.13}$$

Since the action of GL_{n-1} does not affect the vector e_n, we obtain a structure of GL_{n-1}-module on the bottom row of the bicomplex $\mathcal{P}^n_*$. This defines the canonical homomorphism $H_n(\mathrm{GL}_{n-1}) \to \wp^{n-1}$. On the other hand, the standard GL_{n-1}-resolution of $\mathbb{Z}$ maps to the resolution $P^{n-1}_0 \xleftarrow{d} P^{n-1}_1 \xleftarrow{d} \cdots$. This gives us an alternative way to define the mapping $H_n(\mathrm{GL}_{n-1}) \to \wp^{n-1}$. Repeating almost word by word arguments from the proof of [15, Lemma 3.4], we can see that both constructions define the same map and the diagram

$$\begin{tikzcd} H_n(\mathrm{GL}_{n-1}) \arrow[rr, "i_*"] \arrow[dr] & & H_n(\mathrm{GL}_n) \arrow[dl, "\xi"] \\ & \wp^{n-1}_{\mathrm{cl}} & \end{tikzcd} \tag{5.14}$$

commutes.

The homomorphism ξ constructed in this section allows us to make some estimations of kernels of maps $H_n(\mathrm{GL}_{n-1}) \xrightarrow{i_*} H_n(\mathrm{GL}_n)$. It can be considered as a generalization of [15, Proposition 3.1] for higher dimensions. (See also Corollary 4.9.) The following proposition holds.

Proposition 5.3 *For* $3 \le n \le 5$ *the kernel of natural homomorphism* $H_n(\mathrm{GL}_{n-1}) \xrightarrow{i_*} H_n(\mathrm{GL}_n)$ *maps into the decomposable part of the group* $H_n(\mathrm{GL}_{n-1}, \mathrm{GM}_{n-1})$.

(Here i_* *denotes, as usual, the homomorphism induced on homology by the natural group inclusion.)*

Proof Consider the following diagram:

$$\begin{tikzcd} & H_n(\mathrm{GM}_{n-1}) \arrow[d] & \\ & H_n(\mathrm{GL}_{n-1}) \arrow[r, "i_*"] \arrow[d] & H_n(\mathrm{GL}_n) \arrow[d, "\xi"] \\ \bigoplus_{i+j=n} H_i(n-2) \otimes H_j(F^*) \arrow[r] & H_n(\mathrm{GL}_{n-1}, \mathrm{GM}_{n-1}) \arrow[r] & \tilde{\wp}^{n-1}(F), \end{tikzcd} \tag{5.15}$$

where $\tilde{\wp}^i$ denotes $\wp^4/\wp^2$ for $i = 4$ and $\wp^i$ otherwise. Diagram (5.14) and results of Section 4 imply commutativity of this diagram and exactness of its rows and columns. Diagram chasing now shows that an element belonging to $\mathrm{Ker}\big(H_n(\mathrm{GL}_{n-1}) \to H_n(\mathrm{GL}_n)\big)$ should either come from $H_n(\mathrm{GM}_{n-1})$ or map to the decomposable part of $H_n(\mathrm{GL}_{n-1}, \mathrm{GM}_{n-1})$. ■

Remark 5.4 One can easily see that for $n = 3$ there is no nonzero decomposable elements in $H_3(CL_2, \mathrm{GM}_2)$, and our result coincides with Corollary 4.9.

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Max Planck Institute für Mathematik
Vivatsgasse 7
53111 Bonn
Germany
email: yagunov@mpim-bonn.mpg.de

Steklov Mathematical Institute (St. Petersburg)
Fontanka 27
191011 St. Petersburg
Russia
email: yagunov@pdmi.ras.ru