Notes on Truncated Displays

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Introduction

Introduction

The aim of this paper is to clarify the relation between truncated displays and truncated Barsotti-Tate groups.

Let us recall the notion of a truncated display as used in [L2]. We fix a prime number p. Let R be a commutative ring with unit element such that $pR = 0$. We denote by $W(R)$ the ring of p-typical Witt vectors and by $W_n(R)$ the truncated ones. We write ${}^F\xi$ and ${}^V\xi$ for the Frobenius and Verschiebung of an element $\xi \in W(R)$.

A display $\mathcal{P}$ over R may be given by the following data: Two locally free finitely generated $W(R)$-modules T and L and a Frobenuis linear isomorphism

$$\Phi : T \oplus L \to T \oplus L.$$

For this introduction we will assume that $T \cong W(R)^d$ and $L \cong W(R)^c$ are free $W(R)$-modules. Then Φ is given by an invertible block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{d+c}(W(R)).$$

$$\Phi(\begin{pmatrix} t \\ l \end{pmatrix}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^F t \\ {}^F l \end{pmatrix},$$

where $t \in W(R)^d$ and $l \in W(R)^c$. The height of the display is $h = d + c$.

Assume a second display $\mathcal{P}'$ is given by a block matrix. Then a morphism $\mathcal{P} \to \mathcal{P}'$ is the same as a matrix:

$$\begin{pmatrix} X & \mathfrak{J} \\ Z & Y \end{pmatrix} \in M(h' \times h, W(R)))$$

of size $h' \times h$ such that the following relation holds:

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} {}^F X & \mathfrak{J} \\ p\ {}^F Z & {}^F Y \end{pmatrix} = \begin{pmatrix} X & {}^V \mathfrak{J} \\ Z & Y \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

This is the description of the category of displays in terms of matrices.

To define truncated displays of level n we take all matrices with coefficients in $W_n(R)$. Since $pR = 0$ we have a Frobenius $F : W_n(R) \to W_n(R)$. Therefore the definition of a morphism (1) makes perfectly sense if we take V as the composition $W_n(R) \stackrel{V}{\to} W_{n+1}(R) \stackrel{Res}{\to} W_n(R)$.

Let $\mathcal{P}$ be a truncated display of level n. Let $m \leq n$. If we apply the restriction morphism $W_n(R) \to W_m(R)$ to the matrix of $\mathcal{P}$ we obtain a truncated display $\mathcal{P}(m)$ of level m. This is called a truncation of $\mathcal{P}$.

By [L1] we have a functor

$$\Phi_{n,R} : \mathscr{BT}_n(R) \to \mathcal{D}_n(R)$$

from the category of truncated BT-groups (or in another terminology "p-divisible groups")of level n over R to the category of truncated displays over R.

This functor is compatible with truncations. If G is a finite group scheme we denote by $G[p^m]$ the kernel of the endomorphism $p^m \operatorname{id}_G$. If G is a truncated BT-group of level $n \geq m$ then $G[p^m]$ is a truncated BT-group of level m, which is also denoted by $G(m)$.

Let $G \in \mathscr{BT}_n(R)$. We denote by $\hat{G}$ the Cartier dual of G. Consider the e-th iterate of the Verschiebung for some number e:

$$\operatorname{Ver}^e : \hat{G}^{(p^e)} \to \hat{G}. \tag{2}$$

We say that G has order of nilpotence e is e is the smallest number such that the morphism (2) induces zero on the tangent spaces. The order of nilpotence may also be defined in terms of displays. By restriction we obtain from $\Phi_{n,R}$ a functor

$$\Phi_{n,R} : \mathscr{BT}_n^{(e)}(R) \to \mathcal{D}_n^{(e)}(R)$$

from the category of truncated BT-groups of order of nilpotence e to the category of truncated displays of order of nilpotence e.

Theorem 1. *Assume that $pR = 0$. Let $n, e, m \geq 1$ be natural numbers such that $n \geq 2m(e + 2)$. There is a functor a functor*

$$BT_{m,R} : \mathcal{D}_n^{(e)}(R) \to \mathscr{BT}_m^{(e)}(R)$$

such that we have an isomrphism of functors

$$BT_{m,R}(\Phi_{n,R}(G)) \cong G[p^m]$$

Remark: We note that for two rational numbers n and t the conditions $p^n\mathcal{W}_t(S) = 0$ und $p^n W_t(S) = 0$ are equivalent.

Theorem 0.1. *Let S be a ring and let m ba a natural number such that $p^{m+1}S = 0$. Let t and e be natural numbers such that $t > 2e(e+2)(m(e+1)+2)$. Let s be a natural number such that $t \geq 2e(e+2)(s+m)$. Then there is a functor*

$$BT_s : \mathcal{D}_t^e(S) \to \mathscr{BT}_s^{(e)}(S).$$

Let n be a natural number such that $p^n W_t(S) = 0$. Then the composite $BT_s \circ \Phi_n$ restricted to $\mathscr{BT}_n^{(e)}(S)$ is the truncation functor for BT-groups.

Proof. We set $S = R/pR$. Consider to truncated p-divisible groups G_1 and G_2 of level $u > m$. We denote there reductions of R by $\bar{G}_1$ and $\bar{G}_2$. If the reductions of two morphisms $\phi, \psi : G_1 \to G_2$ agree then $\phi(u-m)$ and $\psi(u-m)$ agree.

We define the functor as follows: Let $\mathcal{T} \in \mathcal{D}_t^e(S)$. We prolong $\mathcal{T}$ to a display $\mathcal{P}$ over S. Let X be the p-divisible group of. We show that the isomorphism class of $X(s+m)$ depends only on $\mathcal{T}$. By the remark and Proposition ??? in the beginning this suffices if we only take isomorphism as morphisms in the categories. But a standard argument shows that this suffices.

Assume we are given a second display $\mathcal{P}'$ that prolongs $\mathcal{T}$ and let X' be the corresponding BT-group. Then by the case $pR = 0$ we have an isomorphism of the reductions

$$\bar{X}(s+m) \cong \bar{X}'(s+m). \tag{3}$$

We choose a BT-group Z_0 over R whose truncation is isomorphic to (3). By Illusie we find BT-groups Z and Z' which lift Z and such that

$$Z(s+m) \cong X(s+m), \quad Z'(s+m) \cong X'(s+m).$$

It we apply the functor Φ_{s+m} to these isomorphisms we see that value of the crystals of Z and Z' with the Hodge filtration are deduced from $\mathcal{T}_1$. But this means by Grothendieck-Messing that Z and Z' are isomorphic. □

We generalize the notion of a truncated display to rings R where p and develop a deformation theory for truncated displays. This leads to a construction of crystals associated to truncated displays: Let $\mathrm{Cris}_m(R)$ be the category of all pd-thickenings $S \to R$ with kernel $\mathfrak{a}$ such that $p^m\mathfrak{a} = 0$. Let $n, e, m \geq 1$ be natural numbers such that $n > m(e+1)+1$. Let $\mathcal{P}$ be a truncated display of level n over R of order of nilpotence $\leq e$. We construct a crystal $\mathbb{D}_{\mathcal{P}}$ of locally free $\mathcal{O}$-modules on $\mathrm{Cris}_m(R)$.

Theorem 2. *Let $t \geq n$ such that $p^t W_{n+1}(R) = 0$. Then there is a functor*

$$\Phi_n : \mathscr{BT}_t(R) \to \mathcal{D}_n(R)$$

Let X be a p-divisible group R such that $X_{R/pR}$ has order of nilpotence e. We set $\mathcal{P} = \Phi_n(X[p^t])$. Let $S \to R$ an object of $\mathrm{Cris}_m(R)$ where $n > m(e+1)+1$. Then we have a canonical isomorphism

$$\mathbb{D}_X(S) \cong \mathbb{D}_{\mathcal{P}}(S)$$

where the left hand side is the Grothendieck-Messing crystal evaluated in S.

1 The category of truncated displays

We fix a prime number p. Let R be a ring such that p is nilpotent in R. For fixed $n \in \mathbb{N}$ let $W_n(R)$ be the ring of truncated Witt vectors. We consider the ring homomorphism induced by the restriction $\mathrm{Res} : W_{n+1}(R) \to W_n(R)$ and the Frobenius $F : W_{n+1}(R) \to W_n(R)$:

$$(\mathrm{Res}, F) : W_{n+1}(R) \to W_n(R) \times W_n(R). \tag{4}$$

The image of this ring homomorphism will be denoted by $\mathcal{W}_n(R)$. The kernel consists of the elements ${}^{V^n}[s]$, where $s \in R$ and $ps = 0$. It follows easily that $R \mapsto \mathcal{W}_n(R)$ is a sheaf for the f.p.q.c.-topology; see the Appendix.

The two projections will be denoted by

$$\mathrm{Res} : \mathcal{W}_n(R) \to W_n(R), \qquad F : \mathcal{W}_n(R) \to W_n(R).$$

If $pR = 0$ then $\mathrm{Res} : \mathcal{W}_n(R) \to W_n(R)$ is an isomorphism.

Let $I_{n+1} = {}^V W_n(R) \subset W_{n+1}(R)$. This is a $\mathcal{W}_n(R)$-module by

$$\xi \, {}^V\eta = {}^V({}^F\xi\eta), \qquad \text{for } \xi \in \mathcal{W}_n(R),\ \eta \in W_n(R).$$

The inverse of the Verschiebung defines a bijective map $V^{-1} : I_{n+1} \to W_n(R)$, which is F-linear with respect to $F : \mathcal{W}_n(R) \to W_n(R)$. We denote by $\kappa : I_{n+1} \to \mathcal{W}_n(R)$ the map induced by (4). The cokernel of κ is $\mathbf{w}_0 \circ \mathrm{Res} : \mathcal{W}_n(R) \to W_n(R) \to R$.

Definition 1.1. *A truncated display $\mathcal{P}$ of level n over a ring R in which p is nilpotent consists of $(P, Q, \iota, \epsilon, F, \dot{F})$. Here P and Q are $\mathcal{W}_n(R)$-modules,*

$$\iota : Q \to P, \qquad \epsilon : I_{n+1} \otimes_{\mathcal{W}_n(R)} P \to Q,$$

are $\mathcal{W}_n(R)$*-linear maps, and*

$$F : P \to W_n(R) \otimes_{\mathcal{W}_n(R)} P, \qquad \dot{F} : Q \to W_n(R) \otimes_{\mathcal{W}_n(R)} P$$

are F*-linear maps. (The tensor products are taken with respect to* Res.*)*

The following conditions are required:

(i) P *is a finitely generated projective* $\mathcal{W}_n(R)$*-module.*

(ii) The maps $\iota \circ \epsilon$ *and* $\epsilon \circ (\mathrm{id}_{I_{n+1}} \otimes \iota)$ *are the multiplication maps (via* κ*).*

(iii) The cokernels of ι *and* ϵ *are finitely generated projective* R*-modules.*

(iv) There is a commutative diagram

$$\begin{array}{ccc} I_{n+1} \otimes_{\mathcal{W}_n(R)} P & \xrightarrow{\quad \epsilon \quad} & Q, \\ \Big\downarrow {\scriptstyle \tilde{F}} & \swarrow {\scriptstyle \dot{F}} & \\ W_n(R) \otimes_{\mathcal{W}_n(R)} P & & \end{array} \tag{5}$$

where $\tilde{F}$ *is defined by* $\tilde{F}({}^{V}\eta \otimes x) = \eta F(x)$*.*

(v) $\dot{F}(Q)$ *generates* $W_n(R) \otimes_{\mathcal{W}_n(R)} P$ *as a* $W_n(R)$*-module.*

(vi) We have an exact sequence

$$0 \to Q/\operatorname{Im} \epsilon \xrightarrow{\iota} P/\kappa(I_{n+1})P \to P/\iota(Q) \to 0.$$

(Only the injectivity of the second arrow is a requirement.)

Truncated displays of level n *over* R *form an additive category in an obvious way, which we denote by* $\mathcal{D}_n(R)$*.*

The surjective R*-linear map* $P/\kappa(I_{n+1})P \to P/\iota(Q)$ *is called the Hodge filtration of the truncated display* $\mathcal{P}$*.*

When $pR = 0$ we have just F-linear maps $F : P \to P$, $\dot{F} : Q \to P$ as in the case of displays, but Q is not a submodule of P.

It follows from the above axioms that

$$F(\iota(y)) = p\dot{F}(y), \quad \text{for } y \in Q$$

using $\eta = 1$ and $x = \iota(y)$ in (iv).

Definition 1.2. *Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be a truncated diaplay of level n over R. A normal decomposition for $\mathcal{P}$ consists of (T, L, u, v) where T and L are finitely generated projective $\mathcal{W}_n(R)$-modules with isomorphisms*

$$u : P \cong T \oplus L, \qquad v : Q \cong I_{n+1} \otimes_{\mathcal{W}_n(R)} T \oplus L,$$

such that the maps

$$\begin{array}{rcl} \iota : I_{n+1} \otimes_{\mathcal{W}_n(R)} T \oplus L & \to & T \oplus L \\ \epsilon : I_{n+1} \otimes_{\mathcal{W}_n(R)} T \oplus I_{n+1} \otimes_{\mathcal{W}_n(R)} L & \to & I_{n+1} \otimes_{\mathcal{W}_n(R)} T \oplus L \end{array}$$

are given as follows: ι is the multiplicationon the first summand and the identity on the second summand, while ϵ is the identity on the first summand and the multiplication on the second summand.

Unlike in the case of displays, the isomorphism $u : T \oplus L \cong P$ does not in general determine v.

Proposition 1.3. *Every truncated display has a normal decomposition.*

Proof. Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be a truncated display of level n over R.

We take a projective $\mathcal{W}_n(R)$-module T which lifts $P/\iota(Q)$ and a projective $\mathcal{W}_n(R)$-module L which lifts $Q/\operatorname{Im}\epsilon$. We choose liftings $T \to P$ and $L \to Q$ of the natural projections $P \to P/\iota(Q)$ and $Q \to Q/\operatorname{Im}\epsilon$. We consider the homomorphism

$$T \oplus L \to P$$

which is on the second summand induced by ι. This becomes by (vi) an isomorphism if we tensor it by $R \otimes_{\mathcal{W}_n(R)}$ and is therefore an isomorphism.

Now we consider the homomorphism

$$\nu : I_{n+1} \otimes_{\mathcal{W}_n(R)} T \oplus L \to Q, \tag{6}$$

which is induced on the first factor by ϵ.

We note that $\epsilon : I_{n+1} \otimes L \to Q$ is the multiplication, by (ii). This shows that the image of ν contains the image of ϵ. Therefore the homomorphism (6) is surjective. We will show it is injective.

Let us denote by $\Phi : T \to W_n(R) \otimes_{\mathcal{W}_n(R)} P$ the restriction of F to T and by $\dot{\Phi} : L \to W_n(R) \otimes_{\mathcal{W}_n(R)} P$ the composite of $\dot{F}$ with $L \to Q$.

We denote by $\tilde{\Phi} : I_{n+1} \otimes_{\mathcal{W}_n(R)} T \to W_n(R) \otimes_{\mathcal{W}_n(R)} P$ the map defined by $\tilde{\Phi}(\,^V\xi \otimes t) = \xi \otimes \Phi(t)$. Then we obtain a commutative diagram

$$\begin{array}{ccc} I_{n+1} \otimes T \oplus L & \xrightarrow{\quad\nu\quad} & Q \\ & \searrow^{\tilde{\Phi}\oplus\dot{\Phi}} \quad\quad \swarrow_{\dot{F}} & \\ & W_n(R) \otimes_{\mathcal{W}_n(R)} P. & \end{array}$$

We now assume without loss of generality that L and T are free; see Lemma 1.4 below. Let $t_1, \dots t_d$ be a basis of T and $l_1, \dots, l_c$ be a basis of L. Since by the diagram $\tilde{\Phi} \oplus \dot{\Phi}$ is an F-linear epimorphism we conclude that $\Phi(t_1), \dots, \Phi(t_d), \dot{\Phi}(l_1), \dots, \dot{\Phi}(l_c)$ is a basis of $W_n(R) \otimes_{\mathcal{W}_n(R)} P$. *

Consider an element in the kernel of ν:

$$\sum {}^V\xi_i \otimes t_i + \sum \eta_j l_j \in I_{n+1} \otimes T \oplus L, \quad \xi_i \in W_n(R),\ \eta \in \mathcal{W}_n(R). \tag{7}$$

Since $\Phi \oplus \dot{\Phi}$ applied to this element must be zero in $W_n(R) \otimes_{\mathcal{W}_n(R)} P$ we conclude that $\xi_i = 0$.

On the other hand the restriction to ν to $0 \oplus L$ is injective because $\iota \circ \nu$ is the injection $0 \oplus L \subset P$. This proves that the element (7) is zero. □

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Lemma 1.4. *Let $R' = R[S^{-1}]$. Then $\mathcal{W}_n(R') = \mathcal{W}_n(R)[[S]^{-1}]$.*

Proof. The corresponding fact for W_{n+1} is known. The kernel of $W_{n+1}(R) \to \mathcal{W}_n(R)$ is the module of p-torsion elements $R[p]$, considered as an R/pR-module via Frob^n. The lemma follows. □

Remark 1.5. Proposition 1.3 implies that Definition 1.1 coincides with the definition of truncated displays in [L2] if $pR = 0$. Indeed, the conditions on (P, Q, ι, ϵ) imposed here are weaker than those of [L2], which are equivalent to the existence of a normal decomposition. But the difference disappears in the presence of $(F, \dot{F})$. See also Lemma 3.3 below.

Any normal decomposition is obtained as follows: Choose liftings $T' \to P/\iota(Q)$ and $L' \to Q/\operatorname{Im}\epsilon$, to projective $\mathcal{W}_n(R)$-modules and extend them to homomorphisms $T' \to P$ and $L' \to Q$. Then $\iota : L' \to P$ is injective and $P = T' \oplus L'$ is a normal decomposition.

We note that the maps F and $\dot{F}$ are uniquely determined by their linearisations:

$$\begin{array}{rlcl} F^\sharp : & W_n(R) \otimes_{F, \mathcal{W}_n(R)} P & \to & W_n(R) \otimes_{\mathcal{W}_n(R)} P \\ \dot{F}^\sharp : & W_n(R) \otimes_{F, \mathcal{W}_n(R)} Q & \to & W_n(R) \otimes_{\mathcal{W}_n(R)} P \end{array}$$

From the normal decomposition we obtain as in the case of displays an isomorphism of $W_n(R)$-modules

$$F^\sharp \oplus \dot{F}^\sharp : (W_n(R) \otimes_{F, \mathcal{W}_n(R)} T) \oplus (W_n(R) \otimes_{F, \mathcal{W}_n(R)} L) \to W_n(R) \otimes_{\mathcal{W}_n(R)} P. \tag{8}$$

We write the last map as a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{9}$$

where

$$\begin{array}{ll} A : W_n \otimes_{F,\mathcal{W}_n} T \to W_n \otimes_{\mathcal{W}_n} T, & B : W_n \otimes_{F,\mathcal{W}_n} L \to W_n \otimes_{\mathcal{W}_n} T, \\ C : W_n \otimes_{F,\mathcal{W}_n} T \to W_n \otimes_{\mathcal{W}_n} L, & D : W_n \otimes_{F,\mathcal{W}_n} L \to W_n \otimes_{\mathcal{W}_n} L, \end{array}$$

are $W_n(R)$-linear maps.

Conversely, by the following construction, a matrix (9) which is an isomorphism of $W_n(R)$-modules defines a truncated display of level n.

We set $\dot{\sigma} = V^{-1} : I_{n+1} \to W_n(R)$ and consider this as an isomorphism von $\mathcal{W}_n(R)$-modules:

$$I_{n+1} \to W_n(R)_{[F]},$$

where the last index denotes restriction of scalars by F. For an arbitrary $\mathcal{W}_n(R)$-module $\dot{\sigma}$ induces an isomorphism denoted by the same letter

$$\dot{\sigma} : I_{n+1} \otimes_{\mathcal{W}_n(R)} T \to W_n(R) \otimes_{F,\mathcal{W}_n(R)} T.$$

This is F-linear, i.e. with respect to $F : \mathcal{W}_n(R) \to W_n(R)$.

We also use the notation σ for the F-linear maps

$$\begin{array}{rcl} \sigma : T & \to & W_n \otimes_{F,\mathcal{W}_n(R)} T \\ \ell & \mapsto & 1 \otimes \ell. \end{array}$$

To obtain a truncated display of level n from a matrix (9) we set

$$P = T \oplus L, \quad Q = I_{n+1} \otimes T \oplus L.$$

Then we have obvious maps ι and ϵ as in Proposition 1.3. For vectors

$$\begin{pmatrix} t \\ \ell \end{pmatrix} \in T \oplus L, \quad \begin{pmatrix} y \\ \ell \end{pmatrix} \in I_{n+1} \otimes T \oplus L$$

We define F and $\dot{F}$ as follows

$$F \begin{pmatrix} t \\ \ell \end{pmatrix} = \begin{pmatrix} A & pB \\ C & pD \end{pmatrix} \begin{pmatrix} \sigma(t) \\ \sigma(\ell) \end{pmatrix} \tag{10}$$

$$\dot{F} \begin{pmatrix} y \\ \ell \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \dot{\sigma}(y) \\ \sigma(\ell) \end{pmatrix} \tag{11}$$

Definition 1.6. *Let T and L be finitely generated projective $\mathcal{W}_n(R)$-modules. Assume we are given a matrix* (9) *of homomorphisms A, B, C, D which is invertible. If we define F, $\dot{F}$ by the formulas* (10) *and* (11) *we obtain a truncated display $\mathcal{S}(T, L; A, B, C, D)$ level n which we call a standard truncated display of level n over the ring R.*

We will now describe a homomorphism of standard truncated displays:

$$\alpha : \mathcal{S}(T, L; A, B, C, D) \to \mathcal{S}(T', L'; A', B', C', D').$$

We write $P = T \oplus L$ and so on. The morphism α is given by two module homomorphism $\alpha_0 : P \to P'$ and $\alpha_1 : Q \to Q'$ which have to be compatible with the maps ι, ϵ and $F, \dot{F}$. From the compatability with the first two maps we see that there are homomorphisms

$$\begin{array}{ll} X \in \operatorname{Hom}_{\mathcal{W}_n(R)}(T, T'), & U \in \operatorname{Hom}_{\mathcal{W}_n(R)}(L, I_{n+1} \otimes_{\mathcal{W}_n(R)} T'), \\ Z \in \operatorname{Hom}_{\mathcal{W}_n(R)}(T, L'), & Y \in \operatorname{Hom}_{\mathcal{W}_n(R)}(L, L'), \end{array} \tag{12}$$

such that the homomorphisms α_0 and α_1 are given by the formulas:

$$\alpha_1 \begin{pmatrix} \xi \otimes t \\ \ell \end{pmatrix} = \begin{pmatrix} \xi \otimes Xt + U\ell \\ \kappa(\xi) Zt + Y\ell \end{pmatrix} \in I_{n+1} \otimes_{\mathcal{W}_n(R)} T' \oplus L', \tag{13}$$

$$\alpha_0 \begin{pmatrix} t \\ \ell \end{pmatrix} = \begin{pmatrix} X & \hat{U} \\ Z & Y \end{pmatrix} \begin{pmatrix} t \\ \ell \end{pmatrix}, \tag{14}$$

for $t \in T$, $\ell \in L$, $\xi \in I_{n+1}$. Here $\hat{U}$ is the composition of U with the multiplication $I_{n+1} \otimes_{\mathcal{W}_n(R)} T' \to T'$. We consider the map

$$\dot{\sigma} \oplus \sigma : (I_{n+1} \otimes_{\mathcal{W}_n(R)} T') \oplus L' \to W_n(R) \otimes_{F, \mathcal{W}_n(R)} T' \oplus W_n(R) \otimes_{F, \mathcal{W}_n(R)} L'.$$

If we apply this to the vector (13) we obtain:

$$\begin{pmatrix} \sigma(X) & \dot{\sigma}(U) \\ p\sigma(Z) & \sigma(Y) \end{pmatrix} \begin{pmatrix} \dot{\sigma}(\xi)\sigma(t) \\ \sigma(\ell) \end{pmatrix} \in W_n(R) \otimes_{F, \mathcal{W}_n(R)} T' \oplus W_n(R) \otimes_{F, \mathcal{W}_n(R)} L'.$$

Here we use the notation

$$\sigma(X) = \mathrm{id}_{W_n(R)} \otimes_{\mathcal{W}_n(R)} X : W_n(R) \otimes_{F, \mathcal{W}_n(R)} T \to W_n(R) \otimes_{F, \mathcal{W}_n(R)} T',$$

and similarly $\sigma(Z)$ and $\sigma(Y)$. The composition

$$L \xrightarrow{U} I_{n+1} \otimes_{\mathcal{W}_n(R)} T' \xrightarrow{\dot{\sigma}} W_n(R) \otimes_{F, \mathcal{W}_n(R)} T'$$

is linear with respect to $F : \mathcal{W}_n(R) \to W_n(R)$. Its linearisation is

$$\dot{\sigma}(U) : W_n(R) \otimes_{F,\mathcal{W}_n(R)} L \to W_n(R) \otimes_{F,\mathcal{W}_n(R)} T'. \tag{15}$$

The pair α_0, α_1 is a morphism of truncated displays iff the following diagram is commutative:

$$\begin{array}{ccc} Q & \xrightarrow{\dot{F}} & W_n(R) \otimes_{\mathcal{W}_n(R)} P \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \alpha_0} \\ Q' & \xrightarrow{\dot{F}'} & W_n(R) \otimes_{\mathcal{W}_n(R)} P' \end{array} \tag{16}$$

We have just computed

$$\dot{F}' \circ \alpha_1 \begin{pmatrix} \xi \otimes t \\ \ell \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & \dot{\sigma}(U) \\ p\sigma(Z) & \sigma(Y) \end{pmatrix} \begin{pmatrix} \dot{\sigma}(\xi)\sigma(t) \\ \sigma(\ell) \end{pmatrix}$$

If we tensor (14) with $W_n(R) \otimes_{\mathcal{W}_n(R)}$ we obtain the matrix

$$\begin{pmatrix} \bar{X} & \bar{U} \\ \bar{Z} & \bar{Y} \end{pmatrix}.$$

Then the commutativity of (16) is equivalent with the equation

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & \dot{\sigma}(U) \\ p\sigma(Z) & \sigma(Y) \end{pmatrix} = \begin{pmatrix} \bar{X} & \bar{U} \\ \bar{Z} & \bar{Y} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{17}$$

Let us summarise the preceding considerations.

Definition 1.7. *We define the category $s\,\mathcal{D}_n(R)$ of standard truncated displays of level n over R as follows. Its objects are data $(T, L; A, B, C, D)$ as above, and a morphism*

$$(T, L; A, B, C, D) \to (T', L'; A', B', C', D').$$

is a matrix of homomorphisms (12) *which satisfies the equation* (17)*.*

Proposition 1.8. *There is a functor*

$$\mathcal{S} : s\,\mathcal{D}_n(R) \to \mathcal{D}_n(R),$$

and this functor is an equivalence of categories. □

Base change and truncation functors

Let $\phi : R \to S$ is a homomorphism of rings in which p is nilpotent. Let $\mathcal{S}(T, L; A, B, C, D)$ be a standard truncated display of level n. We set $\tilde{T} = \mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} T$ and $\tilde{L} = \mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} L$. By tensoring the maps (9) by $W_n(S) \otimes_{W_n(R)}$ we obtain an object of $s\,\mathcal{D}_n(S)$. It is easily checked that this gives a functor

$$\beta_s : s\,\mathcal{D}_n(R) \to s\,\mathcal{D}_n(S),$$

which is the base change functor for standard truncated displays. Therefore by Proposition 1.8 we also get a base change functor

$$\beta : \mathcal{D}_n(R) \to \mathcal{D}_n(S). \tag{18}$$

To make this canonical one can proceed in the standard way. Let $\mathcal{P} \in \mathcal{D}_n(R)$. Then we consider the category $\mathcal{C}$ whose objects are isomorphisms $\mathcal{S} \to \mathcal{P}$, where $\mathcal{S} \in s\,\mathcal{D}_n(R)$, and we define $\beta(\mathcal{P})$ as the projective limit over $\mathcal{C}$:

$$\beta(\mathcal{P}) = \varprojlim \beta_s(\mathcal{S}).$$

If $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ we write $\beta(\mathcal{P}) = (P_S, Q_S, \iota_S, \epsilon_S, F_S, \dot{F}_S)$. We note that there is a canonical isomorphism $P_S = \mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} P$.

In the same way we can define truncation functors

$$\tau_n : \mathcal{D}_{n+1}(R) \to \mathcal{D}_n(R). \tag{19}$$

They are compatible with the base change functors.

One could also define the truncation and base change functors by a universal property without referring to standard representations (but the proof that the functors exist uses Proposition 1.8). Namely, for $\phi : R \to S$ as above and for truncated displays $\mathcal{P}$ over R and $\mathcal{P}'$ over S of level n one defines homomorphisms $\mathcal{P} \to \mathcal{P}'$ over ϕ in the obvious way. Then we have a universal homomorphism $\mathcal{P} \to \beta(\mathcal{P})$ over ϕ. A similar remark applies to the truncation functors.

Matrix description

For simplicity we will often assume that the modules T resp. L are free of rank d and c; see Lemma 1.4. We fix isomorphisms $T \cong \mathcal{W}(R)^d$ and $L \cong \mathcal{W}(R)^c$. Then a standard truncated display with normal decomposition given by T and L is determined by the invertible matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{d+c}(W_n(R))$$

which defines the map (8). For ${}^V\underline{\eta} \in I_{n+1}^d$, $\underline{\zeta} \in \mathcal{W}_n(R)^d$, and $\underline{\xi} \in \mathcal{W}_n(R)^c$, we consider the vectors:

$$\begin{pmatrix} {}^V\underline{\eta} \\ \underline{\xi} \end{pmatrix} \in I_{n+1} \otimes_{\mathcal{W}_n} T \oplus L = Q, \qquad \begin{pmatrix} \underline{\zeta} \\ \underline{\xi} \end{pmatrix} \in T \oplus L = P.$$

Then F and $\dot{F}$ can be written in matrix form

$$F(\begin{pmatrix} \underline{\zeta} \\ \underline{\xi} \end{pmatrix}) = \begin{pmatrix} A & pB \\ C & pD \end{pmatrix} \begin{pmatrix} {}^F\underline{\zeta} \\ {}^F\underline{\xi} \end{pmatrix}.$$

$$\dot{F}(\begin{pmatrix} {}^V\underline{\eta} \\ \underline{\xi} \end{pmatrix}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \underline{\eta} \\ {}^F\underline{\xi} \end{pmatrix}.$$

As for displays we represent morphisms by matrices. Let $\mathcal{P}'$ be a second truncated display with a given normal decomposition

$$P' = T' \oplus L', \quad Q' = I_{n+1} \otimes_{\mathcal{W}_n(R)} T' \oplus L'.$$

We fix isomorphisms $T' \cong \mathcal{W}_n(R)^{d'}$ and $L' \cong \mathcal{W}_n(R)^{c'}$. We assume that $\mathcal{P}'$ is defined by the matrix

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{GL}_{d'+c'}(W_n(R)).$$

A morphism $\alpha : \mathcal{P} \to \mathcal{P}'$ is given by a matrix

$$\begin{pmatrix} X & {}^V\mathfrak{J} \\ Z & Y \end{pmatrix}, \tag{20}$$

where $X \in \mathrm{Hom}_{\mathcal{W}_n(R)}(T, T')$, $Y \in \mathrm{Hom}_{\mathcal{W}_n(R)}(L, L')$, $Z \in \mathrm{Hom}_{\mathcal{W}_n(R)}(T, L')$ ${}^V\mathfrak{J} \in \mathrm{Hom}_{\mathcal{W}_n(R)}(L, I_{n+1} \otimes T')$. The matrices X, Y, Z have coefficients in $\mathcal{W}_n(R)$ and $\mathfrak{J}$ has coefficients in $W_n(R)$.

The maps $Q \to Q'$ and $P \to P'$ induced by α are given by the matrices

$$\begin{pmatrix} X & {}^V\mathfrak{J} \\ Z & Y \end{pmatrix} : I_{n+1} \otimes_{\mathcal{W}_n(R)} T \oplus L \to I_{n+1} \otimes_{\mathcal{W}_n(R)} T' \oplus L'$$

and

$$\begin{pmatrix} X & \kappa({}^V\mathfrak{J}) \\ Z & Y \end{pmatrix} : T \oplus L \to T' \oplus L',$$

where the first matrix needs a little interpretation.

The matrix (20) defines a morphism of truncated displays iff the following equation holds:

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} {}^FX & \mathfrak{J} \\ p\,{}^FZ & {}^FY \end{pmatrix} = \begin{pmatrix} \mathrm{Res}(X) & {}^V\bar{\mathfrak{J}} \\ \mathrm{Res}(Z) & \mathrm{Res}(Y) \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{21}$$

Here $\bar{\mathfrak{J}}$ is the restriction of $\mathfrak{J}$ to a matrix with coefficients in $W_{n-1}(R)$.

This equation shows in particular that $\mathfrak{J}$ is already uniquely determined by X, Y, Z and $\kappa({}^V\mathfrak{J})$. Therefore a morphism of truncated displays is already uniquely determined by the induced $\mathcal{W}_n$-module homomorphism $P \to P'$, i.e. we have proved the following.

Lemma 1.9. *For two truncated displays $\mathcal{P}$ and $\mathcal{P}'$ of level n over R the forgetful homomorphism*

$$\operatorname{Hom}_{\mathcal{D}_n}(\mathcal{P}, \mathcal{P}') \to \operatorname{Hom}_{\mathcal{W}_n(R)}(P, P'). \tag{22}$$

is injective. □

Definition 1.10. *Let $\mathcal{M}_n(R)$ be the category whose objects are invertible block matrices:*

$$\mathfrak{F} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_h(W_n(R)),$$

where A and D are square matrices of arbitrary size and whose morphisms $\mathfrak{F} \to \mathfrak{F}'$ are block matrices $\mathfrak{X}$ of the form (20) *which satisfy the relation* (21)*. If $\mathfrak{X}' : \mathfrak{F}' \to \mathfrak{F}''$ is a second morphism then the composite $\mathfrak{X}' \circ \mathfrak{X}$ is the matrix*

$$\begin{pmatrix} X' & {}^V\mathfrak{J}' \\ Z' & Y' \end{pmatrix} \circ \begin{pmatrix} X & {}^V\mathfrak{J} \\ Z & Y \end{pmatrix} := \begin{pmatrix} X'X + \kappa({}^V\mathfrak{J}')Z & {}^V({}^F X'\mathfrak{J} + \mathfrak{J}'{}^F Y) \\ Z'X + Y'Z & Z'\kappa({}^V\mathfrak{J}) + Y'Y \end{pmatrix}.$$

We have a fully faithful functor $\mathcal{M}_n(R) \to s\,\mathcal{D}_n(R) \xrightarrow{\sim} \mathcal{D}_n(R)$. The essential image consists of the truncated displays such that the modules in the exact sequence (vi) of Definition 1.1 are free R-modules.

Nilpotent truncated displays

Let $\mathcal{P}$ be a truncated display of level n over a ring R. There is a unique homomorphism of $W_n(R)$-modules

$$V^\sharp : W_n(R) \otimes_{\mathcal{W}_n(R)} P \to W_n(R) \otimes_{F, \mathcal{W}_n(R)} P, \tag{23}$$

such that for each $y \in Q$

$$V^\sharp(\dot{F}(y)) = 1 \otimes \iota(y).$$

The existence follows as in the case of displays by using a normal decomposition. One deduces from the last equation that

$$V^\sharp(\xi F(x)) = p\xi \otimes x.$$

We assume now that $pR = 0$. Then we have $W_n(R) = \mathcal{W}_n(R)$ and $I_n = \kappa(I_{n+1})$. The homomorphism (23) takes the form

$$V^\sharp : P \to W_n(R) \otimes_{F,W_n(R)} P.$$

Iterating the last morphism we obtain for each natural number N:

$$(V^N)^\sharp : P \to W_n(R) \otimes_{F^N,W_n(R)} P. \tag{24}$$

In the case $pR = 0$ we say that the truncated display $\mathcal{P}$ is nilpotent if for large N the image of the map (24) is zero modulo I_n. Equivalently we can say that the map induced by (24)

$$(V^N)^\sharp : P/I_R P \to R \otimes_{\mathrm{Frob}^N,R} P/I_R P$$

is zero. In the case of general R we call $\mathcal{P}$ nilpotent if its base change to R/pR is nilpotent. Assume that $pR = 0$. By definition the image of $V^\sharp$ coincides with the image of the homomophism $W_n(R) \otimes_{F,W_n(R)} Q \to W_n(R) \otimes_{F,W_n(R)} P$ induced by ι. This implies that the map $V^\sharp : P \to R \otimes_{\mathrm{Frob},R} P/\iota(Q)$ is zero. Therefore $V^\sharp$ induces a homomorphism $P/I_n P \to R \otimes_{\mathrm{Frob},R} \iota(Q)/I_n P$. By restriction of we obtain the homomorphism

$$\bar{V}^\sharp : \iota(Q)/I_n P \to R \otimes_{\mathrm{Frob},R} \iota(Q)/I_R P. \tag{25}$$

Definition 1.11. *Let $\mathcal{P}$ be a nilpotent truncated display of level n over a ring ring R. If $pR = 0$ the nilpotence order of $\mathcal{P}$ is the smallest natural number $e \geq 1$ such that*

$$(\bar{V}^e)^\sharp = 0,$$

for the iterate of (25). *If R is arbitrary the order of nilpotence of $\mathcal{P}$ is the order of nilpotence of the base change $\mathcal{P}_{R/pR}$.*

The same makes sense for displays.

Lemma 1.12. *Let $\mathcal{P}$ be a truncated display of level n over R, which is given by the block matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We denote the inverse matrix by

$$\begin{pmatrix} \breve{A} & \breve{B} \\ \breve{C} & \breve{D} \end{pmatrix}.$$

Let $\hat{D}_0$ be the image of $\mathbf{w}_0(\breve{D})$ in R/pR.

Then $\mathcal{P}$ has order of nilpotence $\leq e$, if

$$\hat{D}_0^{(p^{e-1})} \cdot \ldots \cdot \hat{D}_0^{(p)} \hat{D}_0 = 0,$$

where the upper index (p^i) means that we take the p^i-power of all entries.

Proof. This is similar to the case of nilpotent displays in [Z1]. □

We note that the order of nilpotence doesn't change under truncation (19). Also it doesn't change by base change with respect to an injective ring homomorphism.

Let G be a formal p-divisible group with display $\mathcal{P}$. Then we say that G has order of nilpotence e if $\mathcal{P}$ has. It is equivalent to say that e is the smallest number such that the Verschiebung of the dual p-divisible group $\hat{G}$

$$\mathrm{Ver}^{e}_{\hat{G}} : \hat{G}^{(p^e)} \to \hat{G} \tag{26}$$

induces zero on the tangent spaces. This makes also sense for a truncated p-divisible group G. Clearly truncation doesn't influence the order of nilpotence.

If R is not necessarily annihilated by p the order of nilpotence of G is by definition that of $G_{R/pR}$.

Let R be a perfect field. Then P is the covariant Dieudonné module and $Q/I_R P$ is equal to VP/pP. Therefore $\mathcal{P}$ has order of nilpotence $\leq e$ iff $V^{e+1}P \subset pP$. If G is the (truncated) p-divisible group to associated to $\mathcal{P}$ then the order of nilpotence is $\leq e$ iff the iterated Frobenius

$$\mathrm{Frob}^{e+1}_{G[p]} : G[p] \to G[p]^{(p^{e+1})} \tag{27}$$

is zero.

Lemma 1.13. *Let R be a ring such that $pR = 0$. Let G be a formal p-divisible group over R and let $\mathcal{P}$ the associated display.*

1. *Then the order of nilpotence of $\mathcal{P}$ is $\leq e$ iff (26) is zero on the tangent spaces.*
2. *If the morphism (27) is zero then the order of nilpotence is $\leq e + 1$.*
3. *If R is reduced then the order of nilpotence is $\leq e$ iff the morphism (27) is zero.*

Proof. We already proved the first assertion. The second follows from the first by taking the Cartier dual of (27). The last assertion may be reduced to the case of a perfect field. In this case it follows from the argument above. □

2 Relative truncated displays

We consider now a surjective ring homomorphism $S \to R$, such that p is nilpotent in S and the kernel $\mathfrak{a}$ is endowed with divided powers. We will say that S/R is a *pd*-thickening. We set

$$\mathcal{J}_{n+1} = W_{n+1}(\mathfrak{a}) + I_{n+1}(S) \subset W_{n+1}(S).$$

Let $\kappa : \mathcal{J}_{n+1} \to \mathcal{W}_n(S)$ be the homomorphism induced by (4).

The divided powers define an isomorphism of $W_{n+1}(S)$-modules

$$W_{n+1}(\mathfrak{a}) = \prod_{i=0}^{n} \mathfrak{a}_{[\mathbf{w}_i]}$$

which is given by the divided Witt polynomials $W_{n+1}(\mathfrak{a}) \to \mathfrak{a}$. The first factor on the right hand side will be also written as $\tilde{\mathfrak{a}} \subset W_{n+1}(\mathfrak{a})$. Since $\tilde{\mathfrak{a}}$ is an S-module it is a fortiori a $\mathcal{W}_n(S)$-module and therefore $\mathcal{J}_{n+1} = \tilde{\mathfrak{a}} \oplus I_{n+1}(S)$ is a $\mathcal{W}_n(S)$-module too.

The map $V^{-1} : I_{n+1}(S) \to W_n(S)$ extends uniquely to:

$$\dot{\sigma} : \mathcal{J}_{n+1} \to W_n(S), \quad \text{where } \dot{\sigma}(\tilde{\mathfrak{a}}) = 0. \tag{28}$$

We will write σ for the Frobenius map $F : \mathcal{W}_n(S) \to W_n(S)$. We can define relative truncated displays of level n with respect to $S \to R$ as before:

Definition 2.1. *A relative truncated display $\mathcal{P}$ of level n for $S \to R$ consists of $(P, Q, \iota, \epsilon, F, \dot{F})$. Here P and Q are $\mathcal{W}_n(S)$-modules,*

$$\iota : Q \to P, \quad \epsilon : \mathcal{J}_{n+1} \otimes_{\mathcal{W}_n(S)} P \to Q$$

are $\mathcal{W}_n(S)$-linear maps, and

$$\begin{array}{rccl} F: & P & \to & W_n(S) \otimes_{\mathcal{W}_n(S)} P \\ \dot{F}: & Q & \to & W_n(S) \otimes_{\mathcal{W}_n(S)} P \end{array}$$

are σ-linear maps. As in Definition 1.1 we define a the map

$$\begin{array}{rccc} \tilde{F}: & \mathcal{J}_{n+1} \otimes_{\mathcal{W}_n(S)} P & \to & W_n(S) \otimes_{\mathcal{W}_n(S)} P \\ & \tau \otimes x & \mapsto & \dot{\sigma}(\tau) \otimes Fx. \end{array}$$

We require that the following properties hold:

(i) The $\mathcal{W}_n(S)$-module P is projective and finitely generated.

(ii) The compositions $\iota \circ \epsilon$ and $\epsilon \circ (\mathrm{id} \otimes \iota)$ are the multiplication maps.

(iii) The cokernels of ι and ϵ are finitely generated projective R-modules.

(iv) The diagram similar to (5) is commutative, i.e. we have $\dot{F} \circ \epsilon = \tilde{F}$.

(v) The image $\dot{F}(Q)$ generates $W_n(S) \otimes_{\mathcal{W}_n(S)} P$ as a $W_n(S)$-module.

(vi) The following sequence is exact:

$$0 \to Q/\operatorname{Im}\epsilon \xrightarrow{\iota} P/\kappa(\mathcal{J}_{n+1})P \to P/\iota(Q) \to 0. \tag{29}$$

Relative truncated displays of level n for $S \to R$ form an additive category in an obvious way, which we denote by $\mathcal{D}_n(S/R)$.

The surjective R-linear map $P/\kappa(\mathcal{J}_{n+1})P \to P/\iota(Q)$ is called the Hodge filtration of the relative truncated display $\mathcal{P}$.

As before one can show that there is a normal decomposition

$$P = T \oplus L, \qquad Q = \mathcal{J}_{n+1} \otimes_{\mathcal{W}_n(S)} T \oplus L,$$

where T and L are $\mathcal{W}_n(S)$-modules. The map

$$F^\sharp \oplus \dot{F}^\sharp : (W_n(S) \otimes_{F,\mathcal{W}_n(S)} T) \oplus (W_n(S) \otimes_{F,\mathcal{W}_n(S)} L) \to W_n(S) \otimes_{\mathcal{W}_n(S)} P$$

is an isomorphism of $\mathcal{W}_n(S)$-modules which we write in matrix form as before; see (9):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

This leads to the notion of a standard relative truncated display of level n with respect to S/R, $\mathcal{S}(T, L; A, B, C, D)$. The data are the same as for standard truncated displays over the ring S, but in the case of relative truncated displays the notion of a morphism changes. A morphism of standard relative truncated displays of level n

$$\mathcal{S}(T, L; A, B, C, D) \to \mathcal{S}(T', L'; A', B', C', D')$$

is given by four homomorphisms of $\mathcal{W}_n(S)$-modules:

$$\begin{array}{ll} X \in \operatorname{Hom}_{\mathcal{W}_n(S)}(T, T'), & U \in \operatorname{Hom}_{\mathcal{W}_n(R)}(L, \mathcal{J}_{n+1} \otimes_{\mathcal{W}_n(S)} T'), \\ Z \in \operatorname{Hom}_{\mathcal{W}_n(S)}(T, L'), & Y \in \operatorname{Hom}_{\mathcal{W}_n(S)}(L, L'), \end{array}$$

which satisfy the following relation:

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & \dot{\sigma}(U) \\ p\sigma(Z) & \sigma(Y) \end{pmatrix} = \begin{pmatrix} \bar{X} & \bar{U} \\ \bar{Z} & \bar{Y} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here in the relative case $\dot{\sigma}$ is induced by the morphism (28) as in (15), and $\bar{U}$ is induced by $\kappa : \mathcal{J}_{n+1} \to \mathcal{W}_n(S)$ as follows:

$$W_n(S) \otimes_{\mathcal{W}_n(S)} L \xrightarrow{\mathrm{id} \otimes U} W_n(S) \otimes_{\mathcal{W}_n(S)} \mathcal{J}_{n+1} \otimes_{\mathcal{W}_n(S)} T' \to W_n(S) \otimes_{\mathcal{W}_n(S)} T'.$$

As in the case of truncated displays (Proposition 1.8) we see that the category of standard relative truncated displays is equivalent to the category of relative truncated displays. Using this, again we define truncation functors:

$$\mathcal{D}_{n+1}(S/R) \to \mathcal{D}_n(S/R).$$

We also have obvious reduction functors which are compatible with truncation

$$\mathcal{D}_n(S) \to \mathcal{D}_n(S/R) \to \mathcal{D}_n(R).$$

For a morphism of *pd*-thickenings

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

we have a base change functor $\mathcal{D}_n(S/R) \to \mathcal{D}_n(S'/R')$.

Matrix description

Assume that T and L are free $\mathcal{W}_n(S)$-modules. If we fix isomorphisms $T \cong \mathcal{W}_n(S)^d$ and $L \cong \mathcal{W}_n(S)^c$, the relative truncated display is given by a matrix in $GL_{d+c}(W_n(S))$ as before:

$$\dot{F}(\begin{pmatrix} \underline{\tau} \\ \underline{\ell} \end{pmatrix}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \dot{\sigma}(\underline{\tau}) \\ \sigma(\underline{\ell}) \end{pmatrix}, \quad \underline{\tau} \in \mathcal{J}_{n+1}^d,\ \underline{\ell} \in \mathcal{W}_n(S)^c. \tag{30}$$

Let $\mathcal{P}' = (P', Q', \iota', \epsilon', F', \dot{F}')$ be a second relative truncated display. Consider a normal decomposition $P' = T' \oplus L'$ with $T' \cong \mathcal{W}_n(S)^{d'}$ and $L' \cong \mathcal{W}_n(S)^{c'}$. Let

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in GL_{d'+c'}(W_n(S))$$

be the matrix of $\mathcal{P}'$. A homomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$ is a matrix

$$\begin{pmatrix} X & J \\ Z & Y \end{pmatrix}, \tag{31}$$

where $X \in \mathrm{Hom}_{\mathcal{W}_n(S)}(T, T')$, $Y \in \mathrm{Hom}_{\mathcal{W}_n(S)}(L, L')$, $Z \in \mathrm{Hom}_{\mathcal{W}_n(S)}(T, L')$, $J \in \mathrm{Hom}_{\mathcal{W}_n(S)}(L, \mathcal{J}_{n+1} \otimes T')$, i.e. the matrices X, Y, Z have coefficients in $\mathcal{W}_n(S)$ and J has coefficients in $\mathcal{J}_{n+1}$, which satisfies the relation

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & \dot{\sigma}(J) \\ p\sigma(Z) & \sigma(Y) \end{pmatrix} = \begin{pmatrix} \bar{X} & \bar{J} \\ \bar{Z} & \bar{Y} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{32}$$

Here the bar denotes the image under the restriction map $\mathcal{W}_n(S) \to W_n(S)$ resp. $\mathcal{J}_{n+1} \to \mathcal{W}_n(S) \to W_n(S)$.

A morphism α given by a matrix (31) induces a homomorphism $P \to P'$ which is given by the matrix

$$\begin{pmatrix} X & \kappa(J) \\ Z & Y \end{pmatrix}.$$

This matrix already determines the matrix (31) uniquely. Indeed, from $\kappa(J)$ one obtains $\bar{J}$. Then the equation (32) determines $\dot{\sigma}(J)$. Since the intersection of the kernels of the two maps $\mathcal{J}_{n+1} \to W_n(S)$ given by $\dot{\sigma}$ and $J \mapsto \bar{J}$ is zero, this determines J.

As for truncated displays we conclude:

Lemma 2.2. *For relative truncated displays $\mathcal{P}$ and $\mathcal{P}'$ of level n for $S \to R$ the forgetful homomorphism*

$$\mathrm{Hom}_{\mathcal{D}_n(S/R)}(\mathcal{P}, \mathcal{P}') \to \mathrm{Hom}_{\mathcal{W}_n(S)}(P, P') \tag{33}$$

is injective.

Let us denote by $\mathcal{D}_n(S/R)$ the category of relative truncated displays with respect to $S \to R$ and by $\mathcal{M}_n(S/R)$ the corresponding category of matrices. The objects in $\mathcal{M}_n(S/R)$ are just invertible block matrices in $GL_h(W_n(S))$. The compositions of morphisms (32) are defined by

$$\begin{pmatrix} X' & J' \\ Z' & Y' \end{pmatrix} \circ \begin{pmatrix} X & J \\ Z & Y \end{pmatrix} := \begin{pmatrix} X'X + \kappa(J')Z & X'J + J'Y \\ Z'X + Y'Z & Z'\kappa(J) + Y'Y \end{pmatrix}.$$

Here the expression $X'J + J'Y$ makes sense because $\mathcal{J}_{n+1}$ is a $\mathcal{W}_n(S)$-module.

Lifting of truncated displays

The following is the main result of this section.

Proposition 2.3. *Let $S \to R$ be a pd-thickening as above with kernel $\mathfrak{a}$. Let m be a natural number such that $p^m \mathfrak{a} = 0$. Let $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ be truncated displays of level n over R. Let $\mathcal{P}_1$ resp. $\mathcal{P}_2$ be two relative truncated displays of level n for $S \to R$ which lift $\bar{\mathcal{P}}_1$ resp. $\bar{\mathcal{P}}_2$. Then each morphism $\bar{\alpha} : \bar{\mathcal{P}}_1 \to \bar{\mathcal{P}}_2$ lifts to a morphism*

$$\alpha : \mathcal{P} \to \mathcal{P}'. \tag{34}$$

Assume moreover that $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ have order of nilpotence $\leq e$. If $n > m(e+1)+1$, then the truncation

$$\alpha[n - m(e+1) - 1] : \mathcal{P}_1[n - m(e+1) - 1] \to \mathcal{P}_2[n - m(e+1) - 1]$$

does not depend on the choice of α but only on $\bar{\alpha}$.

Proof. As in [Z1] we may replace $\bar{\alpha}$ by the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ \bar{\alpha} & 1 \end{smallmatrix}\right)$ of $\bar{\mathcal{P}}_1 \oplus \bar{\mathcal{P}}_2$. Note that if $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ are nilpotent of order $\leq e$, then the same holds for $\bar{\mathcal{P}}_1 \oplus \bar{\mathcal{P}}_2$. Thus is suffices to prove the following assertion.

Let $\bar{\mathcal{P}}$ be a display of level n over R and let $\mathcal{P}$ and $\mathcal{P}'$ be two relative displays of level n for $S \to R$ which lift $\bar{\mathcal{P}}$. Then there is an isomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$ which lifts the identity. If $\bar{\mathcal{P}}$ is nilpotent of order $\leq e$ then the truncation $\alpha[n - m(e+1) - 1]$ is uniquely determined.

We choose a normal decomposition $\bar{P} = \bar{T} \oplus \bar{L}$. For simplicity we will assume that these modules are free with a given basis. Let T and L the free $\mathcal{W}_n(S)$-modules with basis which lift $\bar{T}$ and $\bar{L}$. Then we have normal decompositions:

$$P \cong T \oplus L, \quad P' \cong T \oplus L,$$

which lift the chosen normal decompositions of $\bar{\mathcal{P}}$. We are looking for homomorphisms of the form:

$$\begin{pmatrix} E_d & 0 \\ 0 & E_c \end{pmatrix} + \begin{pmatrix} X & J \\ Z & Y \end{pmatrix} : \mathcal{P} \longrightarrow \mathcal{P}'. \tag{35}$$

The matrix J has coefficients in $W_{n+1}(\mathfrak{a}) \subset \mathcal{J}_{n+1}$ and the matrices X, Y, Z have coefficients in the kernel of $\mathcal{W}_n(S) \to \mathcal{W}_n(R)$.

Let us describe this kernel. An element $\xi \in W_{n+1}(S)$ represents an element of the kernel iff it takes the form $\xi = \eta + {}^{V^n}[s]$, where η lies in $W_{n+1}(\mathfrak{a})$ and where $s \in S$ satisfies $ps \in \mathfrak{a}$. In this case the elements $\bar{\xi} = \operatorname{Res} \xi$ and $\sigma(\xi)$ lie in $W_n(\mathfrak{a})$, and the pairs $(\bar{\xi}, \sigma(\xi)) \in \mathcal{W}_n(S)$ for $\xi = \eta + {}^{V^n}[s]$ as above are exactly the elements in the kernel. We write the logarithmic coordinates of these elements with respect to the divided powers on $\mathfrak{a}$:

$$\bar{\xi} = [a_0, \dots, a_{n-1}], \qquad \sigma(\xi) = [x_1, \dots, x_n].$$

The logarithmic coordinates of $\sigma({}^{V^n}[s]) \in W_n(\mathfrak{a})$ are $[0, \ldots, 0, ps]$. We see that $x_i = pa_i$ for $i \leq n-1$ and that $x_n \in pS \cap \mathfrak{a}$. Thus the elements of the kernel correspond bijectively to vectors

$$< a_0, \ldots a_{n-1}, x_n >, \qquad a_i \in \mathfrak{a}, \quad x_n \in pS \cap \mathfrak{a}$$

such that

$$\begin{array}{l} \sigma(< a_0, \ldots a_{n-1}, x_n >) = [pa_1, \ldots, pa_{n-1}, x_n] \\ \mathrm{Res}(< a_0, \ldots a_{n-1}, x_n >) = [a_0, \ldots a_{n-1}]. \end{array}$$

With these notations we may write the matrices X, Y, Z, J:

$$X =< X(0), \ldots, X(n) >$$

where the $X(i)$ are matrices with coefficients in $\mathfrak{a}$ and moreover $X(n)$ has coefficients in $pS \cap \mathfrak{a}$ and similarly for Y and Z. For the matrix $J \in W_{n+1}(\mathfrak{a})$ we use the logarithmic coordinates

$$J = [J(0), \ldots, J(n)], \qquad \dot{\sigma}(J) = [J(1), \ldots, J(n)].$$

The $J(i)$ are matrices with coefficients in $\mathfrak{a}$.

We assume that $\mathcal{P}$ and $\mathcal{P}'$ are given by matrices as above (30). We set

$$\begin{pmatrix} \eta_A & \eta_B \\ \eta_C & \eta_D \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The condition that (35) is a homomorphism of relative displays becomes:

$$\begin{pmatrix} \eta_A & \eta_B \\ \eta_C & \eta_D \end{pmatrix} + \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & \dot{\sigma}(J) \\ p\sigma(Z) & \sigma(Y) \end{pmatrix} = \begin{pmatrix} \bar{X} & \bar{J} \\ \bar{Z} & \bar{Y} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

This is an equation in the $W_n(S)$-module $W_n(\mathfrak{a})$. We rewrite it in logarithmic coordinates and obtain for $0 \leq i \leq n-2$ the equations:

$$\begin{pmatrix} \eta_A(i) & \eta_B(i) \\ \eta_C(i) & \eta_D(i) \end{pmatrix} + \begin{pmatrix} w_i(A') & w_i(B') \\ w_i(C') & w_i(D') \end{pmatrix} \begin{pmatrix} pX(i+1) & J(i+1) \\ p^2Z(i+1) & pY(i+1) \end{pmatrix} = \begin{pmatrix} X(i) & J(i) \\ Z(i) & Y(i) \end{pmatrix} \begin{pmatrix} w_i(A) & w_i(B) \\ w_i(C) & w_i(D) \end{pmatrix} \tag{36}$$

and for $i = n-1$ we obtain the equation

$$\begin{pmatrix} \eta_A(n-1) & \eta_B(n-1) \\ \eta_C(n-1) & \eta_D(n-1) \end{pmatrix} + \begin{pmatrix} w_{n-1}(A') & w_{n-1}(B') \\ w_{n-1}(C') & w_{n-1}(D') \end{pmatrix} \begin{pmatrix} X(n) & J(n) \\ pZ(n) & Y(n) \end{pmatrix}$$
$$= \begin{pmatrix} X(n-1) & J(n-1) \\ Z(n-1) & Y(n-1) \end{pmatrix} \begin{pmatrix} w_{n-1}(A) & w_{n-1}(B) \\ w_{n-1}(C) & w_{n-1}(D) \end{pmatrix}. \tag{37}$$

We see that for arbitrary given $X(n), Y(n), Z(n), J(n)$ there are unique solutions of (37) and of (36) for $0 \leq i \leq n$, which means that for given $X(n), \ldots, J(n)$ there is a unique isomorphism (35) which lifts the identity.

Assume now that $\bar{\mathcal{P}}$ is nilpotent of order ≤ 1. Let us write:

$$H(i) = \begin{pmatrix} X(i) & J(i) \\ Z(i) & Y(i) \end{pmatrix}.$$

We claim that for each $k \geq 1$ and $0 \leq i \leq n-k(e+1)-1$ the reduction of $H(i)$ in $\mathfrak{a}/p^k\mathfrak{a}$ is independent of the choice of $H(n)$. This proves the uniqueness assertion of the proposition. Let

$$\begin{pmatrix} \breve{A} & \breve{B} \\ \breve{C} & \breve{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}.$$

If we multiply (36) by the image of this matrix under w_i, we obtain for $0 \leq i \leq n$ an equation

$$H(i) = R(i) + \begin{pmatrix} w_i(A') & w_i(B') \\ w_i(C') & w_i(D') \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} H(i+1) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_i(\breve{A}) & w_i(\breve{B}) \\ w_i(\breve{C}) & w_i(\breve{D}) \end{pmatrix}$$

where $R(i)$ is a given matrix with coefficients in $\mathfrak{a}$. Let

$$\Delta'_i = \begin{pmatrix} w_i(A') & pw_i(B') \\ w_i(C') & pw_i(D') \end{pmatrix}, \qquad \breve{\Delta}_i = \begin{pmatrix} pw_i(\breve{A}) & pw_i(\breve{B}) \\ w_i(\breve{C}) & w_i(\breve{D}) \end{pmatrix}. \tag{38}$$

Assume that $H(i)_{0 \leq i \leq n}$ and $H'(i)_{0 \leq i \leq n}$ are two solutions of (36) and (37). For their difference $h = H - H'$ we obtain the equations

$$h(i) = \Delta'_i \cdot h(i+1) \cdot \breve{\Delta}_i \tag{39}$$

for $0 \leq i \leq n-2$. If we can show that for $i \geq 0$ the product of $e+1$ factors

$$\breve{\Delta}_{i+e} \cdot \ldots \cdot \breve{\Delta}_{i+1} \breve{\Delta}_i$$

has coefficients in pS, it follows by induction that for $i \geq 0$ and $k \geq 1$ the product of $k(e+1)$ factors

$$\breve{\Delta}_{i+k(e+1)-1} \cdot \ldots \cdot \breve{\Delta}_{i+1} \breve{\Delta}_i$$

has coefficients in $p^k S$. Then (39) implies that $h(i) = 0$ for $i \leq n-k(e+1)-1$, which proves the claim.

Let $\breve{D}_0$ be the first component of the Witt vector matrix $\breve{D}$ modulo p. The assumption that $\bar{\mathcal{P}}$ is nilpotent of order $\leq e$ means that

$$\breve{D}_0^{(p^{e-1})} \cdot \ldots \cdot \breve{D}_0^{(p)} \breve{D}_0 \equiv 0$$

modulo $\mathfrak{a}(S/pS)$; see Lemma 1.12. But $a^p \in pS$ for $a \in \mathfrak{a}$ since $\mathfrak{a}$ has divided powers. Thus we get

$$\breve{D}_0^{(p^e)} \cdot \ldots \cdot \breve{D}_0^{(p^2)} \breve{D}_0^{(p)} = 0.$$

Thus for $i \geq 0$ the lower right block of $\Delta_{i+e} \cdot \ldots \cdot \Delta_{i+1}$ has coefficients in pS, and it follows that all coefficients of $\Delta_{i+e} \cdot \ldots \cdot \Delta_i$ lie in pS as required. This finishes the proof. □

Corollary 2.4. *(Rigidity) Let $S \to R$, $\mathfrak{a}$ and $m \in \mathbb{N}$ be as in Proposition 2.3.*

Let $\alpha_1, \alpha_2 : \mathcal{P} \to \mathcal{P}'$ be two morphisms of truncated displays of level n over S. We denote the truncated displays over R which are obtained by base change with $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}'$ and assume that they are nilpotent of order $\leq e$.

If the two morphisms $\bar{\alpha}_1, \bar{\alpha}_2 : \bar{\mathcal{P}} \to \bar{\mathcal{P}}'$ agree, we have

$$\alpha_1[n - m(e+1) - 1] = \alpha_2[n - m(e+1) - 1]$$

for the truncations.

Proof. Let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}'$ be the relative truncated displays obtained from $\mathcal{P}$ and $\mathcal{P}'$. It suffices to prove the equation of the Corollary on the relative truncated displays. This is a statement of the Proposition. □

The crystal of a nilpotent truncated display

We explain how to associate to a nilpotent truncated display a "truncated crystal". We fix natural numbers n, e, and m such that $n > m(e+1) + 1$. Let R be a ring in which p is nilpotent. Let $\mathrm{Cris}_m(R)$ be the category of all pd-thickenings $S \to R$ with kernel $\mathfrak{a}$ such that $p^m \mathfrak{a} = 0$.

Let $\mathcal{P}$ be a truncated display of level n over R which is nilpotent of order $\leq e$. We construct a locally free S-module $\mathbb{D}_{\mathcal{P}}(S)$ as follows. We choose a lifting of $\mathcal{P}$ to a relative truncated display $\tilde{\mathcal{P}}$ with respect to $S \to R$. Then we define

$$\mathbb{D}_{\mathcal{P}}(S) = S \otimes_{\mathcal{W}_n(S)} \tilde{P} \tag{40}$$

where the tensor product is taken with respect to the projection $\mathcal{W}_n(S) \to S$. In terms of the truncation of level 1 of $\tilde{\mathcal{P}}$ we may write the right hand side of (40) as $S \otimes_{\mathcal{W}_1(S)} \tilde{P}[1]$. Therefore Proposition 2.3 shows that $\mathbb{D}_{\mathcal{P}}(S)$ does not depend on the choice of $\tilde{\mathcal{P}}$ and that $\mathbb{D}_{\mathcal{P}}(S)$ is functorial in $\mathcal{P}$. If $S_1 \to S_2$ is a morphism in $\mathrm{Cris}_m(R)$ we obtain a canonical isomorphism

$$S_2 \otimes_{S_1} \mathbb{D}_{\mathcal{P}}(S_1) \cong \mathbb{D}_{\mathcal{P}}(S_2).$$

Let $\bar{\mathcal{P}}$ be a truncated display of level n over R, which is not necessarily nilpotent. Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be a relative truncated display which lifts $\bar{\mathcal{P}}$. We consider the Hodge filtration of $\bar{\mathcal{P}}$:

$$R \otimes_{\mathcal{W}_n(R)} \bar{P} = \bar{P}/\kappa(I_{n+1})\bar{P} \to \bar{P}/\iota(\bar{Q}).$$

This homomorphism can be identified with the Hodge filtration of $\mathcal{P}$:

$$R \otimes_{\mathcal{W}_n(S)} P = P/\kappa(\mathcal{J}_{n+1})P \to P/\iota(Q).$$

We define a lift of the Hodge filtration of $\mathcal{P}$ to S as a commutative diagram:

$$\begin{array}{ccc} S \otimes_{\mathcal{W}_n(S)} P & \longrightarrow & \breve{T} \\ \downarrow & & \downarrow \\ R \otimes_{\mathcal{W}_n(S)} P & \longrightarrow & \bar{P}/\iota(\bar{Q}), \end{array}$$

where $\breve{T}$ is a finitely generated projective S-module which lifts $\bar{P}/\iota(\bar{Q})$.

Assume that $\mathcal{P}'$ is a truncated display of level n over S which lifts $\mathcal{P}$. We have $\mathcal{P}' = (P, Q', \iota, \epsilon, F, \dot{F})$ with $Q' \subseteq Q$. The Hodge filtration of $\mathcal{P}'$ is a lift of the Hodge filtration of $\mathcal{P}$, and we get back Q' as the kernel of

$$Q \xrightarrow{\iota} S \otimes_{\mathcal{W}_n(S)} P \to P/Q'.$$

Conversely, let $\breve{T}$ be a lift of the Hodge filtration of $\mathcal{P}$ to S. Let Q' be the kernel of

$$Q \xrightarrow{\iota} S \otimes_{\mathcal{W}_n(S)} P \to \breve{T}.$$

We claim that we obtain a truncated display $\mathcal{P}' = (P, Q', \iota, \epsilon, F, \dot{F})$ of level n over S. It is easy to see that the restriction of $\epsilon : \mathcal{J}_{n+1} \otimes P \to Q$ to $I_{n+1,S} \otimes P$ lies in Q', using that $\iota \circ \epsilon$ is the multiplication map. Let $T \subseteq P$ and $L \subseteq Q$ be direct summands which give a normal decomposition of $\mathcal{P}$, which means that $P \cong L \oplus T$ and $Q \cong \mathcal{J}_{n+1} \otimes T \oplus L$. The composition $L \to P \to \breve{T}$ induces a homomorphism $L \to \mathfrak{a}\breve{T}$, which we lift to $\phi : L \to W_{n+1}(\mathfrak{a})T$, for example using the inclusion $\mathfrak{a} \subset W_{n+1}(\mathfrak{a})$ by the first logarithmic coordinate. If we replace the inclusion $i : L \to Q$ by $i - \phi$, then L and T define a normal decomposition for $\mathcal{P}'$. The remaining axioms for truncated displays for $\mathcal{P}'$ follows easily. Thus we have shown that lifts of $\mathcal{P}$ to truncated displays of level n over S correspond to lifts of the Hodge filtration.

Proposition 2.5. *Let $\bar{\mathcal{P}}$ be a truncated display of level n over R which is nilpotent of order $\leq e$. Let $S \to R$ be a divided power extension in* $\mathrm{Cris}_m(R)$ *with $n > m(e+1)+1$. Then the isomorphism classes of liftings of $\bar{\mathcal{P}}$ to* *a truncated display of level n over S correspond bijectively to liftings of the Hodge filtration of $\bar{\mathcal{P}}$ to $\mathbb{D}_{\bar{\mathcal{P}}}(S)$ as in the following diagram:*

$$\begin{array}{ccc}
\mathbb{D}_{\bar{\mathcal{P}}}(S) & \longrightarrow & \breve{T} \\
\downarrow & & \downarrow \\
\mathbb{D}_{\bar{\mathcal{P}}}(R) = R \otimes_{\mathcal{W}_n(R)} \bar{P} & \longrightarrow & \bar{P}/\iota(\bar{Q})
\end{array}$$

Proof. Let $\tilde{\mathcal{P}}$ be a lifting of $\bar{\mathcal{P}}$ to S, and let $\tilde{\mathcal{P}}^{rel}$ be the associated relative truncated display for $S \to R$. By definition we have a well-defined isomorphism $\mathbb{D}_{\bar{\mathcal{P}}}(S) \cong S \otimes_{\mathcal{W}_n(S)} \tilde{P}$. Thus the Hodge filtration of $\tilde{P}$ gives a lift of the Hodge filtration as in the proposition. We obtain a map from the set of isomorphism classes of liftings of $\bar{\mathcal{P}}$ to S to the set of liftings of the Hodge filtration. Since all liftings of $\bar{\mathcal{P}}$ to a relative truncated display for $S \to R$ are isomorphic, the preceding considerations show that this map is bijective. $\square$

Lifting of displays

Let $S \to R$ be a divided power extension of rings in which p is nilpotent with kernel $\mathfrak{a} \subset S$. We want to see what the proof of Proposition 2.3 gives for non-truncated displays.

We recall the definition of relative displays. Let $\mathcal{J}_{S/R}$ be the kernel of $W(S) \to R$. Let $\dot{\sigma} : I_S \to W(S)$ be the inverse of the Verschiebung and let $\dot{\sigma} : \mathcal{J}_{S/R} \to W(S)$ extend this map by $\dot{\sigma}(x) = 0$ if $x \in W(\mathfrak{a})$ is an element with logarithmic coordinates $[a, 0, 0, \ldots]$. A relative display for $S \to R$ consists of $(P, Q, F, \dot{F})$ where $Q \subseteq P$ are $W(S)$ modules and where $F : P \to P$ and $\dot{F} : Q \to P$ are σ-linear maps such that

(i) P is a finitely generated projective $W(S)$-module,

(ii) $\mathcal{J}_{S/R} P \subseteq Q$ and P/Q is a projective R-module,

(iii) $\dot{F}(ax) = \dot{\sigma}(a)F(x)$ for $a \in \mathcal{J}_{S/R}$ and $x \in P$,

(iv) $\dot{F}(Q)$ generates P.

Proposition 2.6. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two relative displays for $S \to R$ and let $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ be their reductions to displays over R. We consider the reduction map*

$$\rho : \mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2) \to \mathrm{Hom}(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2).$$

Then the following hold.

(a) If $\mathfrak{a}$ is an S-module of finite length, the map ρ is surjective,

(b) If $\mathcal{P}$ and $\mathcal{P}'$ are nilpotent, the map ρ is bijective.

Assertion (b) is proved in [Z1]. We recall it here for completeness.

Proof. By passing to $\mathcal{P}_1 \oplus \mathcal{P}_2$ it suffices to prove the following assertion.

Let $\bar{\mathcal{P}}$ be a display over R and let $\mathcal{P}$ and $\mathcal{P}'$ be two relative displays for $S \to R$ which lift $\bar{\mathcal{P}}$. If $\mathfrak{a}$ is an S-module of finite length then there is an isomorphism $\mathcal{P} \cong \mathcal{P}'$ which lifts the identity. If $\bar{\mathcal{P}}$ is nilpotent then there is a unique isomorphism $\mathcal{P} \cong \mathcal{P}'$.

We can assume that $\mathcal{P}$ and $\mathcal{P}'$ have the same underlying modules with normal decomposition $P = T \oplus L$ and $Q = \mathcal{J}_{S/R} T \oplus L$. For simplicity we assume that T and L are free, $T = W(S)^d$ and $L = W(S)^c$. Then $\mathcal{P}$ and $\mathcal{P}'$ are given by matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ with difference in $W(\mathfrak{a})$:

$$\begin{pmatrix} \eta_A & \eta_B \\ \eta_C & \eta_D \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The components of η_A with respect to the isomorphism $\log : W(\mathfrak{a}) \cong \mathfrak{a}^\infty$ are denoted by $\eta_A(n)$ for $n \geq 0$. These are matrices with coefficients in $\mathfrak{a}$. As in (35) we are looking for matrices

$$H(n) = \begin{pmatrix} X(n) & J(n) \\ Z(n) & Y(n) \end{pmatrix}$$

for $n \geq 0$ with coefficients in $\mathfrak{a}$ such that for $n \geq 0$ we have

$$H(n) = R(n) + \Delta'_n H(n+1) \breve{\Delta}_n$$

where $\breve{\Delta}_n$ and Δ'_n are defined as in (38). Here $R(n)$ for $n \geq 0$ are given matrices with coefficients in $\mathfrak{a}$. Let $\mathcal{H} = M_{c+d}(\mathfrak{a})$. For $n \geq 0$ let $U_n : \mathcal{H} \to \mathcal{H}$ be the map $U_n(H) = \Delta'_n H \Delta_n$. Then a solution $H(n)_{n\geq 0}$ exists if and only if the image of $R(n)_{n\geq 0}$ in

$$\varprojlim{}^1 (\mathcal{H} \xleftarrow{U_0} \mathcal{H} \xleftarrow{U_1} \mathcal{H} \xleftarrow{U_2} \cdots)$$

is zero, and the solution is unique if and only if the S-module

$$\varprojlim (\mathcal{H} \xleftarrow{U_0} \mathcal{H} \xleftarrow{U_1} \mathcal{H} \xleftarrow{U_2} \cdots)$$

is zero. If $\mathfrak{a}$ is an S-module of finite lenght, then $\mathcal{H}$ has finite length. Thus $\varprojlim^1$ is zero by the Mittag-Leffler condition. Assume that $\bar{\mathcal{P}}$ is nilpotent. Let $p^k \mathfrak{a} = 0$. We saw in the proof of Proposition 2.3 that for each $n \geq 0$, the product $\breve{\Delta}_{n+k(e+1)-1} \cdot \ldots \cdot \breve{\Delta}_n$ has coefficients in $p^k S$. Thus

$$U_{n+k(e+1)-1} \circ \ldots \circ U_n = 0,$$

which implies that $\varprojlim$ and $\varprojlim^1$ are zero. □

3 Truncated p-divisible groups and displays

The functor from groups to displays

Let $S \to R$ a pd-thickening with kernel $\mathfrak{a}$. We assume that the divided powers on $\mathfrak{a}$ are compatible with the canonical divided powers on pS. In [L2] one of us has defined a functor

$$\Phi_{S/R} : \mathcal{BT}(R) \to \mathcal{D}(S/R)$$

from the category p-divisible groups over R to the category of displays relative to $S \to R$, and in the case $pR = 0$ also a functor

$$\Phi_{n,R} : \mathcal{BT}_n(R) \to \mathcal{D}_n(R)$$

from the category of truncated p-divisible groups over R of level n to the category of truncated displays over R of level n. We will indicate a few modifications to adapt this to the notion of truncated displays which we use here and to the case of relative truncated displays.

We will use the derived category $D^\flat(\mathcal{E})$ of an exact category $\mathcal{E}$. Let $A^\flat(\mathcal{E})$ be the full subcategory of the bounded homotopy category $K^\flat(\mathcal{E})$ which consists of all complexes which split into short exact sequences in the sense of $\mathcal{E}$. By a result of [Th] $A^\flat(\mathcal{E})$ is épaisse as a subcategory if each idempotent $e : E \to E$ which factors as $E \xrightarrow{\alpha} F \xrightarrow{\beta} E$ such that $\alpha \circ \beta = \mathrm{id}_F$, is a split idempotent. In this case the localization with respect to $A^\flat(\mathcal{E})$ defines the derived category.

Let $\mathcal{G}$ be the category of finite locally free group schemes over R which admit an embedding in a p-divisible group. We consider the bounded derived category $D^\flat(\mathcal{BT}(R))$ of p-divisible groups. (Note that every idempotent in $\mathcal{BT}(R)$ is split.) Writing an object of $G \in \mathcal{G}$ as a kernel of an isogeny of p-divisible groups we obtain a fully faithful functor

$$\mathcal{G} \to D^\flat(\mathcal{BT}_R)$$

The image of this functor lies in the full subcategory $D^\flat_{\leq 1}(\mathcal{BT}_R)$ genrated by complexes $X^\cdot$ of p-divisible groups for which $X^i = 0$ for $i \geq 2$. One can check that a morphism $X^\cdot \to Y^\cdot$ in the category $D^\flat_{\leq 1}(\mathcal{BT}_R)$ may be represented by morphisms of complexes

$$X^\cdot \leftarrow Z^\cdot \to Y^\cdot,$$

where $Z^i = 0$ for $i \geq 2$ and where the left arrow is a quasiisomorphism.

Let us formalise the linear data of (relative) truncated displays. Let A be a ring and let $\kappa : \mathfrak{c} \to A$ be a homomorphism of A-modules such that for $x, y \in \mathfrak{c}$ we have

$$\kappa(x)y = \kappa(y)x.$$

Main example: For a surjective ring homomorphism $B \to A$ and an ideal $\mathfrak{c} \subset B$ which is an A-module we take $\kappa : \mathfrak{c} \to B \to A$.

Definition 3.1. *An $(A, \mathfrak{c})$-module consists of (M, N, ι, ϵ), where M and N are A-modules and ι and ϵ are A-module homomorphisms*

$$\mathfrak{c} \otimes_A M \xrightarrow{\epsilon} N \xrightarrow{\iota} M,$$

such that the composition of these two maps is the multiplication, i.e. $c \otimes m$ is mapped to $\kappa(c)m$, and such that the composition of the following maps is the multiplication:

$$\mathfrak{c} \otimes_A N \xrightarrow{\mathrm{id} \otimes \iota} \mathfrak{c} \otimes_A M \xrightarrow{\epsilon} N.$$

The category of $(A, \mathfrak{c})$-modules is in the obvious way abelian.

If $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ is a truncated display over R (resp. relative to $S \to R$) of level n, then (P, Q, ι, ϵ) is a $(\mathcal{W}_n(R), I_{n+1})$ (resp. $(\mathcal{W}_n(S), \mathcal{J}_{n+1})$)-module.

We call a sequence of truncated or relative truncated displays exact if the underlying sequence of $(\mathcal{W}_n(R), I_{n+1})$-modules or $(\mathcal{W}_m(S), \mathcal{J}_{m+1})$-modules is exact. We obtain exact categories in which every idempotent is split; note that all defining properties of (relative) truncated displays pass over to direct summands. Thus again the bounded derived categories exist:

$$D^\flat(\mathcal{D}_m(S/R)), \quad D^\flat(\mathcal{D}_m(R)).$$

Similarly we have the bounded derived categories $D^\flat(\mathcal{D}(R))$ and $D^\flat(\mathcal{D}(S/R))$ of displays and of relative displays. For each natural number m we obtain functors:

$$\mathcal{G} \to D^\flat_{\leq 1}(\mathcal{BT}(R)) \xrightarrow{\Phi_{S/R}} D^\flat_{\leq 1}(\mathcal{D}(S/R)) \to D^\flat_{\leq 1}(\mathcal{D}_m(S/R)). \tag{41}$$

We consider the category $\mathcal{T}_m = \mathcal{T}_m(S/R)$ of all data $(P, Q, \iota, \epsilon, F, \dot{F})$ as in the definition of a relative truncated display of level m, but we no longer require the conditions (i), (iii), and (vi) of Definition 2.1. Let

$$(P_1, Q_1, \iota_1, \epsilon_1, F_1, \dot{F}_1) \to (P_2, Q_2, \iota_2, \epsilon_2, F_2, \dot{F}_2)$$

be a morphism in $\mathcal{T}_m$. Then one easily defines a cokernel $(P, Q, \iota, \epsilon, F, \dot{F})$ such that (P, Q, ι, ϵ) is the cokernel in the category of $(\mathcal{W}_n(S), \mathcal{J}_{n+1})$-modules. For $G \in \mathcal{G}$ we apply (41) and take H^1, which is a cokernel. In this way, for each natural number m we obtain a functor:

$$\Phi_m : \mathcal{G} \to \mathcal{T}_m. \tag{42}$$

Proposition 3.2. *Let $S \to R$ be a pd-thickening and assume that p is nilpotent in S. Let $n \geq m$ be natural numbers such that $p^n \mathcal{W}_m(S) = 0$. Then (42) induces a functor*

$$\Phi_m : \mathcal{BT}_n(R) \to \mathcal{D}_m(S/R)$$

from the category of truncated p-divisible groups of level n over R to the category of truncated relative displays of level m for $S \to R$.

Note that in the case $pS = 0$ we can take $n = m$.

Proof. Let $\mathcal{G}_n$ the full subcategory of $\mathcal{G}$ which consists of truncated p-divisible groups. We claim that the functor Φ_m of (42) restricted to $\mathcal{G}_n$ takes values in the category of relative truncated displays of level m, i.e. that the conditions (i), (iii), and (vi) of Definition 2.1 are satisfied.

For $G \in \mathcal{G}$ let $\Phi_m(G) = (P, Q, \iota, \epsilon, F, \dot{F})$. Here P is a $\mathcal{W}_m(S)$-module of finite presentation. Coker(ι) and Coker(ϵ) are R-modules of finite presentation. These modules are compatible with base change under homomorphisms of *pd*-thickenings from $S \to R$ to $S' \to R'$.

By [Ill] we may lift a truncated p-divisible group over R to a truncated p-divisible group over S. Zariski locally on $\operatorname{Spec} S$, the lifted group can be embedded into a p-divisible group. Therefore, to prove the claim we may assume that $R = S$.

If X is a p-divisible group over R and if $X[n]$ is its truncation, we can use the resolution $0 \to X[n] \to X \xrightarrow{p^n} X$ to compute $\Phi_m(G)$. Since we assumed that $p^n \mathcal{W}_m(S) = 0$, we get that $\Phi_m(G)$ is the m-truncation of the display associated to X. So the claim is proved in this case.

By [Ill], if R is a complete local ring with perfect residue field, each truncated p-divisible group G of level n over R takes the form $X[n]$ for a p-divisible group X. If R is noetherian, for each prime ideal $\mathfrak{p}$ of A we find a faithfully flat ring homomorphism $\hat{A}_{\mathfrak{p}} \to A'$ where A' is a complete local ring with perfect (or algebraically closed) residue field. By base change it follows that for $G \in \mathcal{G}_n$, $\Phi_m(G)$ satisfies the conditions (i), (iii), and (vi) of Definition 2.1. By faithfully flat descent of truncated displays (see Appendix) this proves the claim when R is noetherian.

Since a truncated p-divisible group embeds Zariski locally in a p-divisible group, if R is noetherian we can extend the functor Φ_m from $\mathcal{G}_n$ to all truncated p-divisible groups by descent. Finally we can use base change to define the functor Φ_m over a base which is not noetherian. □

Exactness and Duality

Let us return for a moment to the study of $(A, \mathfrak{c})$-modules. Let $\mathfrak{c} \to A$ be as above. Let $\mathfrak{u}$ be the kernel of the map $\mathfrak{c} \to A$ and let R be the cokernel,

$$0 \to \mathfrak{u} \to \mathfrak{c} \to A \to R \to 0.$$

Here R is a quotient ring of A, and $\mathfrak{u}$ is an R-module since we have $\kappa(\mathfrak{c})\mathfrak{u} = \kappa(\mathfrak{u})\mathfrak{c} = 0$. We assume in the following that all finitely generated projective R-modules lift to finitely generated projective A-modules.

For given finitely generated projective A-modules T and L we define an $(A, \mathfrak{c})$-module $\mathcal{S}(T, L)$ as follows: We set

$$M = T \oplus L, \quad N = \mathfrak{c} \otimes_R T \oplus L,$$

with the obvious maps ι and ϵ; see Definition 1.2. An $(A, \mathfrak{c})$-module which is isomorphic to some $\mathcal{S}(T, L)$ will be called standard projective, and $M = T \oplus L$ is called a normal decomposition.

For an arbitrary $(A, \mathfrak{c})$-module (M, N, ι, ϵ) we have a canonical isomorphism

$$\operatorname{Hom}(\mathcal{S}(T, L), (M, N, \iota, \epsilon)) = \operatorname{Hom}_R(T, M) \oplus \operatorname{Hom}_R(L, N).$$

In particular $\mathcal{S}(T, L)$ is a projective object in the category of $(A, \mathfrak{c})$-modules (this does not need that T and L are finitely generated). Obviously we have projective resolutions in this category.

To each $(A, \mathfrak{c})$-module (M, N, ι, ϵ) we associate the following complex of A-modules:

$$0 \to \mathfrak{u} \otimes_A (M/N) \xrightarrow{\epsilon'} N \to M \to M/N \to 0 \tag{43}$$

where ε' is the restriction of ϵ.

Lemma 3.3. *An $(A, \mathfrak{c})$-module $\breve{M} = (M, N, \iota, \epsilon)$ is standard projective iff the following holds.*

(i) M is a finitely generated projective A-module,

(ii) M/N is a finitely generated projective R-module,

(iii) the sequence (43) is exact.

Proof. (Cf. [L2]) Clearly standard projective modules satisfy (i)-(iii). Assume that (i)-(iii) hold. Since $\operatorname{Im} \epsilon' \subset \operatorname{Im} \epsilon$, the exact sequence (43) implies that the following is exact:

$$0 \to N / \operatorname{Im} \epsilon \to M/\mathfrak{c}M \to M/N \to 0$$

Thus $N/\operatorname{Im}\epsilon$ is a finitely generated projective R-module. Let T and L be finitely generated projective A-modules which lift M/N and $N/\operatorname{Im}\epsilon$. We have a homomorphism $g : \mathcal{S}(T, L) \to \breve{M}$, and the associated homomorphism of exact sequences (43) is an isomorphism on all components, except possibly on N. By the 5-Lemma g is an isomorphism. □

Lemma 3.4. *Let $0 \to \breve{M}_1 \to \breve{M}_2 \to \breve{M}_3 \to 0$ be a short exact sequence of $(A, \mathfrak{c})$-modules. If $\breve{M}_2$ and $\breve{M}_3$ are standard projective, then so is $\breve{M}_1$.*

Proof. We write $\breve{M}_i = (M_i, N_i, \iota_i, \epsilon_i)$. Clearly M_1 is finitely generated projective over A. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

Applying the snake-lemma and taking into account the exact sequences (43) for $\breve{M}_2$ and $\breve{M}_3$ we obtain an exact sequence of projective R-modules:

$$0 \to M_1/N_1 \to M_2/N_2 \to M_3/N_3 \to 0.$$

In particular M_1/N_1 is finitely generated projective over R. Since the last sequence remains exact under $\mathfrak{u} \otimes_R$, it follows that (43) is exact for $\breve{M}_1$. □

Proposition 3.5. *Let $\mathcal{P}_i = (P_i, Q_i, \iota_i, \epsilon_i, F_i, \dot{F}_i)$ for $i = 1, 2$ be two truncated displays of level n over a ring R. Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism such that $P_1 \to P_2$ and $Q_1 \to Q_2$ are surjective.*

Then there is a truncated display of $\mathcal{P}$ level n and a sequence of truncated displays

$$0 \to \mathcal{P} \to \mathcal{P}_1 \to \mathcal{P}_2 \to 0$$

such that the underlying sequence of $(\mathcal{W}(R), I_{n+1})$-modules is exact.

The same statement is true for relative truncated displays.

Remark: One can also show that surjectivity of $P_1 \to P_2$ implies surjectivity of $Q_1 \to Q_2$. *

Proof. We consider the case of relative truncated displays. Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be the kernel of α, taken componentwise. Then (P, Q, ι, ϵ) is a standard projective $(\mathcal{W}_n(S), \mathcal{J}_{n+1})$-module by the Lemma 3.4. The underlying sequence of $(\mathcal{W}_n(S), \mathcal{J}_{n+1})$-modules splits. Now $\mathcal{P}$ is a relative truncated display iff the operator $F^\sharp \oplus \dot{F}^\sharp$ of (8) is an isomorphism. Since a block upper triangular matrix is invertible iff the diagonal blocks are invertible, the fact that $\mathcal{P}_1$ is a relative truncated display implies the same for $\mathcal{P}$. □

Corollary 3.6. *The functor Φ_m of Proposition 3.2 is exact.*

Proof. A given short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ in $\mathcal{BT}_n(R)$ embeds Zariski locally into a short exact sequence of p-divisible groups $0 \to X_1 \to X_2 \to X_3 \to 0$. Let $Y_i = X_i/G_i$. We have exact sequences in $\mathcal{D}_m(S/R)$ which define $\mathcal{M}_i$:

$$0 \to \mathcal{M}_i \to \tau_m \Phi_{S/R}(X_i) \to \tau_m \Phi_{S/R}(Y_i) \to \Phi_m(G_i) \to 0.$$

Here τ_m means truncation to level m. Indeed, the sequence without $\mathcal{M}_i$ is exact by definition. Proposition 3.5 implies that the image and kernel of the middle arrow are relative truncated displays.

By the snake lemma we obtain an exact sequence in $\mathcal{D}_m(S/R)$

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \Phi_m(G_1) \to \Phi_m(G_2) \to \Phi_m(G_3) \to 0.$$

Let $\Phi_m(G_i) = (P_i, \ldots)$. The rank of P_i is the height of G_i. Since the height is additive in short exact sequences, it follows that $M_3 \to P_1$ is the zero map. By Lemma 2.2 it follows that $\mathcal{M}_3 \to \mathcal{P}_1$ is zero. □

Remark 3.7. (Duality) Let G be a truncated BT-group of level n over R. Assume that G is the kernel of an isogeny of p-divisible groups: $0 \to G \to X_0 \to X_1 \to 0$. Let $\alpha : \mathcal{P}_0 \to \mathcal{P}_1$ be the morphism of relative truncated displays given by the functor $\Phi_{S/R}$ followed by truncation. By construction, $\Phi_m(G) = \operatorname{Coker} \alpha$. We claim that there is a natural isomorphism

$$\Phi_m(G) \cong \operatorname{Ker}(\alpha),$$

i.e. one could also define Φ_m using the kernel. First we note that the kernel is a relative truncated display by Proposition 3.5. Now we have an exact sequence in $\mathcal{BT}_n(R)$

$$0 \to G \to X_0[p^n] \to X_1[p^n] \to G \to 0.$$

Since Φ_m is exact and since $\Phi_m(X_i[p^m]) = \mathcal{P}_i$ the claim follows.

One can define the dual of (relative) truncated displays as in the case of (relative) displays. The functor $\Phi_{S/R}$ preserves duality. Using the obove isomorphism one can deduce that the functor Φ_m preserves duality too. We leave the details to the reader.

Let n, e, and m be natural numbers such that $n > m(e+1)+1$. Let X be a p-divisible formal group over R which is nilpotent of order $\leq e$ (26). We choose a natural number t such $p^t \mathcal{W}_n(S) = 0$. By Proposition 3.2 we associate to $X[t]$ a truncated display $\mathcal{P} = \Phi_n(X[t])$ of level n over R. Then we have a canonical isomorphism between the Grothendieck-Messing crystal of X and the crystal of $\mathcal{P}$:

$$\mathbb{D}_X(S) \cong \mathbb{D}_{\mathcal{P}}(S). \tag{44}$$

Indeed, if $\Phi_{S/R}(X) = (\tilde{P}, \tilde{Q}, F, \dot{F})$ denotes the relative display associated to X, both modules in (44) are canonically identified with $S \otimes_{W(S)} \tilde{P}$.

Smoothness

The functors Φ_n over rings R with $pR = 0$ define a morphism

$$\phi_n : \mathcal{BT} \times \operatorname{Spec} \mathbb{F}_p \to \mathcal{D}_n \times \operatorname{Spec} \mathbb{F}_p$$

of smooth algebraic stacks over $\mathbb{F}_p$. By [L2] this morphism is smooth. Using Proposition 2.6 we can simplify the proof. This remark is independent of the notion of relative truncated displays. Let k be a field of characteristic p. We consider the ring homomorphism $S = k[\varepsilon] \to R = k$. To prove that ϕ_n is smooth it suffices to show that the morphism of fpqc stacks

$$\phi : \mathcal{BT} \to \mathcal{D}$$

from p-divisible groups to displays satisfies the lifting criterion of formal smoothness with respect to $S \to R$. We equip the kernel of $S \to R$ with the trivial divided powers. We consider the commutative diagram of functors:

$$\begin{array}{ccccc} \mathcal{BT}(S) & \xrightarrow{f} & \mathcal{BT}(R) & = & \mathcal{BT}(R) \\ \downarrow{\scriptstyle \Phi_S} & & \downarrow{\scriptstyle \Phi_{S/R}} & & \downarrow{\scriptstyle \Phi_R} \\ \mathcal{D}(S) & \xrightarrow{g} & \mathcal{D}(S/R) & \xrightarrow{h} & \mathcal{D}(R) \end{array}$$

Here the left hand square is 2-Cartesian because lifts under f or under g correspond to lifts of the Hodge filtration. For f this is the Grothendieck-Messing theorem, and for g this is trivial. Proposition 2.6 (a) implies that for each display over R all lifts under h are isomorphic. The lifting criterion for $S \to R$ follows easily.

From displays to groups

Let R be a ring with $pR = 0$. We will view formal groups and groups schemes with a nilpotent augmentation ideal as functors on the category Nil_R of nilpotent R-algebras. We will call such group schemes infinitesimal.

Let G be a functor on Nil_R. We recall the definition of the Frobenius of G. For $N \in \mathrm{Nil}_R$ we have the absolute Frobenius $\mathrm{Frob}^n : N \to N_{[p^n]}$. This induces a homomorphism

$$\mathrm{Frob}^n_G : G(N) \to G^{(p^n)}(N) = G(N_{(p^n)})$$

which is called the Frobenius of G. We denote by $G[F^n]$ the kernel of Frob^n_G. Let N' be the kernel of $\mathrm{Frob}^n : N \to N_{[p^n]}$. If G is left exact we have

$$G(N') = G[F^n](N) = G[F^n](N').$$

Let $\mathrm{Nil}^{(n)}_R \subset \mathrm{Nil}_R$ be the category of R-algebras N such that $x^{p^n} = 0$ for all $x \in N$. For left exact G we can view $G[F^{(n)}]$ as the restriction of the functor G to the category $\mathrm{Nil}^{(n)}_R$.

If G is a commutative formal group of dimension d then $G[F^n]$ is a finite locally free infinitesimal group scheme of rank p^{dn} over R. Finite locally free infinitesimal group schemes which arise in this way are called truncated formal groups of level n over R. Let $\mathcal{FG}_n(R)$ be the category of such groups schemes.

Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be a truncated display of level n over R. We chose a normal decomposition

$$P = T \oplus L, \qquad Q = I_{n+1} \otimes T \oplus L.$$

Let $N \in \mathrm{Nil}^{(n)}_R$. Then the $W(R)$-module $W(N)$ is a $W_n(R)$-module since for $x \in W(N)$ and $a \in W(R)$ we have ${}^{V^n}a \cdot x = {}^{V^n}(aF^n(x)) = 0$. Thus we can define

$$\hat{P}_N = \hat{W}(N) \otimes_{W_n(R)} P, \qquad \hat{Q}_N = {}^V\hat{W}(N) \otimes_{W_n(R)} T \oplus \hat{W}(N) \otimes_{W_n(R)} L.$$

Proposition 3.8. *There is an exact sequence of abelian groups*

$$0 \longrightarrow \hat{Q}_N \xrightarrow{\dot{F}-1} \hat{P}_N \longrightarrow FG_n(\mathcal{P})(N) \longrightarrow 0,$$

which defines $FG_n(\mathcal{P})(N)$; the assertion is that the first map is injective. The functor $N \mapsto FG_n(\mathcal{P})(N)$ on the category $\mathrm{Nil}^{(n)}_R$ is a truncated formal group of level n. This defines an additive and exact functor

$$FG_n : \mathcal{D}_n(R) \to \mathcal{FG}_n(R).$$

Proof. Let $\mathcal{P}' = (P', Q', F, \dot{F})$ be a display over R with truncation $\mathcal{P}$ and let $G = BT(\mathcal{P}')$ be the associated formal Lie group. By definition, for each $N \in \mathrm{Nil}_R$ we have an exact sequence

$$0 \to \hat{Q}'_N \xrightarrow{\dot{F}-1} \hat{P}'_N \to G(N) \to 0.$$

If N lies in $\mathrm{Nil}_R^{(n)}$, this sequence can be identified with the sequence of the proposition. Thus that sequence is left exact, and $FG_n(\mathcal{P}) = G[F^n]$ is a truncated formal group of level n. □

Lemma 3.9. *Let G be a truncated BT-group of level n over a ring R such that $pR = 0$. There is a natural isomorphism*

$$FG_n(\Phi_n(G)) \xrightarrow{\sim} G[F^n]. \tag{45}$$

Proof. Cf. Remark 3.7. Assume that G is the kernel of an isogeny of p-divisible groups, $0 \to G \to X_0 \to X_1 \to 0$. We obtain an exact sequence

$$0 \to G \to X_0[p^n] \to X_1[p^n] \to G \to 0.$$

Since the functors Φ_n and FG_n preserve short exact sequences (Corollary 3.6) and since $\Phi_n(X_i[p^n]) = \Phi_R(X_i)$, we obtain an exact sequence of finite group schemes

$$0 \to FG_n(\Phi_n(G)) \to BT(\Phi(X_0))[F^n] \to BT(\Phi(X_1))[F^n]$$

By [L2] for each p-divisible group X over R there is a natural isomorphism

$$BT(\Phi(X)) \cong \hat{X}. \tag{46}$$

This gives an isomorphism (45). The isomorphism does not depend on the chosen resolution $X_0 \to X_1$ of G. Since such resolutions exist Zariski locally, the lemma follows. □

For a truncated display $\mathcal{P}$ of level n over R and a natural number m we define a finite group scheme over R:

$$BT_m(\mathcal{P}) = FG_n(\mathcal{P})[p^m] \tag{47}$$

Proposition 3.10. *Let R a ring with $pR = 0$. Let $\mathcal{P}$ be a truncated display of level n such that the order of nilpotence of $\mathcal{P}$ is $\leq e$. Let m be a positive integer such that $n \geq 2m(e+2)$.*

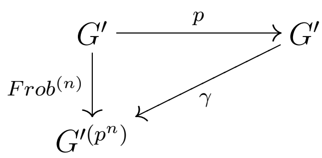
Then the group scheme $BT_m(\mathcal{P})$ is a truncated p-divisible group of level m.

If the ring R is reduced it suffices that $n \geq m(e+1)$.

Proof. Let $\mathcal{P}'$ be a display over R with truncation $\mathcal{P}$ and let $G' = BT(\mathcal{P}')$ be the associated p-divisible formal group. Thus $FG_n(\mathcal{P}) = G'[F^n]$.

We begin with the case where R is not necessarily reduced. It suffices to prove that $G'[p] \subset G'[F^{2(e+2)}]$, because this implies that $G'[p^m] \subset G'[F^{2m(e+1)}] \subset G'[F^n]$. The proposition follows.

We may assume that $n = 2(e+2)$. We need to show the existence of a factorization γ:



This follows from the proof of Proposition 88 in [Z1]. Indeed because the order of nilpotence is e we have a map

$$V^{(e+1)\sharp} : P' \to I_R \otimes_{F^{e+1}, W(R)} P.$$

It is shown in [Z1] that $V^{(e+2)\sharp} = \gamma_0 p$. We set $V^{\sharp}\gamma_0 = \gamma_1$. Then the map

$$\gamma_1 : P \to W(R) \otimes_{F^{e+1}, W(R)} P.$$

respects the Hodge filtration. We have to insure that the maps

$$F\gamma_1 - \gamma_1 F, \quad \dot{F}\gamma_1 - \gamma_1 \dot{F}$$

are zero. By [Z1] this is true after replacing γ_1 by $V^{(e+1)\sharp}\gamma_1$. The proves the Proposition in the nonreduced case.

In the reduced case we have to show that $G'[p] \subset G'[F^{(e+1)}]$. Clearly it is enough to treat the case where R is a perfect field. This follows from (27). □

Proposition 3.11. *Let R be a ring with $pR = 0$. Let G be a truncated BT-group of level n such that the order of nilpotence of G is $\leq e$ (see* (26)*). Let m be a natural number such that $n \geq 2m(e+2)$. Then there is a natural isomorphism*

$$BT_m(\Phi_n(G)) \cong G[m].$$

If the ring R is reduced it suffices that $n \geq m(e+1)$.

Proof. By taking in (45) the kernel of multiplication by p^m on both sides we obtain an isomorphism

$$BT_m(\Phi_n(G)) \cong G[F^n][p^m].$$

We claim that $G[p] \subset G[F^{2(e+2)}]$. Indeed, this question is local for the fpqc-topology. Therefore we may assume that G is the truncation of a p-divisible formal group G'. Then the inclusion holds by the proof of Proposition 3.10. Then we have

$$G[p^m] \subset G[F^{2m(e+2)}] \subset G[F^n].$$

Combining this with the isomorphism above we obtain the Proposition.

If R is reduced we conlude by Lemma 1.13 that $G[p] \subset G[F^{e+1}]$. The rest of the proof is the same. □

Remark 3.12. In the proof of Lemma 3.9 we have used the natural isomorphism (46) for arbitrary p-divisible groups. The proof of this fact in [L2] is complicated because it is difficult to relate directly the functors Φ and BT.

If we want to prove Lemma 3.9 only for infinitesimal truncated BT groups, which is sufficient for Proposition 3.11, we can modify the proof as follows.

a) One can work with resolutions $0 \to G \to X_0 \to X_1 \to 0$ by formal p-divisible groups. Such resolutions exist at least f.p.q.c. locally, because f.p.q.c. locally G extends to a p-divisible group. By f.p.q.c. descent of relative truncated displays this is sufficient to construct Φ_n. In this way we use (46) only for formal p-divisible groups, which is easier than the general case; the proof uses the crystalline comparison of [Z1] and the equivalence $(*)$ between formal p-divisible groups and nilpotent displays over arbitrary rings R in which p is nilpotent ([L1, L2]).

b) In addition one can restrict the relevant base rings R and the f.p.q.c. coverings $R \to R'$ such that $G_{R'}$ extends to a p-divisible group. Namely, w.l.o.g. R is an $\mathbb{F}_p$-algebra of finite type, and we can take $R' = \prod \hat{R}_{\mathfrak{m}}$ where $\mathfrak{m}$ runs through the maximal ideals of R.[1] Over these rings the equivalence $(*)$ is already proved in [Z1], which is sufficient to deduce (46) in the cases necessary for the proof of Lemma 3.9.

c) One can also consider the following variant Φ'_n of the functor Φ_n restricted to infinitesimal groups: Let G be an infinitesimal truncated BT group of level n. If there is a resolution $0 \to G \to X_0 \to X_1 \to 0$ by formal p-divisible groups, let $\mathcal{P}_i$ be the nilpotent display associated to X_i by the equivalence $(*)$, and define $\Phi'_n(G)$ as the kernel of the map of truncations $\mathcal{P}_0[n] \to \mathcal{P}_1[n]$. In general use f.p.q.c. descent to define $\Phi'_n(G)$. Then the proof of Lemma 3.9 shows that $BT_m(\Phi'_n(G)) \cong G[m]$ as before. As explained in b), with appropriate modifications the proof uses only the equivalence $(*)$ in the cases covered by [Z1].

Vanishing homomorphisms

Let G and G' be truncated BT-groups over the reduced ring R. We consider the commutative group scheme of vanishing homomorphisms

$$\underline{\mathrm{Hom}}^o(G, G') = \mathrm{Ker}[\underline{\mathrm{Hom}}(G, G') \to \underline{\mathrm{Hom}}(\mathcal{P}, \mathcal{P}')]$$

and the group scheme of vanishing automorphisms

$$\underline{\mathrm{Aut}}^o(G) = \mathrm{Ker}[\underline{\mathrm{Aut}}(G) \to \underline{\mathrm{Aut}}(\mathcal{P})].$$

We have the following consequence of Proposition 3.11.

[1] One can also use that every truncated BT group extends to a p-divisible group étale locally, but this is more difficult to show.

Corollary 3.13. *With the notations of Proposition 3.10 let G and G' be truncated p-divisible groups of level n over R. We assume that $G_{R_{red}}$ and $G'_{R_{red}}$ have order of nilpotence $\leq e$. Let m be a positive integer with $n \geq m(e+1+u)$.*

Then the reduction map

$$\underline{\mathrm{Hom}}^{o}(G, G') \to \mathrm{Hom}(G[m], G'[m])$$

is zero.

Proof. Proposition 3.11 gives the following commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(G, G') & \xrightarrow{\Phi_n} & \underline{\mathrm{Hom}}(\mathcal{P}, \mathcal{P}') \\ \downarrow & \swarrow_{BT_m} & \\ \underline{\mathrm{Hom}}(G(m), G'(m)) & & \end{array} \tag{48}$$

and the corollary follows. $\square$

Remark 3.14. We have deduced Corollary 3.13 from the existence of a functor Φ_n for infinitesimal groups such that Lemma 3.9 holds, which relies in any of its variants on the equivalence between formal BT groups and nilpotent displays over excellent rings. This logic can be reversed as follows.

We denote by $\mathcal{BT}_n \to \operatorname{Spec} \mathbb{F}_p$ the algebraic stack of truncated BT groups of level n and by $\mathcal{D}_n \to \operatorname{Spec} \mathbb{F}_p$ the algebraic stack of truncated displays of level n. Let $\mathcal{BT}_n^e \subset \mathcal{BT}_n$ be reduced closed substack whose points are the groups which are nilpotent of order $\leq e$, and let $\mathcal{D}_n^e \subset \mathcal{D}_n$ be the reduced closed substack whose points are the truncated displays which are nilpotent of order $\leq e$.

Assume that $n = m(e+1)$ for an integer $m \geq 1$. Consider the following commutative diagram of algebraic stacks over $\operatorname{Spec} \mathbb{F}_p$, where τ are the truncations.

$$\begin{array}{ccc} \mathcal{BT}_n^e & \xrightarrow{\Phi_n} & \mathcal{D}_n^e \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ \mathcal{BT}_m^e & \xrightarrow{\Phi_m} & \mathcal{D}_m^e \end{array} \tag{49}$$

Proposition 3.15. *There is a unique morphism $\Gamma_m : \mathcal{D}_n^e \to \mathcal{BT}_m^e$ with an isomorphism $\Gamma_m \circ \Phi_n \cong \tau$, and we also have an isomorphism $\Phi_m \circ \Gamma \cong \tau$.*

We will prove this proposition using Corollary 3.13 and the geometric properties of the morphism $\Phi_n : \mathcal{BT}_n \to \mathcal{D}_n$ which are explained in [L2].

Proof. We only consider truncated BT groups of a fixed height h and truncated displays of rank h without changing the notation. Let $X = \operatorname{Spec} R \to \mathcal{BT}_n^e$ be a smooth presentation given by the truncated p-divisible group G of level n over R. All arrows in (49) are smooth and surjective. Thus we also get smooth presentations of the other three stacks. Let $\mathcal{P} = \Phi_n(G)$ be the truncated display associated to G. Let G_1, G_2 and $\mathcal{P}_1, \mathcal{P}_2$ over $X \times X$ be the inverse images of G and $\mathcal{P}$ under the two projections. We get a commutative diagram of groupoids over X:

$$\begin{array}{ccc} \underline{\mathrm{Isom}}(G_1, G_2) & \xrightarrow{\Phi_n} & \underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ \underline{\mathrm{Isom}}(G_1[m], G_2[m]) & \xrightarrow{\Phi_m} & \underline{\mathrm{Isom}}(\mathcal{P}_1[m], \mathcal{P}_2[m]) \end{array} \tag{50}$$

The associated diagram of stacks is (49). The morphism Φ_n in (50) is a torsor under the finite flat group scheme $\underline{\mathrm{Aut}}^o(G_1)$. Corollary 3.13 implies that the reduction homomorphism $\underline{\mathrm{Aut}}^o(G_1) \to \underline{\mathrm{Aut}}(G_1[m])$ is trivial. Thus there is a unique morphism of groupoids $\Gamma_m : \underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2) \to \underline{\mathrm{Isom}}(G_1[m], G_2[m])$ such that the upper triangle in (50) commutes. Since Φ_n in (50) is faithfully flat, the lower triangle commutes as well. □

It is easy to see that the functors Γ_m for varying m are compatible in such a way that they form an inverse to the functor

$$\Phi : \varprojlim_n \mathcal{BT}_n^e \to \varprojlim_n \mathcal{D}_n^e,$$

so this is an equivalence. From there one can easily deduce that Φ induces the equivalence $(*)$ between formal BT groups and nilpotent displays over rings in which p is nilpotent.

In order to close the circle it remains to find a direct proof of Corollary 3.13. We will only prove a slightly weaker statement that is sufficient to derive the equivalence $(*)$.

We recall from [L2] that the group scheme $\underline{\mathrm{Aut}}^o(G)$ is a commutative finite flat group scheme of rank p^{ncd}.

Lemma 3.16. *The group scheme $\underline{\mathrm{Hom}}^o(G, G')$ is finite flat of rank $p^{ncd'}$, where c is the codimension of G and d' is the dimension of G'.*

Proof. Clearly $\underline{\mathrm{Hom}}^o(G, G')$ is an affine group scheme over R. It is infinitesimal since the functor Φ_n is an equivalence over perfect fields. We claim that the map

$$\underline{\mathrm{Aut}}^o(G) \to \underline{\mathrm{End}}^o(G), \quad u \mapsto u - 1$$

is an isomorphism of schemes. Indeed, let $f \in \underline{\mathrm{End}}^o(G)(A)$ for some R-algebra A. We have to show that $f+1$ is an automorphism. There is a nilpotent ideal $I \subset A$ such that $f+1 \equiv 1$ modulo I. The assertion follows using the flatness criterion EGA IV 11.3.11 for $f+1 : G \to G$ over S. Thus $\underline{\mathrm{End}}^o(G)$ is finite flat. Using the decomposition

$$\underline{\mathrm{End}}^o(G \oplus G') = \underline{\mathrm{End}}^o(G) \oplus \underline{\mathrm{Hom}}^o(G, G') \oplus \underline{\mathrm{Hom}}^o(G', G) \oplus \underline{\mathrm{End}}^o(G').$$

it follows that $\underline{\mathrm{Hom}}^0(G, G')$ is finite flat. Its rank is locally constant on the stack $\mathcal{BT}_n \times \mathcal{BT}_n$. Since the generic point of $\mathcal{BT}_n$ is ordinary, to compute the rank we may assume that each of G and G' is either $\mathbb{Z}/p^n\mathbb{Z}$ or μ_{p^n}. In those cases the rank is computed in [L2] as desired. □

Let k be an algebraically closed field of characteristic p. For $e \geq 0$ let H_e be the unique unipotent truncated p-divisible group over k of level n, dimension e, and height $e+1$. In particular $H_0 = \mathbb{Z}/p^n\mathbb{Z}$.

Lemma 3.17. *Assume that $n \geq m(e+1)$. Then for each $G \in \mathcal{BT}_n(k)$ the reduction map*

$$\underline{\mathrm{Hom}}^o(G, H_e) \to \underline{\mathrm{Hom}}(G[m], H_e[m])$$

is zero.

Proof. We may assume that $n = m(e+1)$. Consider the truncated p-divisible group $H = H_0 \oplus H_e$ of dimension e and height $e+2$. We have

$$\underline{\mathrm{Aut}}(H) = \begin{pmatrix} \underline{\mathrm{Aut}}(H_0) & 0 \\ \underline{\mathrm{Hom}}(H_0, H_e) & \underline{\mathrm{Aut}}(H_e) \end{pmatrix}.$$

Since $\underline{\mathrm{Aut}}(H_0)$ is an étale group scheme and since $\underline{\mathrm{Aut}}^o(H)$ is infinitesimal and commutative, we get

$$\underline{\mathrm{Aut}}^o(H) = \underline{\mathrm{Hom}}^o(H_0, H_e) \rtimes \underline{\mathrm{Aut}}^o(H_e).$$

Here $\underline{\mathrm{Aut}}(H_e)$ and $\underline{\mathrm{Aut}}(H)$ are finite groups schemes of rank p^{en} and p^{2en}, respectively. Thus $\underline{\mathrm{Hom}}^o(H_0, H_e)$ is a finite group scheme of rank p^{en}. We have

$$\underline{\mathrm{Hom}}^o(H_0, H_e) \subset \underline{\mathrm{Hom}}(H_0, H_e) = H_e,$$

and the finite group scheme H_e has a unique subgroup scheme of each p-power rank. Since the rank of $H_e[n-m]$ is $p^{(e+1)(n-m)} = p^{en}$ it follows that

$$\underline{\mathrm{Hom}}^o(H_0, H_e) = \underline{\mathrm{Hom}}(H_0, H_e[n-m]). \tag{51}$$

This proves the lemma when $G = H_0$.

Now let G be arbitrary and consider an element $u \in \underline{\mathrm{Hom}}^o(G, H_e)(A)$ for some k-algebra A. Let $A \to B$ be a ring homomorphism and let $a \in G(B)$. The composition $(H_0)_B \xrightarrow{a} G_B \xrightarrow{u} (H_e)_B$ lies in $\underline{\mathrm{Hom}}^o(H_0, H_e)(B)$. Using (51) it follows that the reduction of u to level m is trivial as desired. □

Corollary 3.18. *Let $G, G' \in \mathcal{BT}_n(R)$ where R is reduced, and G is infinitesimal of codimension c. Let m be a positive integer with $n \geq m(c+1)$. Then the reduction map*

$$\underline{\mathrm{Hom}}^o(G, G') \to \mathrm{Hom}(G[m], G'[m])$$

is zero.

Proof. We fix again the height h. Let $\mathcal{Y}_n \subset \mathcal{BT}_n$ be the reduced closed substack of truncated BT groups of height h and codimension c. To prove the corollary we may pass to a generic geometric point $\operatorname{Spec} k \to \mathcal{Y}_n$. The fibre of the universal group over k is isomorphic to the dual of $(H_0)^{h-c-1} \oplus H_c$. Thus the corollary follows from Lemma 3.17 by passing to the dual. □

Since an infinitesimal truncated BT group of height h over a reduced ring has order of nilpotence less than h, Corollary 3.18 implies Corollary 3.13 for $e \geq h-1$. This is sufficient to deduce the equivalence $(*)$.

Appendix: Descent for truncated displays

Proposition 3.19. *Let $R \to S$ be a faithfully flat ring homomorphism. Then the Cech complex*

$$\mathcal{W}_n(R) \to \mathcal{W}_n(S) \rightrightarrows \mathcal{W}_n(S \otimes_R S) \overrightarrow{\rightrightarrows} \mathcal{W}_n(S \otimes_R S \otimes_R S) \ldots$$

is acyclic.

Proof: To the simplicial complex above we have also the associated chain complex which will be denoted by $\mathcal{CW}_n(S/R)$.

Let $R[p]$ be the kernel of the multiplication by p. By tensoring with $\otimes_R S$:

$$S[p] = R[p] \otimes_R S.$$

By descent theory we know that the Cech complex of the R-module $R[p]$ relative to the covering $\operatorname{Spec} S \to \operatorname{Spec} R$ is acyclic:

$$R[p] \to S[p] \rightrightarrows (S \otimes_R S)[p] \overrightarrow{\rightrightarrows} (S \otimes_R S \otimes_R S)[p] \ldots.$$

Let $\mathcal{C}(S/R)[p]$ be the associated simple complex. Using the remarks after (4) we obtain an exact sequence of complexes

$$0 \to \mathcal{C}(S/R)[p] \to \mathcal{C}W_{n+1}(S/R) \to \mathcal{C}\mathcal{W}_n(S/R) \to 0.$$

The definition and exactness of the complex in the middle follows from [Z1] Lemma 30. This concludes the proof of the Proposition.

The notion of a W-descent datum [Z1] applies to $\mathcal{W}_n(R) \to \mathcal{W}_n(S)$ and is then called a $\mathcal{W}_n$-descent datum.

Proposition 3.20. *With the assumptions of the last Proposition let P be a finitely generated projective $\mathcal{W}_n(S)$-module with a descent datum*

$$\nabla : \mathcal{W}_n(S \otimes_R S)_{p_1, \mathcal{W}_n(S)} P \to \mathcal{W}_n(S \otimes_R S)_{p_2, \mathcal{W}_n(S)} P. \tag{52}$$

The associated chain complex $\mathcal{C}_{\mathcal{W}_n}(P; S/R)$ (compare: [Z1] (43))

$$P \to \mathcal{W}_n(S \otimes_R S) \otimes_{\mathcal{W}_n(S)} P \to \mathcal{W}_n(S \otimes_R S \otimes_R S) \otimes_{\mathcal{W}_n(S)} P \to \ldots$$

is exact. Here the $\mathcal{W}_n(S)$-module structure on $\mathcal{W}_n(S \otimes_R \ldots \otimes_R S)$ is via the last factor of the tensorproduct.

If P_0 is the kernel of the first arrow we have a canonical isomorphism

$$\mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} P_0 \to P.$$

Proof: We begin with a general remark. Let R' be an R-algebra. We denote by $\mathfrak{n}_{R'} \subset R'$ the ideal of all elements annihilated by p. We set $\mathfrak{n} = \mathfrak{n}_R$. If $R \to R'$ is flat then $\mathfrak{n}_{R'} = \mathfrak{n} \otimes_R R' = \mathfrak{n}R'$.

We denote by $\mathbf{w}_n : \mathcal{W}_n(R') \to R'/\mathfrak{n}R'$ the homomorphism induced by the Witt polynomial of degree p^n. For a $\mathcal{W}_n(R)$-module P_0 we set $\bar{P}_0 = R/\mathfrak{n} \otimes_{\mathbf{w}_n, \mathcal{W}_n(R)} P_o$. We have the isomorphism

$$R'/\mathfrak{n}R' \otimes_{\mathbf{w}_n, \mathcal{W}_n(R')} (\mathcal{W}_n(R') \otimes_{\mathcal{W}_n(R)} P_0) \cong R'/\mathfrak{n}R' \otimes_{R/\mathfrak{n}} \bar{P}_0$$

If we tensor the descent datum (52) with $W_n(S \otimes_R S) \otimes_{\mathcal{W}_n(S \otimes_R S)}$ we obtain a W_n-descent datum on $W_n(S) \otimes_{\mathcal{W}_n(R)} P$ and if we tensor with $(S/\mathfrak{n}S \otimes_{R/\mathfrak{n}} S/\mathfrak{n}S) \otimes_{\mathbf{w}_n, \mathcal{W}_n(S \otimes_R S)}$ we obtain a descent datum on the $S/\mathfrak{n}S$-module $\bar{P} = S/\mathfrak{n}S \otimes_{\mathbf{w}_n, \mathcal{W}_n(S)} P$.

By the definition of $\mathcal{W}_n(S)$ we have an exact sequence of $\mathcal{W}_n(S)$-modules

$$0 \to (S/\mathfrak{n}S)_{[\mathbf{w}_n]} \xrightarrow{V^n} \mathcal{W}_n(S) \to W_n(R) \to 0.$$

By tensoring with P we obtain the exact sequence

$$0 \to \bar{P} \xrightarrow{V^n} P \to W_n(S) \otimes_{\mathcal{W}_n(S)} P \to 0. \tag{53}$$

We have a commutative diagram

$$\begin{array}{ccc} (S/\mathfrak{n}S \otimes_{R/\mathfrak{n}} S/\mathfrak{n}S) \otimes_{\mathbf{w}_n \circ p_1, \mathcal{W}_n(S)} P & \xrightarrow{\mathrm{id} \otimes \nabla} & (S/\mathfrak{n}S \otimes_{R/\mathfrak{n}} S/\mathfrak{n}S) \otimes_{\mathbf{w}_n \circ p_2, \mathcal{W}_n(S)} P \\ {\scriptstyle V^n}\downarrow & & \downarrow{\scriptstyle V^n} \\ \mathcal{W}_n(S \otimes_R S) \otimes_{p_1, \mathcal{W}_n(S)} P & \xrightarrow[\nabla]{} & \mathcal{W}_n(S \otimes_R S) \otimes_{p_2, \mathcal{W}_n(S)} P \end{array}$$

Therefore the exact sequence (53) is compatible with the descent data and yields an exact sequence of complexes:

$$0 \to \mathcal{C}(\bar{P}; (S/\mathfrak{n}S)/(R/\mathfrak{n})) \to \mathcal{C}_{\mathcal{W}_n}(P; S/R) \to \mathcal{C}_{W_n}(P; S/R) \to 0$$

The first complex is the complex associated to the descent datum on the $S/\mathfrak{n}S$-module $\bar{P}$ relative to $R/\mathfrak{n} \to S/\mathfrak{n}S$. By [Z1] and usual descent we know that execpt for the complex in the middle we have $H^i = 0$ for $i \geq 1$. Then this holds also for the complex in the middle. Taking H^0 we obtain the exact cohomology sequence

$$0 \to \bar{P}_0 \to P_0 \to \breve{P}_0 \to 0.$$

By W_n-descent we know that the natural map

$$W_n(S) \otimes_{W_n(R)} \breve{P}_0 \to W_n(S) \otimes_{\mathcal{W}_n(S)} P$$

is an isomorphism. Let E be a finitely generated projective $\mathcal{W}_n(R)$-module which lifts the $W_n(R)$-module $\breve{P}_0$. We find a factorization $E \to P_0 \to \breve{P}_0$. By the Lemma of Nakayama we conclude that

$$\mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} E \to P$$

is an isomorphism. Since the last arrow is compatible with the descent data on both sides we obtain an isomorphism of complexes

$$\mathcal{C}_{\mathcal{W}_n}(\mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} E; S/R) \to \mathcal{C}_{\mathcal{W}_n}(P; S/R).$$

It follows that $E \to P_0$ is an isomorphism. This proves the Proposition.

Let $R \to S$ be a faithfully flat ring homomorphism as before. Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be a truncated display of level over R. We denote the base change to S by $\mathcal{P}_S = (P_S, Q_S, \iota_S, \epsilon_S, F_S, \dot{F}_S)$. There is a natural homomorphism $\mathcal{P} \to \mathcal{P}_S$ which is obvious in terms of a normal decomposition. We obtain a simplicial complex

$$\mathcal{P} \to \mathcal{P}_S \rightrightarrows \mathcal{P}_{S \otimes_R S} \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \mathcal{P}_{S \otimes_R S \otimes_R S} \ldots. \tag{54}$$

Proposition 3.21. *The simplicial complex* (54) *induces exact chain complexes*

$$0 \to P \to P_S \to P_{S\otimes_R S} \to P_{S\otimes_R S\otimes_R S} \cdots$$
$$0 \to Q \to Q_S \to Q_{S\otimes_R S} \to Q_{S\otimes_R S\otimes_R S} \cdots.$$

Proof: We know that $P_S = \mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} P$. Therefore we obtain the first exact sequence from the first Proposition. To obtain the second exact sequence we choose a normal decomposition $P = T \oplus L$. Then $Q_S = I_{n+1}(S) \otimes_{\mathcal{W}_n(R)} T \oplus \mathcal{W}_n(S) \otimes_{\mathcal{W}_n(R)} L$. The exactness of the second sequence amounts therefore to that of

$$I_{n+1}(R) \otimes_{\mathcal{W}_n(R)} T \to I_{n+1}(S) \otimes_{\mathcal{W}_n(R)} T \to I_{n+1}(S \otimes_R S) \otimes_{\mathcal{W}_n(R)} T \to \ldots.$$

But this is clear.

We have the notion of descent datum relative to S/R for a truncated display $\tilde{\mathcal{P}}$ over S. The Proposition shows that the functor which associates to a truncated display over R the base change to a truncated display over S with a descent datum is fully faithful.

Proposition 3.22. *Let $R \to S$ be faithfully flat. The base change functor induces an equivalence of the category $\mathcal{D}_n(R)$ of truncated displays of level n over R with the category of truncated displays of level n over S endowed with a $\mathcal{W}_n$-descent datum relative to S/R (see:* (52)*).*

Proof: It follows from the last Proposition that this functor is fully faithful. Therefore it suffices to show that a $\mathcal{W}_n$-descent datum is always effective.

Following [L2] we begin to prove a related descent result. We call a $(\mathcal{W}_n(R), I_{n+1})$-module (P, Q, ι, ϵ) which admits a normal decomposition a truncated pair. In particular the R-modules $P/\iota(Q)$ and $Q/\operatorname{Im}\epsilon$ are projective finitely generated R-modules.

The first lines of the proof of Proposition 1.3 show that a second truncated pair $(P', Q', \iota', \epsilon')$ is isomorphic to (P, Q, ι, ϵ) iff there are isomorphisms of R-modules

$$P/\iota(Q) \cong P'/\iota(Q'), \quad Q/\operatorname{Im}\epsilon \cong Q'/\operatorname{Im}\epsilon'.$$

More precisely, if two such isomorphisms are given, they are induced by an isomorphism

$$(P, Q, \iota, \epsilon) \to (P', Q', \iota', \epsilon').$$

We fix projective finitely generated R-modules $\bar{T}$ and $\bar{L}$. Let $\mathcal{F}$ be the cofibered category over the category of R-algebras S_1, such that an object of

$\mathcal{F}_{S_1}$ is a truncated pair (P, Q, ι, ϵ) over S_1 endowed with isomorphisms

$$P/\iota(Q) \cong S_1 \otimes_R \bar{T}, \quad Q/\operatorname{Im}\epsilon \cong S_1 \otimes_R \bar{L}.$$

By the remark above any two objects in $\mathcal{F}_{S_1}$ are isomorphic.

We fix an object $(P_0, Q_0, \iota_0, \epsilon_0) \in \mathcal{F}_R$. We denote by $\mathcal{A}_{S_1}$ the automorphism of the base change $(P_0, Q_0, \iota_0, \epsilon_0)_{S_1}$. By [M] Chapt.III §4 the set of isomorphism classes of descent data on $(P_0, Q_0, \iota_0, \epsilon_0)_S$ is bijective to the nonabelian Cech cohomology set $\check{H}^1(S/R, \mathcal{A})$. We will show below that this pointed set is trivial. Equivalently this says that any desent datum in $\mathcal{F}$ relative to S/R is effective.

We can now prove the Proposition. Let $\mathcal{P} = (P, Q, \iota, \epsilon, F, \dot{F})$ be a truncated display over S which is endowed with a descent datum. We denote by $\breve{P} = (P, Q, \iota, \epsilon)$ the associated truncated pair. The descent datum induces a descent datum on the S-modules $P/\iota(Q)$ and $Q/\operatorname{Im}\epsilon$. We find projective finitely generated projective R-modules $\bar{T}$ and $\bar{L}$ and isomorphism which are compatible with the descent data on both sides

$$P/\iota(Q) \cong S \otimes_R \bar{T}, \quad Q/\operatorname{Im}\epsilon \cong S \otimes_R \bar{L}.$$

This makes $\breve{\mathcal{P}}$ an object in $\mathcal{F}_S$ and the descent datum a morphism in $\mathcal{F}_{S\otimes_R S}$. Therefore we know that the descent datum is effective. Because of the fully faithfulness of descent for truncated pairs the morphisms F and $\dot{F}$ descent too.

It remains to show the triviality $\check{H}^1(S/R, \mathcal{A})$. We vary now n and we set $\mathcal{F}_n = \mathcal{F}$. Assume that $n = 1$. In this case we consider the image $\bar{\mathcal{A}}_1$ by the map $\mathcal{A}_1 \to \operatorname{Aut} P_0 \otimes_{\mathcal{W}_1(R)} R$. This is just the additive group of an R-module and therefore the Cech cohomology of $\bar{\mathcal{A}}_1$ is trivial.The matrix representation of an element in the kernel of $\mathcal{A}_1 \to \bar{\mathcal{A}}_1$ has the form $E + \mathcal{X}$ where E is the unit matrix and $\mathcal{X}$ a matrix with coefficients in R-modules.

In the case $pR = 0$ we have $(E + \mathcal{X})(E + \mathcal{X}') = E + \mathcal{X} + \mathcal{X}'$. This shows the the Cech cohomology of the kernel is trivial. By the exact cohomology sequence for Cech cohomology of presheaves we obtain that $H^1(S/R, \mathcal{A}_1)$ is trivial. In the general case we consider the filtration of R by $p^m R$ and obtain the triviality too.

Let now $n > 1$. We denote by $(P_0', Q_0', \iota_0', \epsilon_0') \in \mathcal{F}_{n-1}$ the truncation of $(P_0, Q_0, \iota_0, \epsilon_0)$. We denote by $\mathcal{A}_{n-1}$ its automorphism group. Let $\mathcal{K}$ be the kernel of the natural surjection of presheaves $\mathcal{A}_n \to \mathcal{A}_{n-1}$. By induction it suffices to show that the Cech cohomology of $\mathcal{K}$ is trivial. Again we look at the matrix interpretation of $\mathcal{K}$. The matrices in $\mathcal{K}$ are of the form $E + \mathcal{X}$ where $\mathcal{X}$ has coefficients in an R-module. In the case $pR = 0$ we have simply the additive group of this module and therefore the Cech cohomology is trivial. In the general case we consider the filtration above. By the exact cohomology sequence we obtain the the triviality of $H^1(S/R, \mathcal{A}_n)$. This proves the Proposition.

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