# Gauß-Manin Connection via Witt-Differentials

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We generalize ideas of Bloch [Bl] about de Rham-Witt connections to a relative situation, and apply this to construct the Gauss-Manin connection in crystalline cohomology using Witt differentials.

Let $S$ be a scheme such that $p$ is nilpotent on $S$. Let $X$ be a scheme over $S$. Then the Witt differentials $W_t\Omega^1_{X/S}$ are defined in [L-Z] for each natural number $t$. These sheaves on $X$ satisfy the same exact sequences and functorialities as the usual Kähler differentials: Let $X \to Y \to S$ be morphisms of schemes. Then there is an exact sequence of sheaves on $X$.

$$W_t(\mathcal{O}_X) \otimes_{W_t(\mathcal{O}_Y)} W_t\Omega^1_{Y/S} \to W_t\Omega^1_{X/S} \to W_t\Omega^1_{X/Y} \to 0$$

The reader will deduce this easily from the isomorphism (4.1).

Assume moreover that $X \to Y$ is a closed immersion defined by a sheaf of ideals $\mathfrak{a} \subset \mathcal{O}_Y$. One can proof in the same way that there is an exact sequence:

$$W_t(\mathfrak{a})/W_t(\mathfrak{a})^2 \xrightarrow{d} W_t(\mathcal{O}_X) \otimes_{W_t(\mathcal{O}_Y)} W_t\Omega^1_{Y/S} \to W_t\Omega^1_{X/S} \to 0$$

The structure of $W\Omega^1_{X/S}$ in the case where $X$ is an affine space over $S$ is determined in [L-Z] Prop. 2.17.

Let $E$ be a locally free $W_t(\mathcal{O}_X)$ module on $X$. A de Rham-Witt connection on $E$ is a homomorphism of sheaves of abelian groups on $X$

(0.1) $$\nabla : E \to W_t\Omega^1_{X/S} \otimes_{W_t(\mathcal{O}_X)} E,$$

which satisfies the following relations:

$$\nabla(\xi x) = d\xi \otimes x + \xi\nabla x,$$

where $\xi$ and $x$ are sections of $W_t(\mathcal{O}_X)$ and $E$ respectively. The operator $\nabla$ induces maps $W_t\Omega^l_{X/S} \otimes_{W_t(\mathcal{O}_X)} E \to W_t\Omega^{l+1}_{X/S} \otimes_{W_t(\mathcal{O}_X)} E$, which we denote by the same letter $\nabla$. We call the de Rham-Witt connection (0.1) integrable if $\nabla^2 = 0$. A de

1

Rham-Witt connection induces a connection on the $\mathcal{O}_X$-module $E \otimes_{W_t(\mathcal{O}_X)} \mathcal{O}_X$. If this connection is nilpotent [K], we call $\nabla$ nilpotent.

We denote by $W_t(S)$ the topological space $S$ endowed with the structure sheaf $W_t(\mathcal{O}_S)$. This is a scheme. We consider the crystalline situs $Crys(X/W_t(S))$ with respect to the canonical divided powers on $W_t(S)$.

**Theorem.** *Let $X/S$ be a smooth scheme. Then the category of locally free crystals of finite rank is equivalent to the category of locally free $W_t(\mathcal{O}_X)$-modules of finite rank endowed with a nilpotent, integrable de Rham-Witt connection.*

This was proved by Bloch if $S$ is the spectrum of a perfect field. Let $\mathbb{E}$ be a crystal on $X$ as indicated. The starting point of the proof is the construction of Etesse [E] of the de Rham-Witt complex with coefficients in $\mathbb{E}$ over a perfect base $S$ and more recently over arbitrary $S$ in [L–Z].

Let $E$ be the value of the crystal $\mathbb{E}$ at the pd-thickening $X \to W_t(X)$. The crystal defines an integrable nilpotent de Rham-Witt connection (0.1) on $E$. Hence we obtain a complex:

$$E \xrightarrow{\nabla} W_t\Omega^1_{X/S} \otimes_{W_t(\mathcal{O}_X)} E \xrightarrow{\nabla} W_t\Omega^2_{X/S} \otimes_{W_t(\mathcal{O}_X)} E \xrightarrow{\nabla} \cdots$$

This is called the de Rham-Witt complex with coefficients in $E$.

We obtain a functor

$$(0.2) \qquad\qquad\qquad \mathbb{E} \mapsto (E, \nabla)$$

from the category of finite locally free crystals to the category of finite locally free modules with a de Rham-Witt connection.

It follows that this functor is fully faithful. Indeed, let $\mathbb{E}'$ be a second crystal. We want to assign to a homomorphism $(E, \nabla) \to (E', \nabla')$ a homomorphism of crystals. This is a local question on $X$. Therefore we may assume that there is a Witt-lift $Y/W_t(S)$,i.e. $Y$ is a smooth scheme over $W_t(S)$ which lifts $X$, with a morphism $W_t(X) \to Y$ satisfying some properties ([L–Z] Definition 3.3). By the Poincaré lemma [B–O] (compare [L–Z] 3.56) we have an resolution of $\mathbb{E}$ by the complex of crystals $\mathbf{L}_Y(\Omega_{Y/W_t(S)} \otimes \mathbb{E}_Y)$. The comparison with the de Rham-Witt complex yields a quasi-isomorphim:

$$(0.3) \qquad\qquad\qquad \mathbb{E} \cong \mathbf{L}_Y(W_t\Omega_{X/S} \otimes_{W_t(\mathcal{O}_X)} E).$$

A homomorphism $(E, \nabla) \to (E', \nabla')$ induces a homomorphism of the de Rham-Witt complexes with coefficients in $E$ resp. $E'$. Applying the functor $\mathbf{L}_Y$ one obtains a morphism crystals $\mathbb{E} \to \mathbb{E}'$ from (0.3). Hence the functor (0.2) is fully faithful. The assertion of the theorem is therefore local for the Zariski topology on

$X$. For the proof we assume that a Witt-lift $Y$ of $X$ is given. Let $(E, \nabla)$ be an integrable de Rham-Witt connection. The arguments of Bloch with modifications in the relative case yield a locally free $\mathcal{O}_Y$-module $(M, \nabla_M)$ with an integrable connection that induces $(E, \nabla)$ by the Proposition below. If $\nabla$ is nilpotent then $\nabla_M$ is nilpotent and therefore defines the desired crystal $\mathbb{E}$. This proves the theorem.

We formulate the proposition in terms of rings rather than schemes. Let $R$ be a ring. Let $A$ be an $R$-algebra which is étale over a polynomial ring over $R$. We choose a Witt lift $B/W_t(R)$ as in the proof of [L–Z] Proposition 3.2. The map $B \to W_t(A)$ induces a map of complexes:

$$(0.4) \qquad \Omega^\bullet_{B/W_t(R)} \to W_t\Omega^\bullet_{A/R}$$

This map is injective. Indeed, for a polynomial algebra $B$ this follows from loc.cit. Proposition 2.17. The general case is deduced by étale base change for the de Rham-Witt complex.

**Proposition 1.** *Let $M$ be a free $W_t(A)$-module with an integrable connection:*

$$(0.5) \qquad \nabla : M \to W_t\Omega^1_{A/R} \otimes_{W_t(A)} M.$$

*Then there is a basis $m_1, \ldots, m_n$ of $M$ such that*

$$(0.6) \qquad \nabla \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \Theta \otimes \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix},$$

*where the entries $\theta_{ij}$ of the $n \times n$-matrix $\Theta$ are in the image of the map (0.4).*

Let us assume that $S$ is noetherian and that the Frobenius morphism $Frob : S/pS \to S/pS$ is finite. This implies that the schemes $W_t(X), W_t(Y), W_t(S)$ are noetherian.

Let $X \to Y \to S$ be smooth morphisms such that $Y/S$ is of relative dimension 1. Then we construct an exact triangle

$$(0.7) \qquad W_t\Omega^1_{Y/S} \overset{\mathbb{L}}{\otimes}_{W_t(\mathcal{O}_Y)} W_t\Omega^{\bullet-1}_{X/Y} \longrightarrow W_t\Omega^\bullet_{X/S} \longrightarrow W_t\Omega^\bullet_{X/Y} \ ,$$

in the derived category of Zariski sheaves on $X$. The reader may wonder where the third side of this triangle is. The more precise statement is as follows: The multiplication in $W_t\Omega^\bullet_{X/S}$ induces a natural map

$$W_t\Omega^1_{Y/S} \otimes_{W_t(\mathcal{O}_Y)} W_t\Omega^{\bullet-1}_{X/Y} \longrightarrow W_t\Omega^\bullet_{X/S}$$

whose image is clearly in the kernel of the second map of (0.7). We choose a complex $C$ which represents the first complex in (0.7). From the last map above

we obtain a map $C \to W_t\Omega^{\bullet}_{X/S}$ whose image is a fortiori in the kernel of the second map of (0.7). Hence we obtain a map to the mapping cone of the last morphism in (0.7):

$$C[1] \to W_t\Omega^{\bullet}_{X/S}[1] \oplus W_t\Omega^{\bullet}_{X/Y}.$$

The exact meaning of the words "exact triangle" is that the last morphism is a quasi-isomorphism. We note that this statement is local for the Zariski topology on $X$. Therefore the global statement (0.7) follows from the local statement Proposition 2 which we prove below.

Applying the projection formula ([H] page 106, 5.6) to the morphism $\alpha$ : $W_t(X) \to W_t(Y)$ we obtain an isomorphism

$$\mathrm{R}\alpha_*(W_t\Omega^1_{Y/S} \overset{\mathbb{L}}{\otimes}_{W_t(\mathcal{O}_Y)} W_t\Omega^{\bullet-1}_{X/Y}) = W_t\Omega^1_{Y/S} \overset{\mathbb{L}}{\otimes}_{W(\mathcal{O}_Y)} \mathrm{R}\alpha_* W_t\Omega^{\bullet-1}_{X/Y}$$

Hence the triangle (0.7) induces a morphism

$$\mathrm{R}\alpha_* W_t\Omega^{\bullet}_{X/Y} \to W_t\Omega^1_{Y/S} \overset{\mathbb{L}}{\otimes}_{W(\mathcal{O}_Y)} \mathrm{R}\alpha_* W_t\Omega^{\bullet}_{X/Y}$$

By the comparison theorem [L–Z] Theorem 3.5 the hypercohomology groups of $\mathrm{R}\alpha_* W_t\Omega_{X/Y}$ are the crystalline cohomology groups $\mathrm{H}^i_{\mathrm{cris}}(X/W_t(Y))$. Let us make the assumption that these groups are locally free $W_t(\mathcal{O}_Y)$-modules for all $i$. Then we deduce the Gauß-Manin connection:

$$\nabla\colon \mathrm{H}^i_{\mathrm{cris}}(X/W_t(Y)) \longrightarrow W_t\Omega^1_{Y/S} \otimes_{W(\mathcal{O}_Y)} \mathrm{H}^i_{\mathrm{cris}}(X/W_t(Y)) \ .$$

The nilpotence of this connection follows from [K] since it is by definition a question for $t = 1$. Since we are in the case of relative dimension 1 it is automatically integrable.

# 1 Witt–Differentials modulo $I_R$

Let $A$ be an $R$–algebra. Let $t$ be a natural number. We set $I_R = {}^V W_{t-1}(R) \subset W_t(R)$. To prove the proposition we filter the Witt-differentials $W_t\Omega_{A/R}$ by powers of the ideal $I_R$. We assume that $p$ is nilpotent in $R$. Then the ideal $I_R$ is nilpotent.

We define

$$\mathcal{W}_t\Omega^{\bullet}_{A/R} = W_t\Omega^{\bullet}_{A/R}/I_R W_t\Omega^{\bullet}_{A/R}$$

This is a differential graded algebra. In particular we have:

$$\mathcal{W}_t\Omega^0_{A/R} = \mathcal{W}_t(A) = W_t(A)/I_R W_t(A) \ .$$

We remark that the Teichmüller representative $x \mapsto [x]$ induces a ring homomorphism $R \to \mathcal{W}_t(A)$. Hence $\mathcal{W}_t\Omega^\bullet_{A/R}$ is a complex of $\mathcal{W}_t(A)$–modules with an $R$–linear differential

$$d \colon \mathcal{W}_t\Omega^l_{A/R} \longrightarrow \mathcal{W}_t\Omega^{l+1}_{A/R}$$

We denote the image of

$$^V W\Omega^\bullet_{A/R} \longrightarrow \mathcal{W}_t\Omega^\bullet_{A/R}$$

by $\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R}$ and in degree 0 simply $\mathcal{I}_t = \mathcal{I}\mathcal{W}_t\Omega^0_{A/R} = {}^V W_{t-1}(A)/I_R W_t(A)$ (note that there is no Verschiebung on $\mathcal{W}_\bullet\Omega_{A/R}$, and we have an action of $F$ only if $pR = 0$). $\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R}$ is an ideal in the differential graded algebra $\mathcal{W}_t\Omega^\bullet_{A/R}$. We obviously have

$$(\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R})^2 = 0 \,,$$

because $^V\omega \cdot {}^V\eta = p \cdot {}^V(\omega\eta) = {}^V 1 \cdot {}^V(\omega\eta)$.

We define a second ideal $\mathcal{F}\mathcal{W}_t\Omega^\bullet_{A/R}$ as the kernel of the homomorphism of differential graded algebras

(1.1)  $$0 \longrightarrow \mathcal{F}\mathcal{W}_t\Omega^\bullet_{A/R} \longrightarrow \mathcal{W}_t\Omega^\bullet_{A/R} \longrightarrow \Omega^\bullet_{A/R} \longrightarrow 0 \,.$$

We have

(1.2)  $$\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R} + d\,\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R} \subseteq \mathcal{F}\mathcal{W}_t\Omega^\bullet_{A/R} \,.$$

If $A/R$ is smooth, we will see that (1.2) is a direct sum decomposition and that the differential $d$ is injective, i.e. we have

$$\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R} \oplus d\,\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R} = \mathcal{F}\mathcal{W}_t\Omega^\bullet_{A/R}$$

and there is an exact sequence
(1.3)
$$0 \longrightarrow \mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R} \overset{d}{\longrightarrow} \mathcal{W}_t\Omega^\bullet_{A/R}/\mathcal{I}\mathcal{W}_t\Omega^\bullet_{A/R} \longrightarrow \Omega^\bullet_{A/R} \longrightarrow 0 \,.$$

A Witt-lift $B/W_t(R)$ with $B \to W_t(A)$ defines a map $\Omega^\bullet_{B/W_t(R)} \to W_t\Omega^\bullet_{A/R}$ which induces – by reduction modulo $I_R$ – a map $\Omega^\bullet_{A/R} \to \mathcal{W}_t\Omega^\bullet_{A/R}$, hence a Witt-lift defines a splitting of the sequence (1.1). In particular a Witt-lift defines (for $A/R$ smooth) a direct decomposition

(1.4)  $$\begin{aligned} \mathcal{W}_t\Omega^l_{A/R} &= \mathcal{I}\mathcal{W}_t\Omega^l_{A/R} \oplus d\,\mathcal{I}\mathcal{W}_t\Omega^{l-1}_{A/R} \oplus \Omega^l_{A/R} \\ \omega &= \quad \eta_1 \quad + \quad d\eta_2 \quad + \quad \eta_3 \quad . \end{aligned}$$

We write $\delta\omega = \eta_2$. Let $\mathrm{pr} \colon \mathcal{W}_t\Omega^l_{A/R} \to \mathcal{W}_t\Omega^l_{A/R}$ be the projection on $\Omega^l_{A/R}$. Then $\mathrm{pr}$ is homotopic to the identity:

$$(\mathrm{id} - \mathrm{pr})(\omega) = d\delta\omega + \delta d\omega \,.$$

Hence the complexes $\mathcal{W}_t\Omega^\bullet_{A/R}$ and $\Omega^\bullet_{A/R}$ are homotopy equivalent.

*Remark.* The differential $d\colon \mathcal{IW}_t\Omega^\bullet_{A/R} \to \mathcal{FW}_t\Omega^\bullet_{A/R}/\mathcal{IW}_t\Omega^\bullet_{A/R}$ is a homomorphism of $\mathcal{W}_t\Omega^\bullet_{A/R}$–modules for the action from the right:

$$d(^V\eta\,\omega) = d\,^V\eta\,\omega + (-1)^{\deg\,\eta}\,{}^V\eta\,d\omega \equiv d\,^V\eta\,\omega \quad \mathrm{mod}\ \mathcal{IW}_t\Omega^\bullet_{A/R}\,.$$

It remains to prove the following.

**Lemma 1.** *If $A/R$ is smooth, then we have*

$$\mathcal{IW}_t\Omega^\bullet_{A/R} \oplus d\,\mathcal{IW}_t\Omega^\bullet_{A/R} = \mathcal{FW}_t\Omega^\bullet_{A/R}\,.$$

*Proof.* By [L–Z] (2.43) it suffices to show that the sum is a direct sum. For this it suffices to show that the $W_t(A)$–module homomorphism

(1.5) $$d\colon \mathcal{IW}_t\Omega^\bullet_{A/R} \longrightarrow \mathcal{FW}_t\Omega^\bullet_{A/R}/\mathcal{IW}_t\Omega^\bullet_{A/R}$$

is an isomorphism.

Let $P = R[T_1, \ldots, T_d]$ be a polynomial algebra. Assume that $A$ is étale over $P$. We obtain (1.5) from the corresponding morphism for $P$ by tensoring with the étale morphism $W_t(P) \to W_t(A)$. Hence we may assume $A = R[T_1, \ldots, T_d]$. In this case each element of $W_t\Omega^l_{A/R}$ is a unique sum of basic Witt differentials $e(\xi, k, I_0, \ldots, I_l)$ by [L–Z] Proposition 2.17. Recall that a weight $k : [1, d] \in \mathbb{Z}_{\geq 0}[1/p]$ in such a sum has denominator $p^u \leq p^t$. We denote by $\overline{e}(\xi, k, I_0, \ldots, I_l)$ the images of the basic Witt differentials in $\mathcal{W}_t\Omega^l_{A/R}$. Then the claim in Lemma 1 follows from the following. $\square$

**Lemma 2.** *Let $\omega \in \mathcal{W}_t\Omega^l_{P/R}$. Then $\omega$ has a unique representation*

$$\omega = \sum_{k\ integral} \overline{e}(\xi_k, k, I_0, \ldots, I_l) + \sum_{k\ frac} \overline{e}(\xi_k, k, I_0, \ldots, I_l)$$

*with $\xi_k \in R$ for $k$ integral and $\xi_k \in {}^{V^u}W_{t-u}(R)/p\,{}^{V^u}W_{t-u}(R)$ for $k$ fractional such that $p^u$ is the denominator of $k$. The elements in $\mathcal{IW}_t\Omega^l_{A/R}$ are the elements with a representation*

$$\omega = \sum_{\substack{I_0\neq\varnothing \\ k\ frac}} \overline{e}(\xi_k, k, I_0, \ldots, I_l)\,.$$

*Proof.* Lemma 2 follows from loc.cit. Prop. 2.17 in conjunction with Proposition 2.5. $\square$

# 2   Integrable connections on $\mathcal{W}_t(A)$–modules

Let $\mathcal{M}$ be a free $\mathcal{W}_t(A)$–module of rank $n$. We consider a connection

$$\nabla \colon \mathcal{M} \longrightarrow \mathcal{W}_t\Omega^1_{A/R} \otimes_{\mathcal{W}_t(A)} \mathcal{M}$$

and call $\nabla$ integrable, if $\nabla^2 \colon \mathcal{M} \to \mathcal{W}_t\Omega^2_{A/R} \otimes_{\mathcal{W}_t(A)} \mathcal{M}$ vanishes. Let $m_1, \dots, m_n$ be a basis of $\mathcal{M}$. We write

$$(2.1) \qquad\qquad \nabla \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \Theta \otimes \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

where $\Theta$ is a $n \times n$–matrix with entries in $\mathcal{W}_t\Omega^1_{A/R}$. We abbreviate (2.1) as $\nabla(\underline{m}) = \Theta \otimes \underline{m}$, and we call $\Theta$ the connection matrix.

**Lemma 3.** *Let $(\mathcal{M}, \nabla)$ be an integrable connection. Then there is a basis $\underline{m}$ of $\mathcal{M}$, such that the connection matrix has coefficients in $\Omega^1_{A/R} \subset \mathcal{W}_t\Omega^1_{A/R}$.*

*Proof.* We first show the analogous claim for the induced connection

$$\overline{\nabla} \colon \mathcal{M} \longrightarrow (\mathcal{W}_t\Omega^1_{A/R}/\mathcal{I}\mathcal{W}_t\Omega^1_{A/R}) \otimes_{\mathcal{W}_t(A)} \mathcal{M} \ .$$

By (1.4) the connection matrix $\overline{\Theta}$ of $\overline{\nabla}$ is of the form $\overline{\Theta} = \omega + d\rho$ with $\omega \in M(n \times n, \Omega^1_{A/R})$ and $\rho \in M(n \times n, \mathcal{I})$. Then define $\underline{m}' = (1 - \rho)\underline{m}$. We get

$$(2.2) \qquad \begin{aligned} \overline{\nabla}\underline{m}' &= d(1 - \rho) \otimes \underline{m} + (1 - \rho)\omega \otimes \underline{m} + (1 - \rho)d\rho \otimes \underline{m} \\ &= \omega \otimes \underline{m}' - \rho d\rho \otimes \underline{m} \equiv \omega \otimes \underline{m}' \quad \mathrm{mod}\ \mathcal{I}\mathcal{W}_t\Omega^1_{A/R} \otimes \mathcal{M} \ . \end{aligned}$$

Hence we can find a basis of $\mathcal{M}$ such thet $\overline{\Theta}$ has coefficients in $\Omega^1_{A/R}$. Then the connection matrix $\Theta$ with respect to this basis has the form $\Theta = \omega + \eta$ with $\omega \in M(n \times n, \Omega^1_{A/R})$ and $\eta \in M(n \times n, \mathcal{I}\mathcal{W}_t\Omega^1_{A/R})$. Then we have

$$\nabla^2\underline{m} = d\omega \otimes \underline{m} + d\eta \otimes \underline{m} + \omega^2 \otimes \underline{m} + \omega\eta \otimes \underline{m} + \eta\omega \otimes \underline{m} + \eta^2 \otimes \underline{m} \ .$$

We consider the matrix of this map $\nabla^2 \colon \mathcal{M} \to \mathcal{W}_t\Omega^2_{A/R} \otimes \mathcal{M}$ corresponding to the decomposition (1.4)

$$(2.3) \qquad\qquad (\omega\eta + \eta\omega) \oplus (d\eta) \oplus (d\omega + \omega^2) \ .$$

We remark that $\eta^2 = 0$ because $(\mathcal{I}\mathcal{W}_t\Omega_{A/R})^2 = 0$. Since $\nabla$ is integrable, all summands in (2.3) are zero, in particular $d\eta = 0$, hence $\eta = 0$. So the connection matrix $\Theta$ has coefficients in $\Omega^1_{A/R}$. Lemma 3 follows. $\qquad\square$

# 3  Proof of Proposition 1

For $A/R$ smooth we consider the filtration $I_R^m W_t \Omega_{A/R}^\bullet$ of the de Rham–Witt complex. We set

$$\mathcal{W}_t^{(l)}\Omega_{A/R}^\bullet = W_t\Omega_{A/R}^\bullet / I_R^l W_t\Omega_{A/R}^\bullet$$

and

$$\mathcal{G}_l^\bullet = I_R^l W_t\Omega_{A/R}^\bullet / I_R^{l+1} W_t\Omega_{A/R}^\bullet \ .$$

Then we have a canonical map

$$(3.1) \qquad\qquad p^l \colon \mathcal{IW}_t\Omega_{A/R}^\bullet \longrightarrow \mathcal{G}_l^\bullet \ .$$

Assume that $B \to W_t(A)$ is a Witt-lift over $W_t(R)$ and $\Omega_{B/W_t(R)}^\bullet \to W_t\Omega_{A/R}^\bullet$ the induced map. Similarly to (1.4) we find a direct sum decomposition

$$(3.2) \qquad \mathcal{G}_l^\bullet = p^l\, \mathcal{IW}_t\Omega_{A/R}^\bullet \oplus p^l d\mathcal{IW}_t\Omega_{A/R}^{\bullet-1} \oplus I_R^l/I_R^{l+1} \otimes_R \Omega_{A/R}^\bullet \ .$$

We start with the connection $\nabla \colon M \to W_t\Omega_{A/R}^1 \otimes_{W_t(A)} M$ of Proposition 1. This induces connections on $\mathcal{M}_l = M/I_R^l M$

$$\nabla^{(l)} \colon \mathcal{M}_l \longrightarrow \mathcal{W}_t^{(l)}\Omega_{A/R} \otimes_{\mathcal{W}_t^{(l)}(A)} \mathcal{M}_l \ .$$

Assume we have a basis $\underline{\widetilde{m}}$ of $\mathcal{M}_l$ such that

$$(3.3) \qquad \nabla^{(l)}\underline{\widetilde{m}} = \Theta \otimes \underline{\widetilde{m}} \quad \text{with} \quad \Theta \in M(n \times n, \Omega_{B/W_t(R)}^1 / I_R^l \Omega_{B/W_t(R)}^1) \ .$$

Since the existence of such a base was shown for $l = 1$ in section 2 it is enough to show the following:

*Claim.* There is a lifting $\underline{m}$ of $\underline{\widetilde{m}}$ to $\mathcal{M}_{l+1}$ such that (3.3) holds for $l + 1$.

*Proof.* Let $\underline{m}$ be any lifting. Then (3.2) implies

$$\nabla^{(l+1)}\underline{m} = \Theta \otimes \underline{m} + p^l \eta \otimes \underline{m} + p^l d\rho \otimes \underline{m}$$

with $\rho, \eta$ having coefficients in $\mathcal{IW}_t\Omega^{0 \text{ or } 1}$ and $\Theta$ in $\Omega_{B/W_t(R)}^1 / I_R^{l+1}\Omega_{B/W_t(R)}^1$. Now we consider the induced connection

$$\widetilde{\nabla}^{(l+1)} \colon \mathcal{M}_{l+1} \longrightarrow \mathcal{W}_t^{(l+1)}\Omega_{A/R}/p^l\, \mathcal{IW}_t\Omega_{A/R}^\bullet \otimes_{\mathcal{W}_t^{(l+1)}(A)} \mathcal{M}_{l+1} \ .$$
$$\widetilde{\nabla}^{(l+1)}(\underline{m}) = \Theta \otimes \underline{m} + p^l d\rho \otimes \underline{m} \ .$$

Choose $\underline{m}' = (1 - p^l\rho)\underline{m}$ as new basis. As in (2.2) we can write

$$\nabla^{(l+1)}\underline{m}' = \Theta \otimes \underline{m}' + p^l \eta \otimes \underline{m}' \ .$$

8

We compute

$$(\nabla^{(l+1)})^2 \underline{m}' = (d\Theta + \Theta^2) \otimes \underline{m}' + p^l(\Theta\eta + \eta\Theta) \otimes \underline{m}' + p^l d\eta \otimes \underline{m}' \ .$$

Since $\nabla^{(l)}$ is integrable, we have $d\Theta + \Theta^2 \in M(n \times n, I_R^l/I_R^{l+1} \otimes_R \Omega_{A/R}^2)$. Since $p^l(\Theta\eta + \eta\Theta)$ has coefficients in $p^l \mathcal{I} \mathcal{W}_t \Omega_{A/R}^2$, the decomposition (3.2) implies that $p^l d\eta = 0$ and hence $p^l \eta = 0$. The claim follows. $\qquad\square$

# 4 The Gauß–Manin connection in the smooth case

Consider ring homomorphisms $R \to S \to T$. In the graded pro-algebra $W_t\Omega_{T/R}^\bullet$ we can consider the ideal $\mathcal{I}_t$ generated by all $ds$ for $s \in W_t(S)$. Then $\mathcal{I}_t$ is invariant under $d$, and the pro-ideal $\{\mathcal{I}_t\}$ is invariant under $F$ and $V$. We show the invariance under $F$. A general element of $\mathcal{I}_t$ is a sum of elements of the form $\omega ds$ with $\omega \in W_t\Omega_{T/R}$ and $s \in W_t(S)$. Let $s = (x_0, x_1, x_2, \dots)$ and $\rho = (x_1, x_2, x_3, \dots)$. Then ${}^F ds = [x_0^{p-1}]d[x_0] + d\rho$ and we have

$$^F(\omega\, ds) = {}^F\omega\, {}^F ds = {}^F\omega\, [x_0^{p-1}]d[x_0] + {}^F\omega\, d\rho \ .$$

This is an element in $\mathcal{I}_{t-1}$.

The canonical epimorphism $W_t\Omega_{T/R}^\bullet \to W_t\Omega_{T/S}^\bullet$ factors obviously through a map

$$(4.1) \qquad\qquad W_t\Omega_{T/R}^\bullet/\mathcal{I}_t \longrightarrow W_t\Omega_{T/S}^\bullet \ .$$

The properties of $\mathcal{I}_t$ show that the left hand side is a $F$–$V$–pro-complex in which $d$ becomes a $W(S)$–linear map. Hence the map (4.1) has a section by the universal property of $W_t\Omega_{T/S}^\bullet$. This section is surjective, because it maps $dt$, for $t \in W_t(T)$, to the same element in $W_t\Omega_{T/R}^\bullet$, and these differentials generate $W_t\Omega_{T/R}^\bullet$. Therefore the map (4.1) is an isomorphism and we have an exact sequence

$$0 \longrightarrow \mathcal{I}_t \longrightarrow W_t\Omega_{T/R}^\bullet \longrightarrow W_t\Omega_{T/S}^\bullet \longrightarrow 0 \ .$$

We also have a sequence of maps, in fact a complex for each $k \geq 1$

$$(4.2) \qquad W_t\Omega_{S/R}^1 \otimes_{W_t(S)} W_t\Omega_{T/R}^{k-1} \longrightarrow W_t\Omega_{T/R}^k \longrightarrow W_t\Omega_{T/S}^k \longrightarrow 0 \ .$$

Let now $S/R$ be smooth of relative dimension 1, hence $W\Omega_{S/R}^i = 0$ for $i \geq 2$. This implies that the maps $W_t\Omega_{S/R}^1 \otimes_{W_t(S)} \mathcal{I}_t \to W_t\Omega_{T/R}^\bullet$ are zero (because $ds_1 ds_2 = 0$ for $s_1, s_2 \in W_t(S)$). The sequence (4.2) factors through a sequence of complexes

$$(4.3) \qquad W_t\Omega_{S/R}^1 \otimes_{W(S)} W_t\Omega_{T/S}^\bullet[-1] \longrightarrow W_t\Omega_{T/R}^\bullet \longrightarrow W_t\Omega_{T/S}^\bullet \ .$$

From now on we assume that $p$ is nilpotent in $R$, and that the Frobenius morphism on $R/pR$ is a finite map. Moreover all morphisms $R \to S \to T$ will be smooth and $S/R$ will be of relative dimension 1. In the following we will prove that (4.3) becomes a distinguished triangle when we replace the tensor product on the left hand side by the derived tensor product. More precisely we show:

**Proposition 2.** *Under the above assumptions the sequence (4.3) defines a distinguished triangle in the derived category of $W_n(R)$-modules.*

$$(4.4) \qquad W_t\Omega^1_{S/R} \overset{\mathbb{L}}{\otimes}_{W(S)} W_t\Omega^\bullet_{T/S}[-1] \longrightarrow W_t\Omega^\bullet_{T/R} \longrightarrow W_t\Omega^\bullet_{T/S} \,,$$

The question is local for the étale topology on $\mathrm{Spec}\, T$. To see this we choose a number $m$ such that $p^m$ annihilates $W_t(R)$ and consequently all modules in the triangle above. The individual modules in the complexes involved are $W_t(T)$-modules. We consider them as $W_{t+m}(T)$-modules via restriction of scalars $F^m : W_{t+m}(T) \to W_t(T)$. Then the complexes involved are complexes of $W_{t+m}(T)$-modules with linear differentials. Let $T \to T'$ be a faithfully flat étale morphism. Then the same holds for $W_{t+m}(T) \to W_{t+m}(T')$. The étale base change for the de Rham-Witt complex shows that

$$W_{t+m}(T') \otimes_{W_{t+m}(T)} W_t\Omega^\bullet_{T/R} \cong W_t\Omega^\bullet_{T'/R}$$

as complexes in the category of $W_{t+m}(T')$-modules. Together with the corresponding fact for $T/R$ replaced by $T/S$ this shows that the assertion is local with respect to the étale topology on $\mathrm{Spec}\, T$. Hence we may assume that the map $S \to T$ of $R$-algebras fits into a square

$$(4.5) \qquad \begin{array}{ccc} T & \longleftarrow & S \\ \uparrow & & \uparrow \\ R[T_1, \dots T_d, X] & \longleftarrow & R[X], \end{array}$$

where the vertical maps are étale. Let $S' = R[X]$. We first reduce to the case that $S = S'$. Suppose that (4.4) holds for the diagram of rings $R \to S' \to T$. Since any $W_t(S')$–linear derivation on $W_t(T)$ vanishes on $W_t(S)$ (because $W_t(S)$ is étale over $W_t(S')$) it follows from the universal property of $\{W_t\Omega^\bullet_{T/S}\}$ as $F$–$V$–pro-complex, that $W_t\Omega^\bullet_{T/S} = W_t\Omega^\bullet_{T/S'}$. The left hand side in (4.4) may be rewritten as follows:

$$W_t\Omega^1_{S/R} \overset{\mathbb{L}}{\otimes}_{W(S)} W_t\Omega^\bullet_{T/S} = (W_t\Omega^1_{S'/R} \otimes_{W_t(S')} W_t(S)) \overset{\mathbb{L}}{\otimes}_{W(S)} W_t\Omega^\bullet_{T/S}$$

$$= W_t\Omega^1_{S'/R} \overset{\mathbb{L}}{\otimes}_{W(S')} W_t\Omega^\bullet_{T/S} = W_t\Omega^1_{S'/R} \overset{\mathbb{L}}{\otimes}_{W(S')} W_t\Omega^\bullet_{T/S'} \,.$$

10

Hence we may assume that $S = R[X]$. By the étale base change property of the de Rham–Witt complex we may also assume $T = S[Y_1, \ldots, Y_d]$ is a polynomial algebra as well. We will show that the derived tensor product in the triangle (4.4) is quasi-isomorphic to the kernel of the second map.

Let $\widetilde{T} = W_t(S)[Y_1, \ldots, Y_d]$ be the canonical Witt–lift of $T$ with the map $\widetilde{T} \xrightarrow{\sigma} W_t(T)$, $Y_i \mapsto [Y_i]$. By [L–Z] (3.8) this induces a quasi-isomorphism $\Omega^\bullet_{\widetilde{T}/W_t(S)} \to W_t\Omega^\bullet_{T/S}$. Hence we obtain a morphism:

$$(4.6) \qquad W_t\Omega^1_{S/R} \otimes_{W_t(S)} \Omega^\bullet_{\widetilde{T}/W_t(S)}[-1] \to W_t\Omega^1_{S/R} \otimes_{W_t(S)} W_t\Omega^\bullet_{T/S}[-1]$$

This arrow represents the derived tensor product $W_t\Omega^1_{S/R} \overset{\mathbb{L}}{\otimes}_{W(S)} W_t\Omega^\bullet_{T/S}[-1]$. From 4.6 we obtain a morphism

$$(4.7) \qquad W_t\Omega^1_{S/R} \otimes_{W_t(S)} \Omega^\bullet_{\widetilde{T}/W_t(S)}[-1] \to W_t\Omega^\bullet_{T/R},$$

which factors through $\mathcal{I}_t$. Therefore we have to show that 4.7 induces a quasi-isomorphism.

$$(4.8) \qquad W_t\Omega^1_{S/R} \otimes_{W_t(S)} \Omega^\bullet_{\widetilde{T}/W_t(S)}[-1] \to \mathcal{I}_t.$$

We consider Witt differentials in $\mathcal{I}_t$ of the following type: To denote basic Witt differentials in $W_t\Omega^\bullet_{T/R}$ we use weight functions $k : [0, d] \to \mathbb{Z}_{\geq 0}[1/p^{t-1}]$, where the argument $0$ corresponds to the variable $X$. We will consider basic Witt differentials $e(\xi, k, \mathcal{P}) = e_t(\xi, k, \mathcal{P})$ as in [L–Z] Prop. 2.17.

**Definition 1.** *Let $T$ be a polynomial algebra over $R$. We say a differential $\omega \in W_t\Omega^\bullet_{T/R}$ is of weight $k$ if it is a sum of basic Witt differentials of the type $e(\xi, k, \mathcal{P})$.*

The $W_t(R)$-module $W_t\Omega^\bullet_{T/R}$ is the direct sum of its graded pieces $W_t\Omega(k)^\bullet_{T/R}$. The operators $F$, $V$, $d$ respect this graduation as follows ([L–Z] Prop. 2.5):

$$\begin{array}{rcl}
F : W_t\Omega(k)^\bullet_{T/R} & \to & W_t\Omega(pk)^\bullet_{T/R} \\
V : W_t\Omega(k)^\bullet_{T/R} & \to & W_t\Omega(k/p)^\bullet_{T/R} \\
d : W_t\Omega(k)^\bullet_{T/R} & \to & W_t\Omega(k)^\bullet_{T/R}
\end{array}$$

A differential $\omega$ in the integral part may be indentified with a differential of the de Rham complex $\Omega_{W(R)[X,\underline{T}]/W(R)}$. In this case the concept of weight above coincides with the usual concept for the de Rham complex of a polynomial algebra (compare [L–Z] §2.1 and 3.3).

**Lemma 4.** *Let $\omega_1, \omega_2 \in W_t\Omega^\bullet_{T/R}$ be differentials of weight $k_1$ respectively $k_2$. Then the product $\omega_1\omega_2$ has weight $k_1 + k_2$.*

*Proof.* By the remark above this is clear if both weights $k_1$ and $k_2$ are integral. In the general case we assume that $\omega_i$ for $i = 1, 2$ are basic Witt differentials. We write $\omega_i = {}^{V^{u_i}}\omega_i'$ or $d\,{}^{V^{u_i}}\omega_i'$ for $i = 1, 2$ and $\omega_1'$ and $\omega_2'$.

Let us begin with the case $\omega_1 = {}^{V^{u_1}}\omega_1'$ and $\omega_2 = {}^{V^{u_2}}\omega_2'$. Without loss of generality we may assume that $u_1 \geq u_2$. Our assertion follows from the formula:

$$\omega_1\omega_2 = {}^{V^{u_1}}\big(\omega_1' p^{u_2}\,{}^{F^{u_1-u_2}}\omega_2'\big)$$

Next we show that the expression ${}^{V^{u_1}}\omega_1' d\,{}^{V^{u_2}}\omega_2'$ is of weight $k_1 + k_2$. Using the Leibniz rule and what we already proved, we may assume that $u_1 \geq u_2$. Then the expression becomes:

$${}^{V^{u_1}}\big(\omega_1'\,{}^{F^{u_1-u_2}}d\omega_2'\big).$$

The assertion follows from the integral case. The rest of the proof is similiar. $\qquad\square$

From now on we assume $k(0) = 0$, i.e. $X$ doesn't appear in the basic Witt differentials. We will denote by $l \in \mathbb{Z}[1/p^{t-1}]$ a number with denominator $p^{u(l)}$, such that $u(l) \geq 0$.

Then we consider Witt differentials of the following 3 types:

a) $e(\xi, k, \mathcal{P})\,{}^{F^r}d[X^s]$, where $r, s \in \mathbb{Z}_{\geq 0}$, $s$ not divisible by $p$, $\xi \in W_t(R)$.

b) $e(\xi, k, \mathcal{P})d\,{}^{V^{u(l)}}[X^{p^{u(l)}l}]$, where $u(k) \geq u(l) > 0$

c) $e(1, k, \mathcal{P})d\,{}^{V^{u(l)}}(\eta[X^{p^{u(l)}l}])$, where $u(k) < u(l)$, $\eta \in W_{t-u(l)}(R)$.

**Lemma 5.** *Each element in $\mathcal{I}_t$ is a unique sum of elements of the type above.*

*Proof.* We begin showing that an element of $\mathcal{I}_t$ may be expressed as a sum of elements of type a),b),c). $\mathcal{I}_t$ is generated as an abelian group by elements of the form $\omega ds$ where $\omega \in W_t\Omega^{\bullet}_{T/R}$ is a basic Witt differential and $s \in W_t(S)$. If $\omega$ doesn't contain the variable $X$ we are almost done, because an element $ds$ may be written by the one-dimensional case of [L–Z] loc.cit. in the form

$$(4.9) \qquad\qquad \xi\,{}^{F^r}d[X^s], \quad \text{or} \quad d\,{}^{V^{u(l)}}(\eta[X^{p^{u(l)}l}])$$

Assume for example that $ds$ is of the second type in (4.9) and $u(k) \geq u(l)$. We may write $\omega = {}^{V^{u(k)}}\omega_1$. We write:

$$\omega ds = {}^{V^{u(l)}}\big({}^{V^{u(k)-u(l)}}\omega_1 d\eta[X^{p^{u(l)}l}]\big)$$

This permits to place the constant $\eta$ in the Witt differential $\omega$. Hence we obtain the type b) above. The type c) is obtained in a similiar way.

Next we consider the case, where $\omega$ contains the variable $X$. Let us assume that $\omega$ is integral, i.e. is in the de Rham complex of the lifting $W_t(R)[X, \underline{T}]$. In this case $\omega = [X]^e \omega'$ or $\omega = {}^{F^t}d([X]^e[\underline{Y}]^{\underline{f}})\omega'$. If we have the first equality we just have to write $[X]^e ds$ as a sum of basic Witt differentials. If the second equality holds we find by the Leibniz rule:

$$ {}^{F^t}d([X]^e[\underline{Y}]^{\underline{f}})ds = {}^{F^t}d([\underline{Y}]^{\underline{f}}) \, {}^{F^t}[X]^e ds $$

This expression may be written as a sum of Witt differentials of type a). Now assume that $\omega$ is not integral and write $\omega = {}^{V^u}\omega'$ of $d \, {}^{V^u}\omega'$ for some integral $\omega'$. If the first equation holds we write:

$$ \omega ds = {}^{V^u}(\omega' \, {}^{F^u}ds) $$

We apply what we proved to the expression in brackets. Then we remark that $V^u$ applied to an element of type a),b),c). is again an element of this type by standard formulas [L–Z] Definition 1.4 and (1.16). By the former case this expression may be written in the form $\omega_1 ds_1$, where $\omega_1$ doesn't contain the variable $X$. The case $\omega = d \, {}^{V^u}\omega'$ is similiar.

To show the uniqueness we remark that the elements in the list a), b), c) above are all homogenous for the weight graduation by Lemma 4. Therefore we have to show that for a fixed weight a sum of these elements over different weights can't be zero, if not all members of this sum are zero.

Consider the case a) and assume that $k$ is integral. Note that $r, s$ are fixed in the sum. Since we are in the de Rham complex of a polynomial algebra the multiplication by ${}^{F^r}d[X^s]$ can't kill a differential form, which doesn't depend on $X$. By the independence of the $e(\xi, k, \mathcal{P})$ we conclude that all members of the sum must be zero. If $k$ is not integral we have a sum of basic Witt differentials of type ${}^{V^u}e(\xi, k', \mathcal{P})$ and $d \, {}^{V^u}e(\xi, k', \mathcal{P})$ with $k'$ primitive integral multiplied with ${}^{F^r}d[X^s]$. Assume that this expression is zero and apply $d$. Then we obtain for the remaining sum:

$$ d \, {}^{V^{u(k)}}(\sum_{\mathcal{P}} e(\xi_{\mathcal{P}}, k', \mathcal{P}) \, {}^{F^{(r+u)}}d[X^s]) = 0 $$

We claim that the sum is zero. Indeed by Lemma this sum is a sum of basic Witt elements of primitive integral weight. There the operator $d \, {}^{V^{u(k)}}$ acts injectively (compare: [L–Z] Theorem 2.24). Hence by the integral case all terms in this sum are zero. Going back to the situation before applying $d$, we see that no terms ${}^{V^{u(k)}}e(\xi, k', \mathcal{P})$ can appear in our sum. Doing our argument again the case a) follows.

The same argument applies if we want to show our assertion in case b).

Finally we consider the case of a sum of elements of type c). We use the Leibniz rule:

$$e(1,k,\mathcal{P}) d\, ^{V^{u(l)}}(\eta[X^{p^{u(l)}l}]) =$$
$$\pm d(e(1,k,\mathcal{P})\, ^{V^{u(l)}}(\eta[X^{p^{u(l)}l}])) \mp\, ^{V^{u(l)}}(\eta[X^{p^{u(l)}l}]) e(1,k,\mathcal{P})$$

The right hand side is a sum of two basic Witt differentials for different partitions. Indeed for the first summand the first intervall of the partition is zero but for the second not. Therefore in this case the assertion follows from the independence of basic Witt elements. □

We will now finish the proof of Proposition 2. It suffices to show that the map (4.8) is a quasi-isomorphism. We know by [L–Z] Prop. 2.1 that the elements $e(1,k,\mathcal{P}) \in \Omega^\bullet_{\tilde{T}/W_t(S)}$ form a basis over $W_t(S)$, where $k : [1,d] \to \mathbb{Z}_{\geq 0}$ runs over all integral weights. Hence any element in the abelian group $W_t\Omega^1_{S/R} \otimes_{W_t(S)} \Omega^\bullet_{\tilde{T}/W_t(S)}$ is a unique sum of elements of the following type:

1) $\xi\, ^{F^r} d[X^s] \otimes e(1,k,\mathcal{P})$, where $r,s \in \mathbb{Z}_{\geq 0}$, $s$ not divisible by $p$, $\xi \in W_t(R)$.

2) $d\, ^{V^{u(l)}}(\eta[X^{p^{u(l)}l}]) \otimes e(1,k,\mathcal{P})$, where $0 < u(l)$, $\eta \in W_{t-u(l)}(R)$.

These elements are mapped by (4.8) exactly to elements of type a) and c) above with $k$ integral. Therefore an element in the cokernel $\overline{\mathcal{I}_t}$ of (4.8) is a unique sum of elements of type a), b), c) with $k$ not integral. By [L–Z] Prop.2.6 the cycles in the complex $\overline{\mathcal{I}_t}$ are generated by those elements, where the first intervall of the partition $\mathcal{P}$ is empty. But these elements are also boundaries. This proves Proposition 2.

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