Grothendieck-Messing deformation theory for varieties of $K3$-type

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Introduction

Displays were introduced in [Z1] to classify formal p-divisible groups over a ring R such that p is nilpotent. They form a subcategory of an exact tensor category of higher displays constructed in [LZ2]. Such displays arise naturally for a certain class of projective smooth schemes over R (Abelian schemes, K3-surfaces, complete intersections) and equip the crystalline cohomology with an additional structure, in particular the existence of a divided Frobenius which satisfies a relative version of Fontaine's strong divisibility condition.

If R is an artinian local ring with perfect residue field k, a modified theory of displays, called Dieudonné displays which are defined over the small Witt-ring, was introduced in [Z2] and [L] to classify all p-divisible groups over R. The classification is made possible because a Dieudonné defines a crystal locally free modules on $(R/W(k))_{\mathrm{crys}}$. The Dieudonné display of a p-divisible group then controls the deformation theory. In particular the Grothendieck-Messing criterion for lifting p-divisible groups can be explained in this way.

In [LZ2] we have associated to a projective variety X/R whose cohomology has good base change properties a display of higher degree over $W(R)$. We define in this paper under more restrictive conditions on X listed at the beginning of §2 that a 2-display over the small Witt ring $\hat{W}(R)$ associated to X. This can be regarded as an additional structure on the crystalline cohomology $H^2_{\mathrm{crys}}(X/\hat{W}(R))$ (Proposition 18). Let $R' \to R$ be a pd-thickening in the category of local artinian rings with residue field k. We define the notion of a relative 2-display with respect to such a pd-thickening. We obtain a crystal of relative 2-displays which may be regarded as an additional structure on the crystal

$$R' \mapsto H^2_{\mathrm{crys}}(X/\hat{W}(R')). \tag{1}$$

In §3 we define schemes of $K3$-type. The main examples are the Hilbert schemes of zero-dimensional subschemes of $K3$-surfaces denoted by $K3n$ in the literature. We introduce for a scheme of $K3$-type $X \to T$ a Beauville-Bogomolov form (Definition 21) on the de Rham cohomology $H^2_{DR}(X/T)$. It coincides with the usual Beauville-Bogomolov form if $T = \operatorname{Spec}\mathbb{C}$. We prove under mild conditions that this form is horizontal for the Gauss-Manin connection (Proposition 24). In the notations above we obtain for a scheme X of $K3$-type over the artinian ring R perfect pairing on the crystalline cohomology

$$H^2_{\mathrm{crys}}(X/R').$$

In analogy to the Grothendieck-Messing lifting theory we have Theorem 27:

Theorem: The liftings of X to R' correspond to selfdual liftings of the Hodge filtration.

This is proved in the case $R = k$ and $R' = W_n(k)$ for $K3$-surfaces in [De1]. In this case the Beauville-Bogomolov form coincides with the cup product. The Beauville-Bogomolov form makes the crystal 2-displays (1) selfdual.

Let X_0/k be a scheme of $K3$-type such that the Frobnius induces a Frobnius linear bijection on $H^2(X_0/k)$. We say that X_0 is F-ordinary. Let $f : X \to \operatorname{Spec} R$ be a deformation of X_0. We prove that there is a unique functorial extension of the 2-display $H^2_{\text{crys}}(X/\hat{W}(R))$ to a crystal of relative 2-displays (1). In particular the crystal $Rf_{\text{crys},*}\mathcal{O}_X^{\text{crys}}$ in $(X/W(k))$ can be constructed from the 2-display. Then we obtain from the Grothendieck-Messing criterion Theorem 31:

Theorem: Assume that X_0 is F-ordinary. The functor which associates to a deformation X/R of X_0 the Dieudonné 2-display $H^2_{\text{crys}}(X/\hat{W}(R))$ with its Beauville-Bogomolov form is an equivalence to the category of selfdual deformations of the 2-display $H^2_{\text{crys}}(X_0/W(k))$ endowed with the Beauville-Bogomolov form.

In the final chapter we exhibit the second crystalline cohomology of an ordinary K3-surface X over the usual Witt ring $W(R)$ and its associated display and prove (Theorem 34) a Hodge-Witt decomposition which induces a decomposition of the display into a direct sum of displays attached to the formal Brauer group $\hat{\mathrm{Br}}_X$, the étale part of the extended Brauer group and the Cartier dual of $\hat{\mathrm{Br}}_X$, shifted by -1. The proof uses the relative de Rham-Witt complex of [LZ2]. We show that the hypercohomology spectral sequence of the relative de Rham-Witt complex attached to the universal family X_B over the deformation ring B degenerates, the same result holds for X itself if the ring R is reduced. In the latter case, the above decomposition of displays follows directly from the relative de Rham-Witt complex, in the general case it is obtained by base change from B to R.

1 Displays

We fix a prime number p. Consider a frame $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma})$. There is a unique element $\theta \in W$, such that $\sigma(\eta) = \theta\dot{\sigma}(\eta)$, for all $\eta \in J$. We will assume that $\theta = p$. We always assume that W is local and that $J + pW$ is contained in the maximal ideal of W.

If $f : M \to N$ is a σ-linear map of W-modules. Then we define a new σ-linear map

$$\tilde{f} : J \otimes_S M \to N, \quad \tilde{f}(\eta \otimes m) = \dot{\sigma}(\eta) f(m), \quad \text{for } \eta \in J.$$

Definition 1. *An $\mathcal{F}$-predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ consists of the following data.*

1. *A sequence of W-modules P_i for $i \geq 0$.*
2. *Two sequences of S-module homomorphisms*

$$\iota_i : P_{i+1} \to P_i, \quad \alpha_i : J \otimes_S P_i \to P_{i+1}, \textit{for } i \geq 0.$$

3. *A sequence of σ-linear maps for $i \geq 0$*

$$F_i : P_i \to P_0.$$

These data satisfy the following properties:

(i) Consider the following morphism:

$$\begin{array}{ccc} J \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \\ {\scriptstyle \mathrm{id}_J \otimes \iota_{i-1}} \downarrow & & \downarrow {\scriptstyle \iota_i} \\ J \otimes P_{i-1} & \xrightarrow[\alpha_{i-1}]{} & P_i \end{array}$$

The composites $\iota_i \circ \alpha_i$ and $\alpha_{i-1} \circ \mathrm{id}_J \otimes \iota_{i-1}$ are the multiplication $J \otimes P_i \to P_i$, for each i where the composites make sense.

(ii) $F_{i+1} \circ \alpha_i = \tilde{F}_i$.

If we have only the data $\mathcal{P} = (P_i, \iota_i, \alpha_i)$ such that the property (ii) holds we say that $\mathcal{P}$ is an $\mathcal{F}$-module.

We will denote the morphisms in the category of $\mathcal{F}$-predisplays resp. in the category of $\mathcal{F}$-modules by

$$\mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, \mathcal{P}'), \quad \textit{resp.} \quad \mathrm{Hom}_{\mathcal{F}-mod}(\mathcal{P}, \mathcal{P}').$$

We require the same compatibilities as in Definition 2.1 of [LZ2]. It is a consequence of these compatabilities that

$$F_i(\iota_i(x)) = pF_{i+1}(x), \quad \text{for } x \in P_{i+1}.$$

If $i, k \geq 0$ we will denote the map

$$\iota_{i+k-1} \circ \ldots \circ \iota_i : P_{i+k} \to P_i$$

simply by ι^{iter}.

We are going to associate a frame to the following situation. Let R and S be a p-adic rings. Let

$$S \to R \tag{2}$$

be a surjective ring homomorphism such that the kernel $\mathfrak{a}$ is endowed with divided powers. We will assume that $\mathfrak{a}$ becomes nilpotent in the ring S/pS.

Let $W(S) \stackrel{\mathbf{w}_0}{\to} S \to R$ be the composite with the Witt polynomial $\mathbf{w}_0$. Let $\mathcal{J}$ be the kernel of this composite. We set $I_S = VW(S)$ and we denote by $\tilde{\mathfrak{a}} \subset W(S)$ the logarithmic Teichmüller representatives of elements of $\mathfrak{a}$. Then we have a direct decomposition of $\mathcal{J}$ as a sum of two ideals of $W(S)$:

$$\mathcal{J} = \tilde{\mathfrak{a}} \oplus I_S$$

We will denote the Frobenius endomorphism F of the ring of Witt vectors $W(S)$ also by σ. We have

$$\sigma(\tilde{\mathfrak{a}}) = 0, \quad I_S \cdot \tilde{\mathfrak{a}} = 0.$$

We define a map

$$\dot{\sigma} : \mathcal{J} \to W(S), \quad \dot{\sigma}(a + {}^V\xi) = \xi, \quad a \in \tilde{\mathfrak{a}},\ \xi \in W(S).$$

We note that the ideal $\mathcal{J}$ inherits from $\mathfrak{a}$ divided powers which extend the natural divided powers on the $I_S \subset W(S)$.

We call $\mathcal{W}_{S/R} = (W(S), \mathcal{J}, R, \sigma, \dot{\sigma})$ the relative Witt frame. If $S = R$ we call it the Witt frame and write $\mathcal{W}_R$. It $S \to R$ is fixed as above we call a $\mathcal{W}_R$-predisplay simply a predisplay and a $\mathcal{W}_{S/R}$-predisplay a relative predisplay.

It S and R are artinian local rings with perfect residue field of characteristic p and if $p \geq 3$ we can also use the small Witt ring. Then we obtain the small relative Witt frame $\hat{\mathcal{W}}_{S/R} = (\hat{W}(S), \hat{\mathcal{J}}, R, \sigma, \dot{\sigma})$, where $\hat{\mathcal{J}}$ is the kernel of the homomorphism $\hat{W}(S) \rightarrow R$.

These frames are endowed with a Verjüngung: Let $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma})$ be a frame. The structure of a Verjüngung on $\mathcal{F}$ consists of two W-module homomorphisms:

$$\nu : J \otimes_W J \rightarrow J, \quad \pi : J \rightarrow J,$$

such that ν is associative. We will also write

$$\nu(y_1 \otimes y_2) = y_1 * y_2, \quad y_1, y_2 \in J.$$

The iteration of ν is well-defined:

$$\nu^{(k)} : J \otimes_W \ldots \otimes_W J \rightarrow J,$$

where the tensor product on the left hand side has k-factors. We have $\nu^{(2)} = \nu$ and $\nu^{(1)} = \mathrm{id}_J$. The image of ν_k is an ideal $J_k \subset W$. We require that $\nu^{(k+1)}$ factors through a map

$$J \otimes_W J_k \rightarrow J_{k+1}, \tag{3}$$

which is necessarily unique.

We also require, that the following properties hold:

$$\begin{array}{ll} 1) & \pi(y_1 * y_2) = y_1 y_2, \qquad \text{where} \quad y_1, y_2 \in J. \\ 2) & \dot{\sigma}(y_1 * y_2) = \dot{\sigma}(y_1)\dot{\sigma}(y_2). \\ 3) & \dot{\sigma}(\pi(y_1)) = \sigma(y_1). \\ 4) & (\operatorname{Ker} \dot{\sigma}) \cap (\operatorname{Ker} \pi) = 0. \end{array} \tag{4}$$

In the case of the frame $\mathcal{W}_{S/R}$ we define the Verjüngung as follows:

$$(a_1 + {}^V\xi_1) * (a_2 + {}^V\xi_2) = a_1 a_2 + {}^V(\xi_1 \xi_2), \qquad \pi(a + {}^V\xi) = a + p\, {}^V\xi. \tag{5}$$

Then we have

$$J_i = \tilde{\mathfrak{a}}^i + VW(S).$$

The map (3) is given by the first formula of (5).

In the same way we obtain a Verjüngung for the frames $\hat{\mathcal{W}}_{S/R}$. These are the only examples we are interested in.

We define the notion of a standard display over a frame $\mathcal{F}$ with Verjüngung ν, π. In the case of $\mathcal{W}_R$ it coincides with the notion given in [LZ2].

A standard datum consists of a sequence of finitely generated projective W-modules $L_0, \ldots, L_d$ and σ-linear homomorphisms

$$\Phi_i : L_i \to L_0 \oplus \ldots \oplus L_d.$$

We assume that

$$\Phi_0 \oplus \ldots \oplus \Phi_d : L_0 \oplus \ldots \oplus L_d \to L_0 \oplus \ldots \oplus L_d.$$

is a σ-linear isomorphism.

We set:

$$P_i = J_i L_0 \oplus J_{i-1} \ldots \oplus J L_{i-1} \oplus L_i \oplus \ldots \oplus L_d.$$

The map ι is defined by the following diagram:

$$\begin{array}{ccccccccc} J_{i+1}L_0 & \oplus & J_iL_1 & \oplus \ldots \oplus & JL_i & \oplus & L_{i+1} & \oplus \ldots \oplus & L_d. \\ \pi \downarrow & & \pi \downarrow & & \mathrm{id} \downarrow & & \mathrm{id} \downarrow & & \mathrm{id} \downarrow \\ J_iL_0 & \oplus & J_{i-1}L_1 & \oplus \ldots \oplus & L_i & \oplus & L_{i+1} & \oplus \ldots \oplus & L_d. \end{array}$$

We remark that $\pi(J_{i+1}) \subset J_i$ because of the formula $\pi(y_1 * y_2 * \ldots * y_{i+1}) = y_1(y_2 * \ldots * y_{i+1})$.

The homomorphism $\alpha_i : \mathcal{J} \otimes P_i \to P_{i+1}$ is defined as follows

$$\begin{array}{ccccccccc} J \otimes J_iL_0 & \oplus & J \otimes J_{i-1}L_1 & \oplus \ldots \oplus & J \otimes L_i & \oplus & J \otimes L_{i+1} & \ldots \oplus & J \otimes L_d \\ \nu \downarrow & & \nu \downarrow & \ldots & \nu \downarrow & & mult \downarrow & & \ldots mult \downarrow \\ J_{i+1}L_0 & \oplus & J_iL_1 & \oplus \ldots \oplus & JL_i & \oplus & L_{i+1} & \ldots \oplus & L_d. \end{array}$$

Here the arrows denoted by ν are induced by the maps (3), and mult denotes the multiplication.

Finally we define σ-linear maps $F_i : P_i \to P_0$:

$$\begin{array}{cccccccccc} J_iL_0 & \oplus \ldots \oplus & JL_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \ldots \\ \tilde{\Phi}_0 \downarrow & \ldots & \tilde{\Phi}_{i-1} \downarrow & & \Phi_i \downarrow & & p\Phi_{i+1} \downarrow & & p^2\Phi \downarrow & \ldots \\ L_0 & \oplus \ldots \oplus & L_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \ldots \end{array}$$

The maps $\tilde{\Phi}_j$ are by definition:

$$\tilde{\Phi}_j(\eta \ell_j) = \dot{\sigma}(\eta)\Phi_j(\ell_j), \quad \text{for } \eta \in J_j,\ \ell_j \in L_j\ j < i.$$

The data $(P_i, \iota_i, \alpha_i, F_i)$ meet the requirements for a predisplay. This predisplay is called the $\mathcal{F}$-display of a standard datum.

Definition 2. *Let $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$ be a frame with Verjüngung. An $\mathcal{F}$-display $\mathcal{P}$ is an $\mathcal{F}$-predisplay, which is isomorphic to the display of a standard datum. The choice of such an isomorphism is called a normal decomposition of $\mathcal{P}$.*

We call an $\mathcal{F}$-predisplay $\mathcal{P}$ separated if the following commutative diagram

$$\begin{tikzcd} P_i \arrow[r, "F_i"] & P_0 \\ P_{i+1} \arrow[u, "\iota_i"] \arrow[r, "F_{i+1}"'] & P_0 \arrow[u, "p"'] \end{tikzcd}$$

induces an injective map form P_{i+1} to the fibre product $P_i \times_{F_i, P_0, p} P_0$.

An $\mathcal{F}$-display is separated because of condition (4).

Proposition 3. *Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be a separated $\mathcal{F}$-predisplay. Let $\mathcal{P}'$ be an $\mathcal{F}$-predisplay. Then the natural map*

$$\mathrm{Hom}_{\mathcal{F}-dsp}(\mathcal{P}', \mathcal{P}) \to \mathrm{Hom}_{W-modules}(P'_0, P_0)$$

from the Hom-*group of homomorphisms of predisplays to the* Hom-*group of homomorphisms of W-modules is injective.*

A $\mathcal{F}$-display $\mathcal{P}$ is a separated $\mathcal{F}$-predisplay.

Let $\mathcal{P}$ be a $\mathcal{F}$-predisplay. Iterating the homomorphisms α_i in Definition 1 we obtain W-module homomorphisms for $i, k \geq 0$.

$$\alpha_i^{(k)} : J \otimes_W J \ldots \otimes_W J \otimes_W P_i \to P_{i+k}. \tag{6}$$

By definition we have $\alpha_i^{(0)} = \mathrm{id}_{P_i}$ and $\alpha_i^{(1)} = \alpha_i$. We say that $\mathcal{P}$ satisfies the condition **alpha** if the map (6) factors through a homomorphism $\bar{\alpha}_i^{(k)}$

$$(\textbf{alpha}): \qquad J \otimes_W \ldots \otimes_W J \otimes_W P_i \xrightarrow{\nu^{(k)} \otimes \mathrm{id}} J_k \otimes_W P_i \xrightarrow{\bar{\alpha}_i^{(k)}} P_{i+k}. \tag{7}$$

Obviously $\bar{\alpha}_i^{(k)}$ is uniquely determined. A display satisfies the condition **alpha**.

Proposition 4. *Let $\mathcal{Q}$ be the display associated to a standard datum (L_i, Φ_i), $i = 0, \ldots d$. Let $\mathcal{P}$ be a predisplay which satisfies the condition (***alpha***).*

Let $\rho : \mathcal{Q} \to \mathcal{P}$ be a morphism of $\mathcal{F}$-modules. We denote for $i \leq d$ the restriction of $\rho_i : Q_i \to P_i$ to $L_i \subset Q_i$ by $\rho_{|i} : L_i \to P_i$. Conversely arbitrary W-module homomorphisms $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$ define uniquely a morphism of $\mathcal{F}$-modules $\rho : \mathcal{Q} \to \mathcal{P}$.

Moreover the morphism of $\mathcal{F}$-modules ρ defined by a sequence of homomorphisms $\rho_{|i} : L_i \to P_i$ is a morphism of predisplays iff the following diagrams are commutative

$$\begin{tikzcd} L_i \arrow[r, "\Phi_i"] \arrow[d, "\rho_{|i}"'] & Q_0 \arrow[d, "\rho_0"] \\ P_i \arrow[r, "F_i"'] & P_0 \end{tikzcd} \tag{8}$$

We remark that the morphism $\rho_0 : Q_0 = \oplus_{i=1}^{d} L_i \to P_0$ is given on the summand L_i as the composite $L_i \subset Q_i \to P_i \overset{\iota^{iter}}{\to} P_0$, where the composition of the first two arrows is $\rho_{|i}$ and the last arrow is the composition $\iota^{iter} = \iota_0 \circ \ldots \circ \iota_{i-1}$.

Proof. We have

$$Q_i = J_i L_0 \oplus \ldots \oplus J_{i-k} L_k \oplus \ldots \oplus J L_{i-1} \oplus L_i \oplus \ldots$$

We will define $\rho : Q_i \to P_i$. We do this for each of the summands above separately. For $k < i$ we obtain by tensoring $\rho_{|k}$ with J_{i-k} a homomorphism

$$J_{i-k} L_k \to J_{i-k} \otimes_W P_k.$$

Composing the last arrow with $\bar{\alpha}_k^{i-k}$ from the condition (**alpha**) we obtain ρ_i on the summand $J_{i-k} L_k$.

For $j \geq i$ the map $\iota^{iter} : Q_j \to Q_i$ induces the identity on L_j. Therefore we define the restriction of ρ_i to the summand L_j as the composite

$$L_j \xrightarrow{\rho_{|j}} P_j \xrightarrow{\iota^{iter}} P_i.$$

One checks that the ρ_i define a morphism of $\mathcal{F}$-modules and if (8) commute a morphism of $\mathcal{F}$-predisplays. □

We will now define the base change of displays. We consider a morphism of frames with Verjüngung $u : \mathcal{F} \to \mathcal{F}'$. Let $\mathcal{P}'$ be an $\mathcal{F}'$-predisplay. This may be regarded as an $\mathcal{F}$-predisplay with the same P'_i but regarded as W-modules. Only the maps α_i need a definition:

$$\alpha_i : J \otimes_W P'_i \to J' \otimes_{W'} P'_i \xrightarrow{\alpha'_i} P'_{i+1}.$$

We denote the $\mathcal{F}$-predisplay display obtained in this way by $u^\bullet \mathcal{P}'$. Let $\mathcal{P}$ be an $\mathcal{F}$-display. We say that an $\mathcal{F}'$-display $u_\bullet \mathcal{P}$ is a base change of $\mathcal{P}$ if there exist for each $\mathcal{F}'$-display $\mathcal{P}'$ a bijection

$$\operatorname{Hom}_{\mathcal{F}'}(u_\bullet \mathcal{P}, \mathcal{P}') \cong \operatorname{Hom}_{\mathcal{F}}(\mathcal{P}, u^\bullet \mathcal{P}').$$

which is functorial in $\mathcal{P}'$.

Proposition 5. *(base change) Let $u : \mathcal{F} \to \mathcal{F}'$ be a morphism of frames with Verjüngung. Then the base change of an $\mathcal{F}$-display $\mathcal{P}$ exist. Moreover for $\mathcal{F}'$-predisplay $\mathcal{P}'$ which satisfy the condition (***alpha***) we have a functorial bijection*

$$\operatorname{Hom}_{\mathcal{F}'}(u_\bullet \mathcal{P}, \mathcal{P}') \cong \operatorname{Hom}_{\mathcal{F}}(\mathcal{P}, u^\bullet \mathcal{P}').$$

Proof. We choose a normal decomposition (L_i, Φ_i) of $\mathcal{P}$. Then a morphism $\rho : \mathcal{P} \to u^\bullet \mathcal{P}'$ is given by a set of W-module homomorphisms

$$\rho_{|i} : L_i \to P'_i,$$

such that the analogues of (8) are commutative. From the standard datum (L_i, Φ_i) we obtain a standard datum $(L'_i = W' \otimes_W L_i, \Phi'_i = \sigma' \otimes \Phi_i)$ for the frame $\mathcal{F}'$ which defines an $\mathcal{F}'$-display $\mathcal{Q}$. From $\rho_{|i}$ we obtain W'-module homomorphisms

$$\rho'_{|i} : W' \otimes_W L_i \to P'_i.$$

These homomorphisms define a morphism of $\mathcal{F}'$-predisplays $\mathcal{Q} \to \mathcal{P}'$ by Proposition (4). This shows that $u_\bullet \mathcal{P} := \mathcal{Q}$ is a direct image and has the claimed property. □

We apply the base change to the following obvious morphisms of frames with Verjüngung

$$\mathcal{W}_S \to \mathcal{W}_{S/R} \to \mathcal{W}_R.$$

More generally let

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & R' \end{array}$$

be a morphism of pd-extension of the type (2). We obtain a morphism of frames with Verjüngung $\mathcal{W}_{S/R} \to \mathcal{W}_{S'/R'}$. We have this for small Witt frames too.

We will give now an intrinsic characterization of a display which doesn't use a normal decomposition. Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay. Then we denote the image of the homomorphism

$$P_i \xrightarrow{\iota^{iter}} P_0 \longrightarrow P_0/JP_0 \tag{9}$$

by E^i or more precisely by $\mathrm{Fil}^i_{\mathcal{P}}$. This is called the Hodge filtration on the R-module P_0/JP_0:

$$\ldots E^{i+1} \to E^i \to \ldots \to E^0 = P_0/JP_0, \tag{10}$$

If $\mathcal{P}$ is a display this is a filtration by direct summands.

Proposition 6. *Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay with Hodge filtration E^i such that the following properties hold:*

1. *$\mathcal{P}$ is separated and satisfies the condition (**alpha**).*

2. *P_0 is a finitely generated projective W-module.*

3. *The Hodge filtration consists of direct summands $E^i \subset P_0/\mathcal{J}P_0$.*

4. *There is an exact sequence*

$$\mathcal{J} \otimes P_i \xrightarrow{\alpha_i} P_{i+1} \to E_{i+1} \to 0$$

5. *The subgroups F_iP_i for $i \geq 0$ generate the W-module P_0.*

Then $\mathcal{P}$ is an $\mathcal{F}$-display.

We omit the proof. We note that the E_i in this Proposition coincide with the Hodge filtration defined for any predisplay.

Let $\mathcal{F} = \mathcal{W}_{S/R}$ or $\mathcal{F} = \hat{\mathcal{W}}_{S/R}$. Let $\mathcal{P}$ be an $\mathcal{F}$-display. A lifting of the Hodge filtration of $\mathcal{P}$ is a sequence of split injections of projective finitely generated S-modules

$$\ldots \tilde{E}^{i+1} \to \tilde{E}^{i} \to \ldots \to \tilde{E}^{0} = P_0/I_S P_0, \tag{11}$$

which coincides with (10) when tensored with R.

We will now discuss the notion of an extended display. Let $\mathcal{F}$ be a frame with Verjüngung. Let (L_i, Φ_i) be a standard datum. If we replace in the definition of the display associated to this datum all J_i simply by J we obtain an $\mathcal{F}$-predisplay $\tilde{\mathcal{P}}$. We consider this notion only for the frames $\mathcal{W}_{S/R}$ or $\hat{\mathcal{W}}_{S/R}$. Let $\tilde{\mathcal{P}}$ be an extended display and let $E^i i$ be its Hodge filtration. Then $\tilde{\mathcal{P}}$ satisfies all conditions of Proposition 6 except for the last condition 4.

We note that there is no difference between displays and extended displays in the case $S = R$ because then $J = J_i$.

Let $\mathcal{Q}$ be a predisplay relative of $\mathcal{W}_{S/R}$ or $\hat{\mathcal{W}}_{S/R}$. For this discussion we denote by $\bar{Q}_i \subset Q_i$ the intersection of all images of maps

$$Q_{i+k} \xrightarrow{\iota^{iter}} Q_i.$$

If $\mathcal{Q}$ is a display then $\bar{Q}_i = 0$ for all i because $W(S)$ is a p-adic ring. If $\mathcal{Q}$ is an extended display we have that the map $\iota^{iter} : Q_i \to Q_0$ induces an isomorphism

$$\iota^{iter} : \bar{Q}_i \to \tilde{\mathfrak{a}} Q_0. \tag{12}$$

Note that for $k > i$ we have that $\tilde{\mathfrak{a}} L_k \subset (\tilde{\mathfrak{a}} \oplus I_S) L_k \subset Q_i$ is a direct summand of $\bar{Q}_i$ which is mapped by (12) isomorphically to $\tilde{\mathfrak{a}} L_k \subset \tilde{\mathfrak{a}} Q_0$.

We note that an extended display satisfies the condition (**alpha**). We have the following version of Proposition 4.

Proposition 7. *We consider predisplays for the frames $\mathcal{F} = \mathcal{W}_{S/R}$ or $\mathcal{F} = \hat{\mathcal{W}}_{S/R}$. Let $\tilde{\mathcal{Q}}$ be the extended display associated to a standard datum (L_i, Φ_i), $i = 0, \ldots d$. Let $\mathcal{P}$ be a predisplay which satisfies the condition (***alpha***) and (12).*

Let $\rho : \tilde{\mathcal{Q}} \to \mathcal{P}$ be a morphism of $\mathcal{F}$-modules. We denote for $i \leq d$ the restriction of $\rho_i : Q_i \to P_i$ to $L_i \subset Q_i$ by $\rho_{|i} : L_i \to P_i$. Conversely arbitrary $W(S)$-module homomorphisms $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$ define uniquely a morphism of $\mathcal{F}$-modules $\rho : \tilde{\mathcal{Q}} \to \mathcal{P}$.

Moreover the morphism of $\mathcal{F}$-modules ρ defined by a sequence of homomorphisms $\rho_{|i} : L_i \to P_i$ is a morphism of predisplays iff the following diagrams are commutative

$$\begin{array}{ccc} L_i & \xrightarrow{\Phi_i} & \tilde{Q}_0 \\ {\scriptstyle \rho_{|i}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_0} \\ P_i & \xrightarrow[F_i]{} & P_0 \end{array} \tag{13}$$

Corollary 8. *Let $\mathcal{Q}$ be the display associated to the standard datum (L_i, Φ_i). Then we have a canonical bijection*

$$\mathrm{Hom}_{\mathcal{F}}(\tilde{\mathcal{Q}}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P}).$$

We conclude that we have a functor $\mathcal{Q} \mapsto \tilde{\mathcal{Q}}$ which does not depend on the normal decomposition.

Let $(L_i \Phi_i)$ be a standard datum for the frame $\mathcal{W}_{S/R}$, let $\mathcal{Q}$ the associated display and $\tilde{\mathcal{Q}}$ the extended display.

Let $\hat{\mathcal{Q}}$ be the $\mathcal{W}_R$-display associated to $(W(R) \otimes_{W(S)} L_i, \sigma \otimes \Phi_i)$. Then $\hat{\mathcal{Q}}$ is the base change of $\mathcal{Q}$ via $\mathcal{W}_{S/R} \to \mathcal{W}_R$. Since a $\mathcal{W}_R$-display $\mathcal{P}$ regarded as a $\mathcal{W}_{S/R}$-predisplay satifies the conditions (**alpha**) and (12) we obtain

$$\mathrm{Hom}_{\mathcal{W}_R}(\hat{\mathcal{Q}}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{W}_{S/R}}(\tilde{\mathcal{Q}}, \mathcal{P}).$$

This shows that we have also a functor $\tilde{\mathcal{Q}} \mapsto \hat{\mathcal{Q}}$. Therefore we have functors

$$(\mathcal{W}_{S/R} - \text{displays}) \to (\mathcal{W}_{S/R} - \text{extended displays}) \to (\mathcal{W}_R - \text{displays}),$$

such that the composition of these functors is base change. The same functors exists if the small Witt frame $\hat{\mathcal{W}}_{S/R}$ is defined.

We have defined what is a lifting of the Hodge filtration for a $\mathcal{W}_{S/R}$-display $\mathcal{P}$. We will now construct the functor

$$\left(\begin{array}{c} \text{extended} - \mathcal{W}_{S/R} - \text{displays} \\ \text{\& a lift of the Hodgefiltration} \end{array} \right) \longrightarrow (\mathcal{W}_S - \text{displays}). \tag{14}$$

Again the construction will be the same for small Witt frames.

Let $\tilde{\mathcal{P}}$ be an extended $\mathcal{W}_{S/R}$-display. Let

$$\ldots \tilde{E}^{i+1} \to \tilde{E}^i \to \ldots \to \tilde{E}^0 = \tilde{P}_0 / I_S \tilde{P}_0,$$

be a lift of the Hodge filtration. We construct in a functorial way a $\mathcal{W}_S$-display $\mathcal{P}$. We denote by $\hat{E}^i \subset \tilde{P}_0/I_S\tilde{P}_0$ the preimage of the Hodge filtration $E^i \subset \tilde{P}_0/J\tilde{P}_0$. By choosing an arbitrary normal decomposition of $\tilde{\mathcal{P}}$ we find that the map

$$\tilde{P}_i \to \tilde{P}_0/I_S\tilde{P}_0$$

has image $\hat{E}^i$.

We choose a splitting of the lifted Hodge filtration and obtain a decomposition into S-submodules of $\tilde{P}_0/I_S\tilde{P}_0$:

$$\tilde{E}^i = \bar{L}_i \oplus \bar{L}_{i+1} \oplus \ldots \oplus \bar{L}_d$$

We choose a finitely generated projective $W(S)$-module L_i which lifts the S-module $\bar{L}_i$ and we choose a commutative diagram:

$$\begin{tikzcd} L_i \arrow[d] \arrow[dr] & \\ \tilde{P}_i \arrow[r] & \hat{E}_i \end{tikzcd}$$

A composite of the ι maps yields $L_i \to \tilde{P}_i \to \tilde{P}_0 = P_0$. We obtain a homomorphism

$$L_0 \oplus L_1 \oplus \ldots \oplus L_m \to P_0.$$

We see that this map is an isomorphism by taking it modulo J.

The maps $F_i : \tilde{P}_i \to \tilde{P}_0$ give by restriction maps $\Phi_i : L_i \to P_i$. We will show that the map

$$L_0 \oplus L_1 \oplus \ldots \oplus L_m \to P_0 \tag{15}$$

is a Frobenius linear isomorphism. Then we obtain standard data (L_i, Φ_i) for $\mathcal{P}$.

To show that (15) is an isomorphism we consider the $\mathcal{W}_R$ display $\bar{\mathcal{P}}$ obtained by base change from $\mathcal{P}$. We have natural maps $P_i \to \tilde{P}_i \to \bar{P}_i$. The images of the L_i in $\bar{P}_i$ give a normal decomposition of that displays. Therefore the map (15) becomes a Frobenius linear isomorphism when tensored with $W(R)$. Then the map itself is a Frobenius linear isomorphism.

The we define the desired $\mathcal{W}_S$-display $\mathcal{P}$ by the standard datum (L_i, Φ_i). Our construction gives that $P_i \subset \tilde{P}_i$ is the preimage of $\tilde{E}^i$ under the map

$$\tilde{P}_i \to P_0/I_SP_0.$$

This shows that the assignment $\tilde{\mathcal{P}} \mapsto \mathcal{P}$ is functorial and does not depend on the construction of the normal decomposition chosen above.

Proposition 9. *The functor (14) defines an equivalence of the category of extended $\mathcal{W}_{S/R}$-displays together with a Hodge filtration and the category of $\mathcal{W}_S$-displays. The same holds for the small Witt frames.*

Proof. Indeed there is an obvious inverse functor. We denote by $u_\bullet : (\mathcal{W}_S - \text{displays}) \to (\text{extended}-\mathcal{W}_{S/R}-\text{displays})$ the functor induced by base change. An extended $\mathcal{W}_{S/R}$-display $\mathcal{P}$ may be regarded a $\mathcal{W}_S$-predisplay. Then we denote it by $u^\bullet\mathcal{P}$. By the Propositions (4) and (7) we have for a $\mathcal{W}_S$-display $\mathcal{Q}$ a functorial bijection:

$$\mathrm{Hom}_{\mathcal{W}_{S/R}}(u_\bullet\mathcal{Q}, \mathcal{P}) \cong \mathrm{Hom}_{\mathcal{W}_S}(\mathcal{Q}, u^\bullet\mathcal{P}).$$

We set $\hat{\mathcal{Q}} = u_\bullet\mathcal{Q}$. The canonical map $\mathcal{Q} \to u^\bullet\hat{\mathcal{Q}}$ induces natural injections $Q_i \to \hat{Q}_i$. This provides a lifting of the Hodge filtration of the extended display $\hat{\mathcal{Q}}$. Clearly this functor is inverse to the functor (14) $\square$

Let $\mathcal{P}$ be a $\mathcal{W}_{S/R}$-display. We say that a lifting of the Hodge filtration $\tilde{E}^i \subset P_0/I_SP_0$ for $i \geq 0$ is admissible if $\tilde{E}^i$ is in the image of $P_i \overset{\iota^{iter}}{\to} P_0/I_SP_0$. If $\mathcal{Q}$ is a $\mathcal{W}_S$-display and $\tilde{\mathcal{Q}}$ is the $\mathcal{W}_{S/R}$-display by base change then we have a natural inclusion $Q_i \to \tilde{Q}_i$. This shows that the induced Hodge filtration on $\tilde{\mathcal{Q}}$ is admissible. From the proof of the last Proposition we obtain:

Corollary 10. *The functor*

$$(\mathcal{W}_S - \textit{displays}) \longrightarrow \begin{pmatrix} \mathcal{W}_{S/R} - \textit{displays with an admissible} \\ \textit{lift of the Hodge filtration} \end{pmatrix}$$

is an equivalence of categories.

We consider a frame with Verjüngung $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$. We consider 2-displays $\mathcal{P}$ and $\mathcal{P}'$ given by standard data $(L_0, L_1, L_2, \Phi_0, \Phi_1, \Phi_2)$, $(L'_0, L'_1, L'_2, \Phi'_0, \Phi'_1, \Phi'_2)$. We assume that the W-modules L_i and L'_i are free. A morphism of displays $\rho : \mathcal{P} \to \mathcal{P}'$ is given by 3 maps

$$\rho_{|i} : L_i \to P'_i = J_iL'_0 \oplus \ldots \oplus JL'_{i-1} \oplus L'_i \ldots,$$

for $i = 0, 1, 2$. We represent each of these maps by a row vector. These are the rows of the following matrix.

$$\begin{pmatrix} X_{00} & Y_{01} & Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}. \tag{16}$$

The X_{ij} are matrices with coefficients in W, the matrices Y_{01} and Y_{12} have coefficients in J and Y_{02} has coefficients in J_2. Since a morphism of 2-displays commutes with ι one can see that the map $P_0 \to P_0'$ is given by the following matrix

$$\begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}.$$

By Proposition 4 the matrix (16) defines a morphism of displays iff the following 3 diagrams are commutative.

$$\begin{array}{ccc} L_i & \xrightarrow{\Phi_i} & P_0 \\ {\scriptstyle \rho_{|i}}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ P_i' & \xrightarrow[F_i']{} & P_0' \end{array} \qquad i = 0, 1, 2. \tag{17}$$

The σ-linear maps Φ_i resp. Φ_i' are given by the row vectors

$$\begin{pmatrix} A_{0i} \\ A_{1i} \\ A_{2i} \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} A_{0i}' \\ A_{1i}' \\ A_{2i}' \end{pmatrix}$$

We write these vectors into matrix. For example the standard data $(L_0, L_1, L_2, \Phi_1, \Phi_2, \Phi_3)$ for $\mathcal{P}$ are equivalent to the block matrix:

$$\mathbf{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \tag{18}$$

We will call this a structure matrix for the display $\mathcal{P}$.

From the definition of F_i' in terms of standard data these σ-linear maps have the following matrix representation:

$$F_0'(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} A_{00}' & pA_{01}' & p^2 A_{02}' \\ A_{10}' & pA_{11}' & p^2 A_{12}' \\ A_{20}' & pA_{21}' & p^2 A_{22}' \end{pmatrix} \begin{pmatrix} \sigma(x_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix}.$$

$$F_1'(\begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} A_{00}' & A_{01}' & pA_{02}' \\ A_{10}' & A_{11}' & pA_{12}' \\ A_{20}' & A_{21}' & pA_{22}' \end{pmatrix} \begin{pmatrix} \dot{\sigma}(y_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix},$$

$$F_2'(\begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} A_{00}' & A_{01}' & A_{02}' \\ A_{10}' & A_{11}' & A_{12}' \\ A_{20}' & A_{21}' & A_{22}' \end{pmatrix} \begin{pmatrix} \dot{\sigma}(y_0) \\ \dot{\sigma}(y_1) \\ \sigma(x_2) \end{pmatrix}$$

The vectors x_i have coefficients in W, the y_i have coefficients in J but in the equation for F_2' the vector y_0 has even coefficients in J_2.

Then the commutativity of the diagram (17) for $i = 0$ amounts to

$$\begin{pmatrix} A_{00}' & pA_{01}' & p^2A_{02}' \\ A_{10}' & pA_{11}' & p^2A_{12}' \\ A_{20}' & pA_{21}' & p^2A_{22}' \end{pmatrix} \begin{pmatrix} \sigma(X_{00}) \\ \sigma(X_{10}) \\ \sigma(X_{20}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{00} \\ A_{10} \\ A_{20} \end{pmatrix}.$$

For $i = 1$ it amounts to

$$\begin{pmatrix} A_{00}' & A_{01}' & pA_{02}' \\ A_{10}' & A_{11}' & pA_{12}' \\ A_{20}' & A_{21}' & pA_{22}' \end{pmatrix} \begin{pmatrix} \dot{\sigma}(Y_{01}) \\ \sigma(X_{11}) \\ \sigma(X_{21}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{01} \\ A_{11} \\ A_{21} \end{pmatrix}.$$

Finally for $i = 2$ it amounts to

$$\begin{pmatrix} A_{00}' & A_{01}' & A_{02}' \\ A_{10}' & A_{11}' & A_{12}' \\ A_{20}' & A_{21}' & A_{22}' \end{pmatrix} \begin{pmatrix} \dot{\sigma}(Y_{02}) \\ \dot{\sigma}(Y_{12}) \\ \sigma(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{02} \\ A_{12} \\ A_{22} \end{pmatrix}.$$

We may write the last 3 equations as a single matrix equation:

$$\mathbf{A}' \begin{pmatrix} \sigma(X_{00}) & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ p\sigma(X_{10}) & \sigma(X_{11}) & \dot{\sigma}(Y_{12}) \\ p^2\sigma(X_{20}) & p\sigma(X_{21}) & \sigma(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \mathbf{A}, \qquad (19)$$

where $\mathbf{A}$ and $\mathbf{A}'$ are the structure matrices (18).

Let $\mathcal{F}$ be a frame with Verjüngung such that any finitely generated projective module is free. Then the category of $\mathcal{F}$-2-displays is equivalent to the following category $\mathcal{M}_{\mathcal{F}}$. The objects are block matrices $\mathbf{A}$ and the morphisms $\mathbf{A} \to \mathbf{A}'$ are block matrices (16) such that the equation (19) is satified. Of course we have to say what is the composite of two matrices, but we omit this. In this direction we make only the following remark: The maps $\rho_i : P_i \to P_i'$ are explicitly given by the following matrix equations:

$$\rho_0(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},$$

$$\rho_1(\begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix},$$

$$\rho_2(\begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} X_{00} & \breve{Y}_{01} & Y_{02} \\ \pi X_{10} & X_{11} & Y_{12} \\ \pi X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}.$$

We need to explain the last equation. The matrix multiplication becomes meaningful with the following definitions:

$$\breve{Y}_{01} y_1 = Y_{01} * y_1, \quad \pi X_{10} y_0 = X_{10} \pi(y_0), \quad \pi X_{20} y_0 = X_{10} \pi(y_0).$$

Note that y_0 is a vector with entries in J_2 and therefore the vectors of the last equations have entries in J_2 too.

Using these expressions for ρ_i we see that (19) amounts to the commutativity of the following diagram:

$$\begin{array}{ccc} P_2 & \xrightarrow{\rho_2} & P_2' \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_2'} \\ P_0 & \xrightarrow{\rho_0} & P_0'. \end{array}$$

Finally we give the description of the dual display in term of standard data. Let $\mathcal{F}$ be a frame with Verjüngung as before. Assume $\mathcal{P}$ is the display associated to the standard data:

$$\Phi : L_0 \oplus L_1 \oplus L_2 \to L_0 \oplus L_1 \oplus L_2. \tag{20}$$

We write Φ in matrix form:

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \sigma(x) \\ \sigma(y) \\ \sigma(z) \end{pmatrix}$$

Then the dual display $\hat{\mathcal{P}}$ is formed from the following standard data. We take the dual modules $L_i^* = \mathrm{Hom}_W(L_i, W)$ but in the order L_2^*, L_1^*, L_0^*. Changing the order in (20) and taking the dual of this σ-linear map we obtain a linear map

$$\Phi^* : L_2^* \oplus L_1^* \oplus L_0^* \to W \otimes_{\sigma,W} (L_2^* \oplus L_1^* \oplus L_0^*). \tag{21}$$

We set $\hat{\Phi} = (\Phi^*)^{-1}$. We regard this as a σ-linear map. We obtain a standard datum

$$(L_2^*, L_1^*, L_0^*, \hat{\Phi})$$

which is by definition the standard datum of $\hat{\mathcal{P}}$. In particular $\hat{P}_0 = L_2^* \oplus L_1^* \oplus L_0^*$.

In matrix form $\hat{\Phi}$ takes the form

$$\hat{\Phi}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} {}^t\!A_{22} & {}^t\!A_{12} & {}^t\!A_{02} \\ {}^t\!A_{21} & {}^t\!A_{11} & {}^t\!A_{01} \\ {}^t\!A_{20} & {}^t\!A_{10} & {}^t\!A_{00} \end{pmatrix}^{-1} \begin{pmatrix} \sigma(x') \\ \sigma(y') \\ \sigma(z') \end{pmatrix}$$

Let us denote by d_0, d_1, d_2 the ranks of the modules L_0, L_1, L_2 respectively. Consider the block matrix

$$\mathbf{B} := \begin{pmatrix} \mathbf{0} & \mathbf{0} & E_{d_0} \\ \mathbf{0} & E_{d_1} & \mathbf{0} \\ E_{d_2} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where E denotes a unit matrix. This matrix defines a bilinear form:

$$<\ ,\ >: P_0 \times \hat{P}_0 \to W. \tag{22}$$

$$< \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} >= \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

In this notation the definition of $\hat{\Phi}$ reads

$$< \Phi(u), \hat{\Phi}(\hat{u}) >= \sigma < u, \hat{u} >, \quad u \in P_0,\ \hat{u} \in \hat{P}_0.$$

One deduces the formula

$$< F_0(u), \hat{F}_0(\hat{u}) >= p^2 \sigma < u, \hat{u} >$$

If we denote by $\mathcal{U}(2)$ the $\mathcal{F}$-2-display associated to the standard datum $(0, 0, W; \sigma)$ we obtain from $<\ ,\ >$ a bilinear pairing of $\mathcal{F}$-displays

$$<\ ,\ >: \mathcal{P} \times \hat{\mathcal{P}} \to \mathcal{U}(2).$$

The complete definition of this is given by the formulas in [LZ2] before Proposition 2.8.

We note that the Hodge filtrations

$$\{0\} \subset L_2/JL_2 \subset L_1/JL_1 \oplus L_2/JL_2 \subset P_0/JP_0,$$

and

$$\{0\} \subset \hat{L}_2/J\hat{L}_2 \subset \hat{L}_1/J\hat{L}_1 \oplus \hat{L}_2/J\hat{L}_2 \subset \hat{P}_0/J\hat{P}_0$$

are dual with respect to $< , >$.

In particular an isomorphism of displays $\mathcal{P} \rightarrow \hat{\mathcal{P}}$ defines a pairing

$$\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{U}(2),$$

such that the Hodge filtration of $\mathcal{P}$ is self dual with respect to this pairing.

Let $S \rightarrow R$ be a surjective ring homomorphism with kernel $\mathfrak{a}$. We assume that $\mathfrak{a}$ is endowed with divided powers. We call $S \rightarrow R$ a *pd*-thickening with kernel $\mathfrak{a}$.

Let R be an artinian local ring with perfect residue field. Then we can consider the small Witt frame $\hat{\mathcal{W}}_{S/R}$ with the natural Verjüngung.

For a $\hat{\mathcal{W}}_{S/R}$-display $\mathcal{P}$ the Frobenius $F_0 : P_0 \rightarrow P_0$ induces a map

$$\bar{F}_0 : P_0/(JP_0 + pP_0 + \iota_0(P_1)) \rightarrow P_0/(JP_0 + pP_0 + \iota_0(P_1)) \tag{23}$$

Definition 11. *We say that a 2-display is $\bar{F}_0$-étale if the map (23) is an isomorphism.*

We denote by $\hat{\mathcal{W}}_{S/R}$ the relative frame with the natural Verjüngung. Let $S' \rightarrow R$ be a second *pd*-thickening with kernel $\mathfrak{a}'$.

Let $S' \rightarrow S$ a morphism of *pd*-thickenings of R. Then the kernel $\mathfrak{b}$ of this morphism is a sub-*pd*-ideal of $\mathfrak{a}'$. We obtain a morphism of frames

$$\hat{\mathcal{W}}_{S'/R} \rightarrow \hat{\mathcal{W}}_{S/R}. \tag{24}$$

Proposition 12. *Let R be an artinian local ring with perfect residue field. Assume we a given a morphism $S' \rightarrow S$ of pd-thickenings of R. We assume that the ideals $\mathfrak{a}$ and $\mathfrak{a}'$ are nilpotent, and that p is nilpotent in S'.*

Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\hat{\mathcal{W}}_{S/R}$-2-displays which are $\bar{F}_0$-étale and such that the dual $\hat{\mathcal{W}}_{S/R}$-2-displays are $\bar{F}_0$-étale too. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be liftings to $\hat{\mathcal{W}}_{S'/R}$-2-displays.

Then each homomorphism $\rho : \mathcal{P} \rightarrow \mathcal{Q}$ lifts to a homomorphism of $\hat{\mathcal{W}}_{S'/R}$-displays $\rho' : \mathcal{P}' \rightarrow \mathcal{Q}'$.

If we assume moreover that $(\mathfrak{a}')^2 = 0$, the homomorphism ρ' is uniquely determined by ρ.

Proof. By the usual argument (compare [LZ1]) we may assume that $\mathcal{P} = \mathcal{Q}$ and that ρ is the identity. We choose such a normal decomposition and a basis in each module of this decomposition. We lift this to a normal decomposition of $\mathcal{P}'$ and $\mathcal{Q}'$ respectively and we also lift the given basis.

Then we may represent the 2-display $\mathcal{P}'$ by the structure matrix

$$\mathbf{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \in \mathrm{GL}(\hat{W}(S')).$$

and similarly $\mathcal{Q}'$ by the structure matrix $\mathbf{A}' = (A'_{ij})$. We will write $\mathbf{A}^{-1} = (\breve{A}_{ij})$ and $(\mathbf{A}')^{-1} = (\breve{A}'_{ij})$. Then our assumption says that the following matrices are invertible:

$$A_{00}, A'_{00}, \breve{A}_{22}, \breve{A}'_{22}.$$

Let $\mathfrak{c}$ be the kernel of $S' \to S$. Decomposing $S' \to S$ in a series of *pd*-morphisms we may assume that $\mathfrak{c}^2 = 0$ and $p\mathfrak{c} = 0$. A morphism $\rho' : \mathcal{P}' \to \mathcal{Q}'$ which lifts the identity may be represented by a matrix

$$\mathbf{E} + \begin{pmatrix} X_{00} & Y_{01} & Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}.$$

The entries of the matrices X_{ij} and Y_{ij} are in $\hat{W}(\mathfrak{c})$ and the entries of $\mathbf{w}_0(Y_{02})$ are moreover in $(\mathfrak{a}')^2$.

We set $C_{ij} = A'_{ij} - A_{ij}$. These are matrices with entries in $\hat{W}(\mathfrak{c})$. Since $\sigma(X_{ij}) = 0$ the equation (19) may be rewritten:

$$\mathbf{C} + \mathbf{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \mathbf{A}. \qquad (25)$$

We used the notation $\mathbf{C} := (C_{ij})$. We have to show that there are matrices X_{ij} and Y_{ij} which satisfy this equation. We write $Y_{02} = \mathfrak{y}_{02} + {}^V Z_{02}$, where $\mathfrak{y}_{02} \in \tilde{\mathfrak{c}} \cap \widetilde{(\mathfrak{a}')^2}$. We note that $\pi Y_{02} = \mathfrak{y}_{02}$. In particular we have $\pi Y_{02} = 0$ if we want to prove the second assertion of the Proposition that the solutions X_{ij}, Y_{ij} are unique.

We set $\mathbf{D} = \mathbf{C}\mathbf{A}^{-1}$. Then the equation (25) becomes

$$\mathbf{D} + \mathbf{A}' \begin{pmatrix} 0 & \dot\sigma(Y_{01}) & \dot\sigma(Y_{02}) \\ 0 & 0 & \dot\sigma(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} \mathbf{A}^{-1} = \begin{pmatrix} X_{00} & Y_{01} & \mathfrak{y}_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}. \tag{26}$$

We have

$$\mathbf{A}' \begin{pmatrix} 0 & \dot\sigma(Y_{01}) & \dot\sigma(Y_{02}) \\ 0 & 0 & \dot\sigma(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} \mathbf{A}^{-1} =$$

$$\begin{pmatrix} * & A'_{00}\dot\sigma(Y_{01})\breve{A}_{11} + A'_{00}\dot\sigma(Y_{02})\breve{A}_{21} + A'_{01}\dot\sigma(Y_{12})\breve{A}_{21} & ?_1 \\ * & * & ?_2 \\ * & * & ?_3 \end{pmatrix},$$

$$\text{where} \qquad \begin{array}{rl} ?_1 = & A'_{00}\dot\sigma(Y_{01})\breve{A}_{12} + A'_{00}\dot\sigma(Y_{02})\breve{A}_{22} + A'_{01}\dot\sigma(Y_{12})\breve{A}_{22} \\ ?_2 = & A'_{10}\dot\sigma(Y_{01})\breve{A}_{12} + A'_{10}\dot\sigma(Y_{02})\breve{A}_{22} + A'_{11}\dot\sigma(Y_{12})\breve{A}_{22} \\ ?_3 = & * \end{array}$$

The entries $*$ are irrelevant for the following and therefore not specified. Since the X_{ij} don't appear on the left hand side of (26) we see that it is enough to satisfy the following equations if we want to solve (26).

$$\begin{array}{c} D_{01} + A'_{00}\dot\sigma(Y_{01})\breve{A}_{11} + A'_{00}\dot\sigma(Y_{02})\breve{A}_{21} + A'_{01}\dot\sigma(Y_{12})\breve{A}_{21} = Y_{01} \\ D_{12} + A'_{10}\dot\sigma(Y_{01})\breve{A}_{12} + A'_{10}\dot\sigma(Y_{02})\breve{A}_{22} + A'_{11}\dot\sigma(Y_{12})\breve{A}_{22} = Y_{12} \\ D_{02} - \mathfrak{y}_{02} + A'_{00}\dot\sigma(Y_{01})\breve{A}_{12} + A'_{01}\dot\sigma(Y_{12})\breve{A}_{22} = -A'_{00}\dot\sigma(Y_{02})\breve{A}_{22} \end{array} \tag{27}$$

In this equation the D_{ij} are matrices with entries in $\hat{W}(\mathfrak{c})$. We note that for any given matrices $\mathfrak{y}_{02}$ and $\dot\sigma(Y_{02})$ there is a unique Y_{02}.

Therefore the Proposition follows if we show that for any given $\mathfrak{y}_{02}$ with entries in $\mathfrak{c} \cup (\mathfrak{a}')^2$ the equation above has a unique solution for the unknowns

$$Z_0 = Y_{01}, \quad Z_1 = Y_{12}, \quad Z_2 = -A'_{00}\dot\sigma(Y_{02})\breve{A}_{22},$$

with entries in $\hat{W}(\mathfrak{c})$. This is because the matrices A'_{00} and $\breve{A}_{22}$ are invertible. The divided powers on $\mathfrak{c}$ allow us to divide the Witt polynomial $\mathbf{w}_n$ by p^n. The divided Witt polynomials $\mathbf{w}'_n$ define an isomorphism

$$\hat{W}(\mathfrak{c}) \to \bigoplus_{n=0}^{\infty} \mathfrak{c}_{[n]} \tag{28}$$

of $\hat{W}(S')$-modules.

For a matrix M with entries in $\hat{W}(\mathfrak{c})$ of suitable size we define the operator

$$\mathcal{L}_{00}(M) = A'_{00} M \breve{A}_{11}.$$

If M has entries in the ideal $\oplus_{i=0}^{n} \mathfrak{c}_{[i]}$ in the sense of (28) then $\mathcal{L}_{00}(M)$ has entries in the same ideal. In this case we write $\operatorname{length} M \leq n$. It follows that $\operatorname{length} \dot{\sigma}(M) \leq n-1$.

With obvious definitions of the operators $\mathcal{L}_{ij}$ we may write the system of equations (27) as follows:

$$\begin{array}{lcl} D_{01} + \mathcal{L}_{00}(\dot{\sigma}(Z_0)) + \mathcal{L}_{01}(\dot{\sigma}(Z_1)) + \mathcal{L}_{02}(Z_2) & = & Z_0 \\ D_{12} + \mathcal{L}_{10}(\dot{\sigma}(Z_0)) + \mathcal{L}_{11}(\dot{\sigma}(Z_1)) + \mathcal{L}_{12}(Z_2) & = & Z_1 \\ D'_{02} + \mathcal{L}_{20}(\dot{\sigma}(Z_0)) + \mathcal{L}_{21}(\dot{\sigma}(Z_1)) & = & Z_2 \end{array}$$

Here we wrote $D'_{02} := D_{02} - \mathfrak{y}_{02}$. We look for solutions in the space of matrices (Z_0, Z_1, Z_2) with entries in $\hat{W}(\mathfrak{c})$. On this space we consider the following operator U:

$$U(\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}) = \begin{pmatrix} \mathcal{L}_{00}(\dot{\sigma}(Z_0)) + \mathcal{L}_{01}(\dot{\sigma}(Z_1)) + \mathcal{L}_{02}(Z_2) \\ \mathcal{L}_{10}(\dot{\sigma}(Z_0)) + \mathcal{L}_{11}(\dot{\sigma}(Z_1)) + \mathcal{L}_{12}(Z_2) \\ \mathcal{L}_{20}(\dot{\sigma}(Z_0)) + \mathcal{L}_{21}(\dot{\sigma}(Z_1)) \end{pmatrix}$$

Clearly it suffices to prove that the operator U is pointwise nilpotent. Assume we are given Z_0, Z_1, Z_2. We set

$$U(\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}) = \begin{pmatrix} Z'_0 \\ Z'_1 \\ Z'_2 \end{pmatrix}, \qquad U(\begin{pmatrix} Z'_0 \\ Z'_1 \\ Z'_2 \end{pmatrix}) = \begin{pmatrix} Z''_0 \\ Z''_1 \\ Z''_2 \end{pmatrix}$$

Let m be a natural number such that $\operatorname{length} Z_0 \leq m$, $\operatorname{length} Z_1 \leq m$ and $\operatorname{length} Z_2 \leq m-1$. Since $\dot{\sigma}$ decreases the length by one, we obtain

$$\operatorname{length} Z'_0 \leq m-1, \quad \operatorname{length} Z'_1 \leq m-1 \quad \operatorname{length} Z_2 \leq m-1.$$

And in the next step we find

$$\operatorname{length} Z''_0 \leq m-1, \quad \operatorname{length} Z''_1 \leq m-1 \quad \operatorname{length} Z''_2 \leq m-2.$$

The nilpotence of U is now obvious. This proves the uniqueness of the solutions. $\square$

Remark: With the assumptions of the last Proposition we assume that the kernel $\mathfrak{c}$ of the pd-morphism satisfies $\mathfrak{c}^2 = 0$ and $p\mathfrak{c} = 0$. Then the group of isomorphisms $\mathcal{P}' \to \mathcal{P}'$ which lift the identity $\mathrm{id}_{\mathcal{P}}$ is isomorphic to the additive group of $\mathfrak{c} \cap (\mathfrak{a}')^2$. (The assumptions ensure that such a lift is the same as a solution for the equation (26).) This is because $\mathfrak{y}_{02}$ determines the lifting uniquely and because one can check that the composite of two endomorphisms of $\mathcal{P}'$ which lift zero is zero.

Corollary 13. *Let $S \to R$ be a surjective morphism of artinian local rings with perfect residue class field. Let $\mathcal{P}$ and $\mathcal{P}'$ be two $\hat{\mathcal{W}}_S$-displays which are together with their duals $\bar{F}_0$-étale. Let $\rho, \tau : \mathcal{P} \to \mathcal{P}'$ be two homomorphisms such that their base change ρ_R and τ_R are equal.*

Then $\rho = \tau$.

Corollary 14. *Let R be an artinian local ring with perfect residue field. R as above. Let $S' \to R$ be a pd-thickening of R with kernel $\mathfrak{a}'$, such that $(\mathfrak{a}')^2 = 0$ and such that p is nilpotent in S'. Let $\mathcal{Q}$ be a $\hat{\mathcal{W}}_R$-2-display over R which is together with its duals $\bar{F}_0$-étale. By the Proposition 12 there is a unique $\hat{\mathcal{W}}_{S'/R}$-2-display $\hat{\mathcal{Q}}$ which lifts $\mathcal{Q}$.*

The category of $\hat{\mathcal{W}}_{S'}$-2-displays which are together with there duals $\bar{F}_0$-étale is equivalent to the category of pairs $(\mathcal{Q}, Fil)$ where $\mathcal{Q}$ is a $\hat{\mathcal{W}}_R$-2-displays which is together with its dual $\bar{F}_0$-étale and where Fil is an admissible lifting of the Hodge filtration of $\hat{\mathcal{Q}}$.

Let k be a perfect field of characteristic $p > 0$. Let $\mathcal{A}rt_k$ be the category of artinian local rings with residue class field k. Let $\mathcal{S}$ be a $\mathcal{W}_k$-2-display which is together with its dual $\bar{F}_0$-étale. Let $\mathcal{D}$ be the functor that associates to $R \in \mathcal{A}rt_k$ the isomorphism classes of pairs $(\mathcal{P}, \iota)$ where $\mathcal{P}$ is a $\hat{\mathcal{W}}_R$-2-display and $\iota : \mathcal{S} \to \mathcal{P}_k$ is an isomorphism. If we have a diagram $R_1 \to R \leftarrow R_2$ then the canonical map

$$\mathcal{D}(R_1 \times_R R_2) \to \mathcal{D}(R_1) \times_{\mathcal{D}(R)} \mathcal{D}(R_2) \tag{29}$$

is easily seen to be surjective. It is injective because of Corollary (13).

By Corollary 14 the tangent space of the functor $\mathcal{D}$ is finite dimensional.

Corollary 15. *The functor $\mathcal{D}$ is prorepresentable by a power series ring over $W(k)$ in finitely many variables.*

We will use the following version of the deformation functor. We take a $\mathcal{W}_k$-2-display $\mathcal{S}$ as above and we assume moreover that $\mathcal{S}$ is endowed with an isomorphism

$$\lambda_0 : \mathcal{S} \to \hat{\mathcal{S}}.$$

We can λ_0 also regard as a pairing $\mathcal{S} \times \mathcal{S} \to \mathcal{U}(2)$. Then we define the deformation functor $\hat{\mathcal{D}} : \mathcal{A}rt_k \to (sets)$. For $R \in \mathcal{A}rt_k$ we define $\hat{\mathcal{D}}(R)$ as the set of isomorphism classes of $\hat{\mathcal{W}}_R$-2-displays $\mathcal{P}$ together with an isomorphism $\lambda : \mathcal{P} \to \hat{\mathcal{P}}$ and an isomorphism $\iota : \mathcal{S} \to \mathcal{P}_k$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\iota"] \arrow[d, "\lambda_0"'] & \mathcal{P}_k \arrow[d, "\lambda_k"] \\ \hat{\mathcal{S}} & \hat{\mathcal{P}}_k. \arrow[l, "\hat{\iota}"] \end{tikzcd}$$

We note that by the diagram and Corollary 13 the morphism λ is uniquely determined if it exists. Therefore we have an inclusion $\hat{\mathcal{D}}(R) \subset \mathcal{D}(R)$. The map (29) for the functor $\hat{\mathcal{D}}$ is also bijective. We will now find the tangent space of $\hat{\mathcal{D}}$. We consider a *pd*-thickening $S' \to R$ in $\mathcal{A}rt_k$ with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. Assume we are given $(\mathcal{P}, \iota, \lambda) \in \hat{\mathcal{D}}(R)$. Then $\mathcal{P}$ lifts uniquely to a $\hat{\mathcal{W}}_{S'/R}$-display $\mathcal{P}_{rel}$ and λ lifts to an isomorphism $\lambda_{rel} : \mathcal{P}_{rel} \to \hat{\mathcal{P}}_{rel}$. Let $\mathcal{Q}$ be a $\hat{\mathcal{W}}_{S'}$-2-display which lifts $\mathcal{P}$. Giving $\mathcal{Q}$ is the same as giving an admissible lifting of the Hodge filtration of $\mathcal{P}_{rel}$. The dual display $\hat{\mathcal{Q}}$ corresponds to the dual filtration of $\hat{\mathcal{P}}_{rel}$. But then $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ are isomorphic iff λ_{rel} takes the filtration $Fil_{\mathcal{Q}}$ given by $\mathcal{Q}$ to the dual filtration, i.e. $Fil_{\mathcal{Q}}$ is selfdual with respect to the bilinear form

$$P_{rel,0}/I_{S'}P_{rel,0} \times P_{rel,0}/I_{S'}P_{rel,0} \to S' \tag{30}$$

induced by λ_{rel}.

Proposition 16. *Let $S' \to R$ be a pd-thickening with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. Let $(\mathcal{P}, \lambda)$ be a $\hat{\mathcal{W}}_R$-2-display which is $\bar{F}_0$-étale endowed with an isomorphism $\mathcal{P} \to \hat{\mathcal{P}}$. We denote by $\mathcal{P}_{rel}$ the unique $\hat{\mathcal{W}}_{S'/R}$ which lifts $\mathcal{P}$.*

The liftings of $(\mathcal{P}, \lambda)$ to a $\hat{\mathcal{W}}_{S'}$-2-display $\mathcal{Q}$ together with a lift of λ to an isomorphism $\mu : \mathcal{Q} \to \hat{\mathcal{Q}}$ are in bijection

with the liftings of $\mathrm{Fil}^2_{\mathcal{P}}$ to a isotropic direct summand of $P_{rel,0}/I_{S'}P_{rel,0}$.

Proof. We know that λ lifts to an isomorphism $\lambda_{rel} : \mathcal{P}_{rel} \to \hat{\mathcal{P}}_{rel}$. It follows from Corollary 14 that the liftings $(\mathcal{Q}, \mu)$ of $(\mathcal{P}, \lambda)$ are in bijections to selfdual admissible liftings of the Hodge filtration of $\mathcal{P}$.

The isomorphism λ_{rel} induces a perfect pairing (30) of S'-modules. We claim that the image of

$$P_{rel,2} \to P_{rel,0}/I_{S'}P_{rel,0}$$

is isotropic under this pairing (30).

To verify this we take a normal decomposition of $\mathcal{P}_{rel}$:

$$P_{rel,0} = L_0 \oplus L_1 \oplus L_2.$$

This induces the dual normal decomposition of $\hat{\mathcal{P}}_{rel}$ (compare (21))

$$\hat{P}_{rel,0} = \hat{L}_0 \oplus \hat{L}_1 \oplus \hat{L}_2,$$

where $\hat{L}_0 = L_2^*$, $\hat{L}_1 = L_1^*$, $\hat{L}_2 = L_0^*$.

We set $L'_i = L_i/I_{S'}L_i \subset P_{rel,0}/I_{S'}P_{rel,0}$, $\hat{L}'_i = \hat{L}_i/I_{S'}\hat{L}_i \subset \hat{P}_{rel,0}/I_{S'}\hat{P}_{rel,0}$. Then the images of the two maps

$$P_{rel,2} \to P_{rel,0}/I_{S'}P_{rel,0}, \quad \text{resp.} \quad \hat{P}_{rel,2} \to \hat{P}_{rel,0}/I_{S'}\hat{P}_{rel,0}$$

are

$$L'_2 \oplus \mathfrak{a}' L_1, \quad \text{resp.} \quad \hat{L}'_2 \oplus \mathfrak{a}'\hat{L}_1.$$

Since $(\mathfrak{a}')^2 = 0$ the last two modules are orthogonal with respect to the perfect pairing

$$P_{rel,0}/I_{S'}P_{rel,0} \times \hat{P}_{rel,0}/I_{S'}\hat{P}_{rel,0} \to S'.$$

induced by (22). Composing this with the isomorphism λ_{rel} we obtain the claim.

On the other hand any isotropic lift $E^2 \subset P_{rel,0}/I_{S'}P_{rel,0}$ is necessarily contained in $L'_2 \oplus \mathfrak{a}'L_1$ and therefore admissible.

From an isotropic lift $E^2 \subset P_{rel,0}/I_{S'}P_{rel,0}$ we obtain a selfdual admissible lift of the Hodge filtration of $\mathcal{P}$, we define E^1 as the orthogonal complement of E^2. By Corollary 14 this gives a lifting $(\mathcal{Q}, \mu)$. □

In particular we see that lifts of $\mathcal{P}$ always exist. Therefore we obtain:

Corollary 17. *The functor $\hat{\mathcal{D}}$ is prorepresentable by a power series ring over $W(k)$ in finitely many variables.*

2 2-Displays of schemes

Let X_0 be a projective and smooth scheme over a perfect field k of characteristic $p > 2$. We make the following assumptions:

1. Let $T_{X_0/k}$ be the tangent bundle of X_0. Then
$$H^0(X_0, T_{X_0/k}) = H^2(X_0, T_{X_0/k}) = 0. \tag{31}$$

2. Let R be a local artinian $W(k)$-algebra with residue class field k and let $f : X \to \operatorname{Spec} R$ be an arbitrary deformation of X_0. Then the R-modules
$$R^j f_* \Omega^i_{X/R} \tag{32}$$
are free and commute with base change for morphisms $\operatorname{Spec} R' \to \operatorname{Spec} R$, where R' be a local artinian $W(k)$-algebra with residue class field k.

3. The spectral sequence
$$E_1^{ij} = R^j f_* \Omega^i_{X/R} \Rightarrow \mathbb{R}^{j+i} f_* \Omega^{\cdot}_{X/R} \tag{33}$$
degenrates for $i + j \leq 2$, i.e. all differentials starting or ending at E_r^{ij} for $i + j \leq 2$, $r \geq 1$ are zero.

We remark that the last two requirements are fulfilled if
$$H^j(X_0, \Omega^i_{X_0/k}) = 0, \quad \text{for } i + j = 1 \text{ or } 3.$$

Assume that X_0 satisfies the 3 conditions above. Let R be a local $W(k)$-algebra whose maximal ideal $\mathfrak{m}$ is nilpotent, and such that $R/\mathfrak{m} = k$. Let $g : Y \to R$ be a deformation of X_0. Then the last 2 conditions are also satisfied for g. Indeed R is the filtered union of local $W(k)$-algebras of finite type and X is automatically defined over a $W(k)$-algebra of finite type.

Assume again that the 3 conditions are fulfilled for X_0. Then there is a universal deformation, i.e. a morphism of formal schemes
$$\mathfrak{X} \to \operatorname{Spf} A. \tag{34}$$

The adic ring A is the ring $W(k)[[T_1, \ldots, T_r]]$ with the ideal of definition $(p, T_1, \ldots, T_r)$. We have $r = \dim H^1(X_0, T_{X_0/k})$. We denote by σ the endomorphism of A, such that $\sigma(T_i) = T_i^p$ and such that σ is the Frobenius on $W(k)$.

Let $n \geq 1$ be an integer. We set $A_n = W(k)[[T_1, \ldots, T_r]]/(T_1^n, \ldots, T_r^n)$ and $R_n = A_n/p^n A_n$. Then σ induces an endomorphism on A_n denoted by the same letter. We obtain that $\mathcal{A}_n = (A_n, p^n A_n, R_n, \sigma, \sigma/p)$ is a frame as in [LZ2] Section 5. An obvious modification of [LZ2] Corollary 5.6 shows that we have the structure of a $\mathcal{W}_{R_n}$-display on

$$H^2_{crys}(\mathcal{X}_{R_n}/W(R_n)).$$

This is obtained from the Lazard morphism $A_n \to W(R_n)$ which induces a morphism of frames $\mathcal{A}_n \to \mathcal{W}_{R_n}$. Indeed, Theorem 5.5. of [LZ2] gives us an $\mathcal{A}_n$-display structure on $H^2_{crys}(\mathcal{X}_{R_n}/A_n)$. Then we can apply base change for frames Proposition (5).

The morphism of frames factors:

$$\mathcal{A} \to \hat{\mathcal{W}}_{R_n} \to \mathcal{W}_{R_n}.$$

Therefore we obtain also the structure of a $\hat{\mathcal{W}}_{R_n}$-display on $H^2_{crys}(\mathcal{X}_{R_n}/\hat{W}(R_n))$.

This is functorial in R_n. If $f : X \to R$ is a deformation as in (32) we obtain for n big enough a unique $W(k)$-algebra homomorphism $R_n \to R$. Therefore we obtain by base change:

Proposition 18. *Let $f : X \to R$ as above. Then the crystalline cohomology $H^2_{crys}(X/\hat{W}(R))$ has a unique structure of a $\hat{\mathcal{W}}_R$-display which is functorial in R.*

The uniqueness follows from the functoriality and the fact that $\hat{W}(A)$ has no p-torsion.

We now show that X/R defines a crystal of displays in the following sense:

Corollary 19. *With the assumptions of the Proposition let $S \to R$ be a pd-thickening where S is an artinian $W(k)$-algebra. Then we have a natural structure of a $\hat{\mathcal{W}}_{S/R}$-display on $H^2_{crys}(X/S)$. More precisely this structure is functorial with respect of morphism of pd-thickenings and uniquely determined by this property.*

Proof. We obtain a $\hat{\mathcal{W}}_{S/R}$-2-display structure by lifting X to a smooth scheme X' over S and then making base change with respect to $\hat{\mathcal{W}}_S \to \hat{\mathcal{W}}_{S/R}$. We show that the result is independent of the chosen lifting X'. Assume we have two liftings X' and X'' which are induced from the universal family (34) by two morphisms $A \to S$. We consider the commutative diagram

$$\begin{array}{ccc} A\hat{\otimes}_{W(k)}A & \longrightarrow & S \\ \Big\downarrow & & \Big\downarrow \\ A & \longrightarrow & R \end{array} \tag{35}$$

The left vertical arrow is the multiplication. Let J be the kernel. We denote the divided power hull of $(B := A\hat{\otimes}_{W(k)}A, J)$ by P. It is obtained as follows: Let $A_0 = W(k)[T_1, \ldots, T_r]$ and J_0 the kernel of the multiplication $B_0 := A_0 \otimes_{W(k)} A_0 \to A_0$. We denote by P_0 the divided power hull of (B_0, J_0). Then P_0 is isomorphic to the divided power algebra of the free A_0-module with r generators. In particular P_0 is a free A_0-module for the two natural A_0-module structures. We have $P = P_0 \otimes_{B_0} B$. Then P is flat as P_0-module and therefore without p-torsion. Then the diagram (35) extends to a diagram

$$\begin{array}{ccc} P & \longrightarrow & S \\ \Big\downarrow & & \Big\downarrow \\ A & \longrightarrow & R \end{array} \tag{36}$$

Let $\hat{P}$ be the p-adic completion of P. Then $\hat{P} \to A$ is a frame $\mathcal{D}$. By [LZ2] Theorem 5.5 the universal family $\mathfrak{X}$ defines an $\mathcal{D}$-display $\mathcal{U}$. We consider also the trivial frame $\mathcal{D}_0 = (A, 0, A, \sigma, \sigma/p)$. Again $\mathfrak{X}$ defines a $\mathcal{D}_0$-display $\mathcal{U}_0$. The two natural sections $A \to \hat{P}$ define two morphisms of frames $\mathcal{D}_0 \to \mathcal{D}$. Since the construction of [LZ2] Theorem 5.5 is compatible with base change we obtain $\mathcal{U}$ from $\mathcal{U}_0$ by base change with respect to both of these two morphisms.

We consider the morphism of frames

$$\mathcal{D}_0 \overset{\rightarrow}{\rightarrow} \mathcal{D} \to \hat{\mathcal{W}}_{S/R}.$$

The two $\hat{\mathcal{W}}_{S/R}$-displays associated with X' and X'' are obtained by base change from $\mathcal{U}_0$ by the two morphisms

$$\mathcal{D}_0 \overset{\rightarrow}{\rightarrow} \hat{\mathcal{W}}_{S/R}.$$

We see that these two $\hat{\mathcal{W}}_{S/R}$-displays are both obtained by base change of $\mathcal{U}$ with respect to $\mathcal{D} \to \hat{\mathcal{W}}_{S/R}$. This shows that the $\hat{\mathcal{W}}_{S/R}$ display does up to canonical isomorphism not depend on the lifting X' of X. $\square$

3 The Beauville-Bogomolov form

Definition 20. *A scheme of $K3$-type is a smooth and proper morphism $f : X \to S$ with the following properties.*

For each geometric point $\eta \to S$ we have:

$$\begin{aligned} H^q(X_\eta, \Omega^p_{X_\eta/\eta}) &= 0, \quad \textit{for}\, p+q=1,\ p+q=3. \\ \dim_{\kappa(\eta)} H^q(X_\eta, \mathcal{O}_{X_\eta}) &= 1, \quad \textit{for}\, q = 0,2 \\ \dim_{\kappa(\eta)} H^0(X_\eta, \Omega^2_{X_\eta/\eta}) &= 1, \end{aligned} \tag{37}$$

We assume that for each η there is a nowhere degenerate $\sigma \in H^0(X_\eta, \Omega^2_{X_\eta/\eta})$, i.e. $\sigma^n \in H^0(X_\eta, \Omega^{2n}_{X_\eta/\eta})$ defines an isomorphism

$$\mathcal{O}_{X_\eta} \to \Omega^{2n}_{X_\eta/\eta}.$$

We assume that for each η that there is a class $\rho \in H^2(X_\eta, \mathcal{O}_{X_\eta})$ such that ρ^n generates $H^{2n}(X_\eta, \mathcal{O}_{X_\eta})$. (We note that this is a 1-dimensional vector space by Serre duality.)

Finally we require that for each η the pairing

$$\begin{array}{ccc} H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \times H^1(X_\eta, \Omega^1_{X_\eta/\eta}) & \longrightarrow & R \\ \omega_1 \times \omega_2 & \mapsto & \int \omega_1 \omega_2 \sigma^{n-1} \rho^{n-1}. \end{array} \tag{38}$$

is perfect.

Remarks: If X is a Hyperkähler variety over $\mathbb{C}$ such that $H^3(X, \mathbb{C}) = 0$, then X is of $K3$ type.

Over any field a K3-surface is of K3-type. Over a field of characteristic 0 the Hilbert scheme of zero-dimensional subschemes of a $K3$-surface is $K3n$ [G].

By [M] and [Ha] III §12 it follows that for $f : X \to S$ of $K3$-type the direct images $R^q f_* \Omega^p_{X/S}$ for $p + q = 2$ are locally free and commute with arbitrary base change and $f_* \mathcal{O}_X = \mathcal{O}_S$. Therefore locally on S we have sections $\sigma \in H^0(S, f_* \Omega^2_{X/S})$ and $\rho \in H^2(X, \mathcal{O}_X)$ which induce in each geometric fibre the classes required in the definition.

It follows from [Ha] that the set of points of S where a smooth and proper morphism $f : X \to S$ is of $K3$-type is open.

We therefore obtain varieties of $K3$-type as follows. Let S be a scheme of finite type and flat over $\mathbb{Z}$. We consider a smooth and proper morphism

$f : X \to S$ such that $f_{\mathbb{Q}}$ is of $K3$-type. Then over an open subset of S the morphism is of $K3$-type. In particular the schemes $K3n$ are for almost all prime numbers p of $K3$-type over a field of characteristic p.

We note that (38) is equivalent saying that the cup product

$$\sigma^{n-1}\rho^{n-1} : H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \to H^{2n-1}(X_\eta, \Omega^{2n-1}_{X_\eta/\eta})$$

is an isomorphism.

In the following we will assume without loss of generality that $S = \operatorname{Spec} R$ and that σ and ρ are globally defined. We note that σ is closed because $H^0(X, \Omega^3_{X/S}) = 0$.

We denote by $\mathcal{T}_{X/S}$ the dual $\mathcal{O}_X$-module of $\Omega^1_{X/S}$. By definition σ induces a perfect pairing

$$\mathcal{T}_{X/S} \times \mathcal{T}_{X/S} \to \mathcal{O}_X. \tag{39}$$

or equivalently an isomorphism $\mathcal{T}_{X/S} \cong \Omega^1_{X/S}$.

Let X be scheme $K3$-type of dimension $2n$ over a ring R. It follows from our assumptions that $\epsilon := \int(\sigma\rho)^n \in R$ is a unit.

We regard $\sigma \in H^2_{DR}(X/R)$ and we choose an arbitrary lifting $\tau \in H^2_{DR}(X/R)$ of ρ. We have

$$\epsilon = \int \sigma^n\tau^n.$$

Definition 21. *We assume that n, $n+1$ and $n-1$ are units in R. We define a quadratic form $\mathbb{B}_{\sigma,\tau}(\alpha)$ on $H^2_{DR}(X/R)$ as follows*

$$\begin{aligned}\mathbb{B}_{\sigma,\tau}(\alpha) = \tfrac{n}{2}\textstyle\int(\sigma\tau)^{n-1}\alpha^2 \quad & +\tfrac{1-n}{\epsilon}(\textstyle\int \sigma^{n-1}\tau^n\alpha)(\int \sigma^n\tau^{n-1}\alpha) \\ & +\tfrac{1}{\epsilon^2}\tfrac{n(n-1)}{2(n+1)}(\textstyle\int \sigma^n\tau^{n-1}\alpha)^2(\int \sigma^{n-1}\tau^{n+1}).\end{aligned} \tag{40}$$

Lemma 22. *This pairing up to a factor in R^* doesn't depend on the choice of σ, ρ, and τ. If $R = \mathbb{C}$ it coincides with the usual Bogomolov-Beauville form up to a constant.*

Proof. Let $\sigma_1 = u\sigma$ and $\tau_1 = v\tau$ where $u, v \in R^*$. We set $\mathbb{B} = \mathbb{B}_{\sigma,\tau}$ and $\mathbb{B}_1 = \mathbb{B}_{\sigma_1,\tau_1}$ We have $\epsilon_1 = (\sigma_1\tau_1)^n = (uv)^n\epsilon$.

$$\begin{aligned}\mathbb{B}_1(\alpha) \;&=\; \frac{n}{2}(uv)^{n-1}\int(\sigma\tau)^{n-1}\alpha^2 + \frac{1-n}{(uv)^n\epsilon}(uv)^{2n-1}\left(\int \sigma^{n-1}\tau^n\alpha\right)\left(\int \sigma^n\tau^{n-1}\alpha\right) \\ &\quad+\frac{1}{(uv)^{2n}\epsilon^2}\frac{n(n-1)}{2(n+1)}\left(\int \sigma^n\tau^{n-1}\alpha\right)^2(u^nv^{n-1})^2\left(\int \sigma^{n-1}\tau^{n+1}\right)u^{n-1}v^{n+1}.\end{aligned}$$

The (uv)-factors in the last summand together yield the factor $\frac{1}{(uv)^{2n}}u^{3n-1}v^{3n-1} = (uv)^{n-1}$.

Hence we get

$$\mathbb{B}_1(\alpha) = (uv)^{n-1}\mathbb{B}(\alpha) \tag{41}$$

So the form $\mathbb{B}(\alpha)$ changes by a unit in R.

From now on let $R = \mathbb{C}$. In this case we have the Hodge-decomposition. We can use for ρ and τ the complex conjugate of σ, i.e. we set $\tau := \bar{\sigma}$. As $\bar{\sigma}^{n+1} = 0$ we get for the Beauville-Bogomolov form

$$\mathbb{B}_{\sigma,\bar{\sigma}}(\alpha) = \frac{n}{2}\int(\sigma\bar{\sigma})^{n-1}\alpha^2 + \frac{1-n}{\epsilon}\left(\int\sigma^{n-1}\bar{\sigma}^n\alpha\right)\left(\int\sigma^n\bar{\sigma}^{n-1}\alpha\right). \tag{42}$$

This is the usual Beauville-Bogomolov form, if we change $\bar{\sigma}$ by a constant such that $\epsilon = \int(\sigma\bar{\sigma})^n = 1$ (see 41).

Now let $\tau = a\sigma + \gamma + \bar{\sigma}$, where γ is a closed 1-1-form and $a \in \mathbb{C}$, so $\tau \in H^2_{DR}(X/\mathbb{C})$ is a lifting of $\bar{\sigma} \in H^2(X, \mathcal{O}_X)$. We evaluate the forms $\mathbb{B}_{\sigma,\bar{\sigma}}$ and $\mathbb{B}_{\sigma,\tau}$ (the form in the Proposition) in an arbitrary form $\alpha \in H^2_{DR}(X/\mathbb{C})$ and show that $\mathbb{B}_{\sigma,\bar{\sigma}}(\alpha) = \mathbb{B}_{\sigma,\tau}(\alpha)$. Without loss of generality let $\alpha = c\sigma + \beta + c'\bar{\sigma}$ with $c' \neq 0$ and after multiplication with a constant we can assume that $\alpha = c\sigma + \beta + \bar{\sigma}$, with β a closed 1-1-form, also without loss of generality $\epsilon = \int(\sigma\bar{\sigma})^n = 1$. Therefore $\alpha^2 = c^2\sigma^2 + 2c\sigma\beta + 2c\sigma\bar{\sigma} + 2\beta\bar{\sigma} + \bar{\sigma}^2 + \beta^2$.

Now we compute $\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2$ for each summand:

1. $c^2\sigma^2$.

$$\frac{n}{2}\int(\sigma\tau)^{n-1}c^2\sigma^2 = \frac{n}{2}\int\sigma^{n-1}(a\sigma + \gamma + \bar{\sigma})^{n-1}c^2\sigma^2$$

The p-degree (with respect to the p, q-Hodge-decomposition) is $> 2n + 2$. So this integral is zero.

2. summand $2c\sigma\beta$.

$$n\int\sigma^{n-1}(a\sigma + \gamma + \bar{\sigma})^{n-1}\sigma\beta = 0$$

because $\sigma^{n-1}\sigma\beta$ has already p-degree $2n + 1$.

3. summand $2c\sigma\bar{\sigma}$

$$nc\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\sigma\bar{\sigma} = nc\int \sigma\bar{\sigma}^n$$

   So the third summand is independent from the choice of τ.

4. summand $2\beta\bar{\sigma}$

$$n\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\beta\bar{\sigma} = n\int \sum_{i+j+k=n-1} \frac{(n-1)!}{i!j!k!}\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta\bar{\sigma}$$

   The form $\sigma^{n-1+i}\bar{\sigma}^{k+1}\gamma^j\beta$ has p-degree $2n-2+2i+j+1$.

   Hence the integral can be non-zero only for $i=0$ and $j=1$ and $k=n-2$. So the above integral is $n\int \frac{(n-1)!}{(n-2)!}\sigma^{n-1}\bar{\sigma}^{n-1}\beta\gamma$.

   This term depends on the choice of τ.

5. summand $\bar{\sigma}^2$

$$\frac{n}{2}\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\cdot\bar{\sigma}^2 = \frac{n}{2}\sum_{i+j+k=n-1}\int \sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^{k+2}\cdot\frac{(n-1)!}{i!j!k!}$$

   The p-degree of $\sigma^{n-1}\sigma^i\gamma^j\bar{\sigma}^{k+2}$ is $2(n-1)+2i+j$ which is $2n$ only for $i=1$, $j=0$ and $k=n-2$ or $i=0$, $j=2$, $k=n-3$.

   In the first case ($i=1$, $j=0$ and $k=n-2$) the integral does not depend on the choice of τ.

   In the second case ($i=0$, $j=2$, $k=n-3$) one gets the summand $\frac{(n-1)!}{2!(n-3)!}\cdot\frac{n}{2}\int \sigma^{n-1}\bar{\sigma}^{n-1}\gamma^2$ which depends on the choice of τ.

6. summand β^2

$$\frac{n}{2}\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\beta^2 = \frac{n}{2}\sum_{i+j+k=n-1}\frac{(n-1)!}{i!j!k!}\int \sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta^2$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta^2$ is $2n-2+2i+j+2$ which is $2n$ only if $i=0$ and $j=0$ and $k=n-1$.

Hence this integral does not depend on the choice of τ.

Adding up all cases we obtain (one may assume $\int(\sigma\bar{\sigma})^n=1$)

$$\begin{aligned}\frac{m}{2}\int(\sigma\bar{\sigma})^{n-1}\alpha^2 &= nc+n(n-1)\int(\sigma\bar{\sigma})^{n-1}\beta\gamma+(n-1)\frac{n}{2}a \\ &\quad+\frac{n(n-1)!}{4(n-3)!}\int(\sigma\bar{\sigma})^{n-1}\gamma^2+\frac{n}{2}\int\sigma^{n-1}\bar{\sigma}^{n-1}\beta^2 \quad (43)\end{aligned}$$

Now we compute the other summands in $\mathbb{B}_{\sigma,\tau}(\alpha)$.
We have

$$\int\sigma^n\tau^{n-1}\alpha=\int\sigma^n(a\sigma+\gamma+\bar{\sigma})^{n-1}(c\sigma+\beta+\bar{\sigma})=\int\sigma^n\bar{\sigma}^n=1.$$

Then

$$\begin{aligned}\int\sigma^{n-1}\tau^n\alpha &= \int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^n(c\sigma+\beta+\bar{\sigma}) \\ &= \sum_{i+j+k=n}\frac{n!}{i!j!k!}\int\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k(c\sigma+\beta+\bar{\sigma})\end{aligned}$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^kc\sigma$ is $2n+2i+j$ which is $2n$ only for $i=j=0$ and $k=n$.
The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta$ is $2n-2+2i+j+1$ which is $2n$ only for $i=0$, $j=1$ and $k=n-1$.
The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\bar{\sigma}$ is $2n-2+2i+j$ which is $2n$ only for $i=1$, $j=0$, $k=n-1$ or for $i=0$, $j=2$, $k=n-2$.
Using this we get the formula

$$\begin{aligned}\int\sigma^{n-1}\tau^n\alpha &= c\int(\sigma\bar{\sigma})^n+n\int\sigma^{n-1}\gamma\bar{\sigma}^{n-1}\beta+n\int a\sigma^n\bar{\sigma}^n \\ &\quad+\frac{n!}{2(n-2)!}\int\sigma^{n-1}\gamma^2\bar{\sigma}^{n-1} \quad (44) \\ &= c+na+n\int(\sigma\bar{\sigma})^{n-1}\beta\gamma+\frac{n}{2}(n-1)\int(\sigma\bar{\sigma})^{n-1}\gamma^2\end{aligned}$$

(43) and (44) then give us the following formula for the first two summands in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$
\begin{aligned}
&\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2+(1-n)\left(\int\sigma^{n-1}\tau^n\alpha\right)\left(\int\sigma^n\tau^{n-1}\alpha\right)\\
&= \; nc+\frac{n}{2}(n-1)a+\frac{n(n-1)(n-2)}{4}\int(\sigma\bar\sigma)^{n-1}\gamma^2 \qquad (45)\\
&\quad +n(n-1)\int(\sigma\bar\sigma)^{n-1}\beta\gamma+\frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\beta^2+(1-n)+(1-n)na\\
&\quad +n(1-n)\int(\sigma\bar\sigma)^{1-n}\beta\gamma+(1-n)\frac{n}{2}(n-1)\int(\sigma\bar\sigma)^{n-1}\gamma^2
\end{aligned}
$$

A small calculation yields

$$\frac{n(n-1)(n-2)}{4}+\frac{(1-n)n(n-1)}{2}=-\frac{n^2}{4}(n-1)$$

Hence we can simplify (45) and get

$$
\begin{aligned}
&\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2+(1-n)\left(\int\sigma^{n-1}\tau^n\alpha\right)\left(\int\sigma^n\tau^{n-1}\alpha\right) \qquad (46)\\
&= \; c+\frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\beta^2-\frac{n}{2}(n-1)a-\frac{n^2}{4}(n-1)\int(\sigma\bar\sigma)^{n-1}\gamma^2
\end{aligned}
$$

Now we compute the last summand in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$\int\sigma^{n-1}\tau^{n+1}=\sum_{i+j+k=n+1}\frac{(n+1)!}{i!j!k!}\int\sigma^{n-1}a^i\sigma^i\gamma^j\bar\sigma^k$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar\sigma^k$ is $2(n-1)+2i+j$ which is $2n$ only for $i=1$, $j=0$, $k=n$ or for $i=0$, $j=2$, $k=n-1$.

So we get for the last summand in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$\frac{n(n-1)}{2(n+1)}\int\sigma^{n-1}\tau^{n+1}=\frac{n}{2}(n-1)a+\frac{n^2}{4}(n-1)\int(\sigma\bar\sigma)^{n-1}\gamma^2 \qquad (47)$$

(We may assume that $\int\sigma^n\tau^{n-1}\alpha=1$ because of $\int(\sigma\bar\sigma)^n=1$.)

Then (46) and (47) yield

$$\mathbb{B}_{\sigma,\tau}(\alpha)=c+\frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\beta^2=\mathbb{B}_{\sigma,\bar\sigma}(\alpha)$$

The last equality follows from ([Hu], 1.9). □

Remark: In the following proofs we will assume that there is a form $\rho \in H^2(X, \mathcal{O}_X)$ such that $1 = \epsilon = \int(\sigma\rho^n$. If R is a strict henselian local ring whose residue characteristic is prime to n such a form $\bar{\sigma}$ always exists. For $\epsilon = 1$ the form (40) doesn't up to a root of unity depend on the choices of σ and τ. For $R = \mathbb{C}$ it coincides then with the usual Bogomolov-Beauville form up to a root of unity of order n.

Lemma 23. *Let X be a scheme of $K3$-type over S. Then the form*

$$\mathbb{B}_{\sigma,\tau} : H^2_{DR}(X/R) \times H^2_{DR}(X/R) \to R$$

is perfect.

Proof. We can reduce to the case where R is a complete local ring. Then we may assume that $\int(\sigma\tau)^n = 1$. We do so to disburden the computation. We consider the Hodge filtration

$$H^0(X, \Omega^2_{X/R}) = \mathrm{Fil}^2 \subset \mathrm{Fil}^1 \subset H^2_{DR}(X/R).$$

We claim that with respect to $\mathbb{B}_{\sigma,\tau}$

$$\mathrm{Fil}^1 \subset (\mathrm{Fil}^2)^{\perp}.$$

Let $\alpha \in \mathrm{Fil}^1$ we have to show that

$$\mathbb{B}_{\sigma,\tau}(\sigma, \alpha) = (1/2)(\mathbb{B}_{\sigma,\tau}(\sigma + \alpha) - \mathbb{B}_{\sigma,\tau}(\sigma) - \mathbb{B}_{\sigma,\tau}(\alpha)) = 0. \tag{48}$$

The second summand on the right hand side is clearly 0. We note that $\sigma^n\alpha = 0$ because $\mathrm{Fil}^{2n} \cup \mathrm{Fil}^1 \subset \mathrm{Fil}^{2n+1} = 0$. We compute

$$\mathbb{B}_{\sigma,\tau}(\sigma + \alpha) = (n/2)(\int (\sigma\tau)^{n-1}\sigma^2 + 2\int (\sigma\tau)^{n-1}\sigma\alpha + \int (\sigma\tau)^{n-1}(\alpha^2)). \tag{49}$$

The other terms on the right hand side of (40) vanish because $\sigma + \alpha \in \mathrm{Fil}^1$. We see that the first two terms on the right hand side of (49) vanish. This shows that (48) vanishes too.

Therefore $\mathbb{B}_{\sigma,\tau}$ induces a bilinear form $\bar{\mathbb{B}}_{\sigma,\tau}$ on $\mathrm{Fil}^1 / \mathrm{Fil}^2 = H^1(X, \Omega_{X/R})$. By the verification above we obtain

$$\bar{\mathbb{B}}_{\sigma,\tau}(\alpha) = (n/4)\int (\sigma\tau)^{n-1}\alpha^2 = (n/4)\int (\sigma\rho)^{n-1}\alpha^2.$$

By the requirement (38) this is perfect and

$$(\mathrm{Fil}^2)^\perp = \mathrm{Fil}^1\,. \tag{50}$$

Finally one has

$$\mathbb{B}_{\sigma,\tau}(\sigma,\tau) = (1/2).$$

which is a unit. We omit the easy verification. Together with the perfectness of $\bar{\mathbb{B}}_{\sigma,\tau}$ this implies perfectness. □

Proposition 24. *Let R be a local ring with perfect residue class field k. Let X be a scheme of $K3$-type over R such that X_k lifts to a smooth projective variety over a discrete valuation ring O of characteristic zero with residue class field k. Assume that σ and τ are chosen such that $\epsilon = 1$. Then the form*

$$\mathbb{B}_{\sigma,\tau} : H^2_{DR}(X/R) \times H^2_{DR}(X/R) \to R$$

is horizontal with respect to the Gauss-Manin connection.

Remark: Using the arguments of [De1] one can show that an arbitrary projective scheme over k lifts to a projective scheme over some O as above, but we skip the details here.

Proof. We begin to prove a complex analytic version of this Proposition. Let $X \to S$ be a proper and smooth morphism of complex analytic manifolds. Let $\Lambda \in H^0(X, R^2 f_*\mathbb{Q})$. Then we have a pairing

$$q_\Lambda : R^2 f_*\mathbb{Q} \times R^2 f_*\mathbb{Q} \to R^{2n} f_*\mathbb{Q} \xrightarrow{\int} \mathbb{Q}_S, \tag{51}$$

defined by

$$q_\Lambda(\alpha,\beta) = \int \Lambda^{2n-2}\alpha\beta, \quad \alpha,\beta \in R^2 f_*\mathbb{Q}.$$

If $\Lambda \in H^0(X, R^2 f_*\mathbb{Q})$ is the class of a relative ample line bundle on X then the pairing (51) is nondegenerate and we have

$$\nu := \nu(\Lambda) := \int \Lambda^{2n} \neq 0.$$

If we tensor (51) with $\mathcal{O}_S$ we obtain a horizontal pairing with respect to the Gauss-Manin connection:

$$q_\Lambda : H^2_{DR}(X/S) \times H^2_{DR}(X/S) \to \mathcal{O}_S. \tag{52}$$

Assume that Λ is a cohomology class such that q_Λ is non degenerate and $\nu(\Lambda) \neq 0$. Then we denote by $(R^2 f_* \mathbb{Q})_0 \subset R^2 f_* \mathbb{Q}$ the local system which is the orthogonal complement on Λ. The vector bundle $H^2_{DR}(X/S)$ decomposes as a vector bundle with connection

$$H^2_{DR}(X/S) = (H^2_{DR}(X/S))_0 \oplus \mathcal{O}_S \Lambda.$$

Lemma 25. *Let $\Lambda \in H^2_{DR}(X/S)$ and $\nu(\Lambda) = \int \Lambda^{2n} \in \mathcal{O}_S$. Then we have the following formula for all $\alpha \in H^2_{DR}(X/S)$:*

$$\nu(\Lambda)^2 \mathbb{B}_{\sigma,\tau}(\alpha) = \mathbb{B}_{\sigma,\tau}(\Lambda) \left[(2n-1)\nu(\Lambda) \int \Lambda^{2n-2}\alpha^2 - (2n-2) \left(\int \Lambda^{2n-1}\alpha \right)^2 \right].$$

Both terms on each side are functions in $\mathcal{O}_S$. We consider them as functions on the complex manifold S. For each $s \in S(\mathbb{C})$ we evaluate the functions at s. The analogous equality in s, namely for $\Lambda_s, \alpha_s, \sigma_s, \tau_s \in H^2_{DR}(X_s/\mathbb{C}) = H^2_{DR}(X/S) \otimes_{\mathcal{O}_S} k(s)$:

$$\begin{aligned} &\nu(\Lambda_s)^2 \mathbb{B}_{\sigma_s,\tau_s}(\alpha_s) = \\ &\quad \mathbb{B}_{\sigma_s,\tau_s}(\Lambda_s) \left[(2n-1)\nu(\Lambda_s) \textstyle\int \Lambda_s^{2n-2}\alpha_s^2 - (2n-2) \left(\textstyle\int \Lambda_s^{2n-1}\alpha_s \right)^2 \right] \end{aligned} \tag{53}$$

was shown in [Beauville] Théorème 5 (c), because the form $\mathbb{B}_{\sigma_s,\tau_s}$ coincides with the Beauville-Bogomolov form by Lemma 22. Hence the functions coincide on $S(\mathbb{C})$. But then the algebraic functions in $\mathcal{O}_S$ coincide as well. This proves the lemma.

The formula (53) shows that for $\alpha \in (H^2_{DR}(X/S))_0$

$$\nu^2 \mathbb{B}_{\sigma,\tau}(\alpha) = \mathbb{B}_{\sigma,\tau}(\Lambda)(2n-1)\nu q_\Lambda(\alpha). \tag{54}$$

Let $s \in S$. Since $\mathbb{B}_{\sigma_s,\tau_s}$ is up to a root of unity of order n the Beauville-Bogomolov form we know that $\mathbb{B}_{\sigma_s,\tau_s}(\Lambda_s)$ is a real number times an n-th root of unity by [Beauville] Théorème 5 (a). From this it follows that the analytic function $\mathbb{B}_{\sigma,\tau}(\Lambda)$ on S is constant. Therefore $\mathbb{B}_{\sigma,\tau}$ is by (54) a horizontal form with respect to the Gauss-Manin connection on the bundle $(H^2_{DR}(X/S))_0$.

We show that $(H^2_{DR}(X/S))_0$ and $\mathcal{O}_S$ are orthogonal for the form $\mathbb{B}_{\sigma,\tau}$ too. We have to show that

$$\nu^2(\mathbb{B}_{\sigma,\tau}(\alpha + \Lambda) - \mathbb{B}_{\sigma,\tau}(\alpha) - \mathbb{B}_{\sigma,\tau}(\Lambda)) = 0 \tag{55}$$

for all $\alpha \in (H^2_{DR}(X/S))_0$. From Lemma (25) we obtain

$$\nu^2 \mathbb{B}_{\sigma,\tau}(\alpha + \Lambda) = \mathbb{B}_{\sigma,\tau}(\Lambda)[(2n-1)\nu \int \Lambda^{2n-2}\alpha + (2n-1)\nu^2 - (2n-2)\nu^2].$$

From this one obtains (55). Therefore it suffices to show that $\mathbb{B}_{\sigma,\tau}$ is horizontal on the subbundle $\mathcal{O}_S\Lambda \subset H^2_{DR}(X/S))$. This is equivalent saying that $\mathbb{B}_{\sigma,\tau}(\Lambda) \in \mathcal{O}_S$ is a constant function. This we have seen above.

Now we can prove Proposition 24. We may assume that R is a complete local ring. Because the universal deformation exists for X_k it is enough to show that the Beauville-Bogomolov form is horizontal for the universal deformation. This is a formal scheme over $\operatorname{Spf} W(k)[[T_1, \ldots, T_r]]$, where the ideal of definition of $W(k)[[T_1, \ldots, T_r]]$ is $(p, T_1, \ldots, T_r)$. We denote this formal scheme simply by X.

On the other hand we can consider the universal deformation of a lifting $\tilde{X}/O$ of X_k which exists by assumption. Then we obtain a formal scheme Y over $\operatorname{Spec} O[[T_1, \ldots, T_d]]$ where the ideal of definition in the last ring is now $(T_1, \ldots, T_d)$. We may assume that O is complete. Then we a have a natural map

$$W(k)[[T_1, \ldots, T_r]] \to O[[T_1, \ldots, T_d]], \tag{56}$$

which corresponds on the tangent spaces to the natural homomorphism

$$H^1(\tilde{X}, \Omega_{\tilde{X}/O}) \to H^1(X_k, \Omega_{X_k/k}). \tag{57}$$

Therefore we may arrange $T_i \mapsto T_i$ after a coordinate transformation. By definition of (56) the p-adic completion of Y is the push-forward of X. Clearly it is enough to show that the Bogomolov-Beauville form of the family Y is horizontal. We take an embedding $O \to \mathbb{C}$. Then we obtain the universal deformation of $\tilde{X}_{\mathbb{C}}$. It suffices to show that $\mathbb{B}_{\sigma,\tau}$ is horizontal for the Gauss-Manin connection of this family. Since we obtain this by completion of the Kuranishi family $f : \mathfrak{X} \to S$ of $\tilde{X}_{\mathbb{C}}$ we are reduced to the case above. We have to ensure that there is a cohomology class $\Lambda \in H^0(S, R^2 f_* \mathbb{Q})$ such that

$$q_\Lambda \text{ is non-degenerate}, \quad \text{and } \nu(\Lambda) \neq 0. \tag{58}$$

Let $s_0 \in S$ the point such that $f^{-1}(s_0) = \tilde{X}_{\mathbb{C}}$. Let

$$\Lambda_0 \in (R^2 f_* \mathbb{Q})_{s_0} = H^2(\tilde{X}_{\mathbb{C}}, \mathbb{Q})$$

be the cohomology class of an ample line bundle on $\tilde{X}_{\mathbb{C}}$. By shrinking S we may assume that $R^2 f_* \mathbb{Q}$ is a constant local system on S. But then Λ_0 extends to a global section Λ of $R^2 f_* \mathbb{Q}$. Then Λ meets the requirements (58). This proves Proposition 24. $\square$

4 Deformations of varieties of K3-type

Let X_0/k be a projective and smooth scheme over of K3-type a perfect field k We consider the universal deformation

$$\mathfrak{X} \to S = \operatorname{Spf} A,$$

where

$$A = W[[T_1, \ldots, T_r]], \quad r = \dim_k H^1(X_0, \mathcal{T}_{X_0/k}).$$

We consider the Gauß-Manin connection

$$\nabla : H^2_{DR}(\mathfrak{X}/S) \to H^2_{DR}(\mathfrak{X}/S) \otimes_A \Omega^1_{S/W}.$$

If we compose this with the natural maps

$$\partial/\partial t_i : \Omega^1_{S/W} \to A, \quad i = 1, \ldots, r.$$

we obtain the maps

$$\nabla_i : H^2_{DR}(\mathfrak{X}/S) \to H^2_{DR}(\mathfrak{X}/S).$$

The de Rham cohomology is endowed with the Hodge filtration

$$0 \subset \operatorname{Fil}^2 H^2_{DR}(\mathfrak{X}/S) \subset \operatorname{Fil}^1 H^2_{DR}(\mathfrak{X}/S) \subset \operatorname{Fil}^0 H^2_{DR}(\mathfrak{X}/S) = H^2_{DR}(\mathfrak{X}/S).$$

We have $\operatorname{Fil}^2 H^2_{DR}(\mathfrak{X}/S) = H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$. We denote by $\operatorname{gr}^t H^2_{DR}(\mathfrak{X}/S)$ the subquotients of this filtration. By Griffith transversality ∇ induces a map

$$\operatorname{gr}^t \nabla : \operatorname{gr}^t H^2_{DR}(\mathfrak{X}/S) \to \operatorname{gr}^{t-1} H^2_{DR}(\mathfrak{X}/S) \otimes_A \Omega^1_{S/W},$$

which is A-linear. We are interested in this map for $t = 1$. By duality we obtain a A-linear map

$$\mathcal{T}_{S/W} \to \operatorname{Hom}_A(H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S}), H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})). \tag{59}$$

It is proved for $K3$-surfaces in [De1] that this is an isomorphism. The same argument works for varieties of K3-type. Indeed, the map (59) factors

$$\mathcal{T}_{S/W} \to H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}) \to \operatorname{Hom}_A(H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S}), H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})).$$

The first arrow is the Kodaira-Spencer map which is an isomorphism and the second map is the cup product. To see that the second map is an isomorphism we choose a generator $\omega \in H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$. The multiplication with ω induces an isomorphism $\mathcal{T}_{\mathfrak{X}/S} \cong \Omega^1_{\mathfrak{X}/S}$. Therefore the cup product with ω is an isomorphism

$$H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}) \cong H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S}).$$

This proves that (59) is an isomorphism. The isomorphism (59) signifies that $\nabla_1(\omega), \ldots, \nabla_r(\omega)$ is a basis of $H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})$.

Let $\alpha : R' \to R$ be a surjective homomorphism of local artinian W-algebras with residue class field k. We set $\mathfrak{a} = \operatorname{Ker} \alpha$. We assume the $\mathfrak{a}\mathfrak{m}_{R'} = 0$, where $\mathfrak{m}_{R'}$ denotes the maximal ideal of R'. Let X/R be a deformation of X_0 and let X' be a deformation of X over R'.

Let Y/R' be another deformation of X. The Gauß-Manin connection makes the de Rham cohomology a crystal. We endow $\mathfrak{a}$ with the trivial divided powers. Then we obtain an isomorphism (see below)

$$H^2_{DR}(X'/R') \to H^2_{DR}(Y/R'). \tag{60}$$

We denote by $F_Y \in H^2_{DR}(X'/R')$. the preimage of

$$H^0(Y, \Omega^2_{Y/R'}) = \operatorname{Fil}^2 H^2_{DR}(Y/R') \subset H^2_{DR}(Y/R')$$

by the isomorphism (60).

Proposition 26. *We assume that $\mathfrak{a}\mathfrak{m}_{R'} = 0$. The direct summand $F_Y \subset H^2_{DR}(X'/R')$ is contained in $\operatorname{Fil}^1 H^2_{DR}(X'/R')$. The map $Y \mapsto F_Y$ is a bijection between isomorphism classes of liftings Y/R' of X/R and direct summands $F \subset \operatorname{Fil}^1 H^2_{DR}(X'/R')$ which lift the direct summand $\operatorname{Fil}^2 H^2_{DR}(X/R) \subset \operatorname{Fil}^1 H^2_{DR}(X/R)$.*

Proof: We set $F' = \operatorname{Fil}^2 H^2_{DR}(X'/R')$. Let $F \subset H^2_{DR}(X'/R')$ be an arbitrary direct summand which lifts $\operatorname{Fil}^2 H^2_{DR}(X/R)$. We call this a lift of the Hodge filtration.

We consider the canonical map:

$$F \to H^2_{DR}(X'/R')/F'. \tag{61}$$

Its image is in $\mathfrak{a}(H^2_{DR}(X'/R')/F') \cong \mathfrak{a} \otimes_k H^2_{DR}(X_0/k)/\operatorname{Fil}^2 H^2_{DR}(X_0/k)$. The map (61) factors through $F \to Fil^2 H^2_{DR}(X_0/k)$. Therefore liftings of the Hodge filtration are classified by homomorphisms of k-vector spaces

$$\varkappa(F) : H^0(X_0, \Omega^2_{X_0/k}) \to \mathfrak{a} \otimes_k (H^2_{DR}(X_0/k)/\operatorname{Fil}^2 H^2_{DR}(X_0/k)). \tag{62}$$

The assertion that $F_Y \subset \operatorname{Fil}^1 H^2_{DR}(X'/R')$ is equivalent to saying that

$$\varkappa(F_Y)(H^0(X_0, \Omega^2_{X_0/k})) \subset \mathfrak{a} \otimes_k (\operatorname{Fil}^1 H^2_{DR}(X_0/k)/\operatorname{Fil}^2 H^2_{DR}(X_0/k)).$$

The deformation X'/R' of X_0 is given by a uniquely determined W-algebra homomorphism $f : A \to R'$ and the deformation Y is given by $g : A \to R'$. We obtain a commutative diagram:

$$A \underset{g}{\overset{f}{\rightrightarrows}} R' \longrightarrow R.$$

The isomorphism (60) is obtained as follows. Let $u \in H^2_{DR}(X'/R')$. We find $\tilde{u} \in H^2_{DR}(\mathfrak{X}/S)$ such that $u = f_*(\tilde{u})$. We set $v = g_*(\tilde{u})$. Then (60) is given as follows:

$$\begin{array}{ccc} H^2_{DR}(X'/R') & \longrightarrow & H^2_{DR}(Y/R') \\ u & \mapsto & v + \sum_{i=1}^r (f(t_i) - g(t_i))\breve{\nabla}_i(\tilde{u}) \end{array} \tag{63}$$

We denote here by $\breve{\nabla}_i(\tilde{u})$ the image of $\nabla_i(\tilde{u})$ in $H^2_{DR}(X_0/k)$. The formula (63) makes sense because $f(t_i) - g(t_i) \in \mathfrak{a}$.

Now we take for $\tilde{u}$ a generator of the free A-module $\operatorname{Fil}^2 H^2_{DR}(\mathfrak{X}/S)$. We deduce from (60) that

$$u - \sum_{i=1}^r (f(t_i) - g(t_i))\breve{\nabla}_i(\tilde{u})$$

is a generator of F_Y. Let $u_0 \in \operatorname{Fil}^2 H^2_{DR}(X_0/k)$ the image of $\tilde{u}$. Then the map $\varkappa(F_Y)$ is given by [De2] Lemma 1.1.2.

$$\varkappa(F_Y)(u_0) = -\sum_{i=1}^r (f(t_i) - g(t_i)) \otimes \breve{\nabla}_i(\tilde{u}) \in \mathfrak{a} \otimes_k \operatorname{gr}^1 H^2_{DR}(X_0/k).$$

This formula shows that $F_Y \subset Fil^1 H^2_{DR}(X/R)$. As we remarked (59) implies that $\breve{\nabla}_i(\tilde{u})$ form a basis of $\mathrm{gr}^1 H^2_{DR}(X_0/k)$. It follows that F_Y determines the elements $a_i := f(t_i) - g(t_i) \in \mathfrak{a}$ for $i = 1, \ldots, r$. Given such elements a_i we define $g(t_i) = f(t_i) - a_i$. The homomorphism $g : A \to R'$ thus defined gives the desired variety of $K3$-type. *Q.E.D.*

We will now extend the Proposition to the case where $R' \to R$ is an arbitrary *pd*-thickening with nilpotent divided powers on $\mathfrak{a}$.

We assume now that k is algebraically closed. We assume that $2n = \dim X_0$ is prime to the characteristic p of k. We also assume that X_0 lifts to a smooth projective scheme over some discrete valuation ring O with residue class field k. We fix generators σ resp. ρ of the 1-dimensional k-modules $H^0(X_0, \Omega^2_{X_0/k})$ resp. $H^2(X_0, \mathcal{O}_{X_0})$ such that $\int(\sigma\rho)^n = 1$. We can lift them to generators $\tilde{\sigma}$ resp. $\tilde{\rho}$ of the cohomology groups $H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$ resp. $H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Then we obtain by Proposition 24 and Lemma 23 a horizontal perfect symmetric pairing

$$(\, , \,) : H^2_{DR}(\mathfrak{X}/S) \times H^2_{DR}(\mathfrak{X}/S) \to S \tag{64}$$

which depends only on σ and ρ. With respect to this pairing the Hodge filtration is self dual:

$$(\mathrm{Fil}^1)^{\perp} = \mathrm{Fil}^2, \quad (\mathrm{Fil}^2)^{\perp} = \mathrm{Fil}^1 .$$

In the situation of the Proposition it is equivalent to say that the lift of the Hodge filtration $F \subset H^2_{DR}(X'/R')$ is in $\mathrm{Fil}^1 H^2_{DR}(X'/R')$ or that $F \subset H^2_{DR}(X'/R')$ is isotropic. Indeed, we take $\tilde{c} \in H^2_{DR}(\mathfrak{X}/S)$ which induces a generator of the A-module $H^2_{DR}(\mathfrak{X}/S)/\mathrm{Fil}^1$. Then $(\tilde{u}, \tilde{c})$ is a unit in A. The image c of $\tilde{c}$ in $H^2_{DR}(X'/R')$ induces a basis of $H^2_{DR}(X'/R')/\mathrm{Fil}^1 H^2_{DR}(X'/R')$.

Any lifting of the Hodge filtration has a generator of the form

$$u + \beta c + \sum_{i=1}^{r} \alpha_i \breve{\nabla}_i(\tilde{u}), \quad \alpha, \beta \in \mathfrak{a}. \tag{65}$$

Assume F is isotropic. Since u is orthogonal to $\breve{\nabla}_i(\tilde{u})$ we obtain $2\beta(u, c) = 0$ which implies $\beta = 0$. This implies $F \subset \mathrm{Fil}^1 H^2_{DR}(X'/R')$. On the other hand the vector (65) is isotropic if $\beta = 0$.

Theorem 27. *Let X_0 be a projective scheme of $K3$-type over an algebraically closed field k of characteristic $p > 0$.*

Let $\alpha : R' \to R$ be a surjective morphism of artinian local W-algebras with residue class field k. We assume that the kernel $\mathfrak{a}$ of α is endowed with nilpotent divided powers which are compatible with the canonical divided powers on pW.

Let X/R be a deformation of X_0 and X'/R' a lifting of X.

If Y/R' is an arbitrary lifting of X the Gauß-Manin connection provides an isomorphism

$$H^2_{DR}(X'/R') \to H^2_{DR}(Y/R') \tag{66}$$

which respect the symmetric bilinear forms on both sides. We denote by F_Y the preimage of $\operatorname{Fil}^2 H^2_{DR}(Y/R')$ by this isomorphism.

The map $Y \mapsto F_Y$ is a bijection between liftings Y/R' of X and liftings of the Hodge filtration $\operatorname{Fil}^2 H^2_{DR}(X/R) \subset H^2_{DR}(X/R)$ to isotropic direct summands $F \subset H^2_{DR}(X'/R')$.

Proof. The assertion that (66) respects the pairing $(\ ,\ )$ follows because the pairing is horizontal. Therefore F_Y is isotropic.

We consider the divided powers of the ideal $\mathfrak{a}$:

$$\mathfrak{a} \supset \mathfrak{a}^{[2]} \supset \ldots \supset \mathfrak{a}^{[t-1]} \supset \mathfrak{a}^{[t]} = 0.$$

If $t = 2$ the Theorem follows from the Proposition. We consider the *pd*-thickenings

$$R' \to R'/\mathfrak{a}^{[t-1]} \to R$$

By induction we may assume that the Theorem holds for the second thickening.

We start with an isotropic lifting $F \subset H^2_{DR}(X'/R')$ of the Hodge filtration. Let $R_1 = R/\mathfrak{a}^{[t-1]}$. Then F induces a filtration $F_1 \subset H^2_{DR}(X'_{R_1}/R_1)$. By induction there is a lifting Z/R_1 of X which corresponds to F_1. We choose an arbitrary lifting Z'/R' of Z. Since Z' is also a lifting of X we have an isomorphism

$$H^2_{DR}(X'/R') \to H^2_{DR}(Z'/R').$$

Let G be the image of F under this isomorphism. Then G is a lifting of the Hodge filtration $\operatorname{Fil}^2 H^2_{DR}(Z/R_1) \subset H^2_{DR}(Z/R_1)$. If the Proposition is applicable to $R' \to R_1$ we find a lifting Y/R' of Z/R_1 which correspond to $G \subset H^2_{DR}(Z'/R')$ and therefore to $F \subset H^2_{DR}(X'/R')$. Therefore our map is surjective.

Therfore it suffices to show our theorem for $R' \to R_1$. The kernel $\mathfrak{b} = \mathfrak{a}^{[t-1]}$ is endowed with the trivial divided powers and we have $\mathfrak{b}^2 = 0$.

Decomposing $R' \to R_1$ in a series of small surjections (as in the Proposition $R' \to R_m \to \ldots \to R_1$ we may argue as above.

The injectivity follows easily in the same manner. $\square$

We may reformulate this in the language of crystals. Let X be deformation of X_0 over an artinian local ring R with residue class field k (or equivalently a continuous homomorphism $A \to R$).

If $R' \to R$ is a *pd*-thickening where $R' \to R$ is a homomorphism of local artinian rings with residue field k. We consider the ccrystalline cohomology

$$H^2_{crys}(X/R').$$

This is a crystal in R' which is incuced from the Gauss-Manin connection on $H^2_{DR}(\mathfrak{X}/S)$. Therefore (64) induces a bilinear form of crystals

$$H^2_{crys}(X/R') \times H^2_{crys}(X/R') \to R'. \tag{67}$$

The Hodgefiltration on $H^2_{DR}(X/R) = H^2_{crys}(X/R)$ is selfdual with respect to this bilinear form.

We may reformulate the last Theorem.

Corollary 28. *Let $R' \to R$ be a surjective homomorphism of artinian local rings with algebraically closed residue class field k whose kernel is endowed with divided powers.*

Let X/R be a deformation of X_0. Then the liftings of X to X' correspond bijectively to liftings of the Hodge filtration to selfdual filtrations of $H^2_{crys}(X/R')$.

Corollary 29. *Let $R' \to R$ a pd-thickening and let X/R as in the last Corollary. Let X'/R' be a lifting of X. Let $\alpha : X \to X$ an automorphism of the R-scheme X (but not necessarily of the deformation).*

Then α lifts to an automorphism $\alpha' : X' \to X'$ iff $\alpha^* : H^2_{crys}(X/R') \to H^2_{crys}(X/R')$ respects the Hodge filtration given by X'.

Proof. The universal deformation space S classifies pairs (X, ρ) where X is a scheme of $K3$-type over R and $\rho : X_0 \to X_k$ is an isomorphism.

Since X is a deformation of X_0 the map ρ is given. Let $\alpha_0 : X_0 \to X_0$ be the automorphism induced by α. The data α is equivalent saying that the two pairs (X, ρ) and $(X, \rho\alpha_0)$ are isomorphic as deformations.

The existence of a lifting α' is equivalent saying that the pairs (X', ρ) and $(X', \rho\alpha_0)$ are isomorphic as deformations. Therefore we conclude by the last Corollary. $\square$

We set $A_0 = k[[T_1, \ldots, T_r]]$ and $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$. The absolute Frobenius $\mathfrak{X}_0 \to \mathfrak{X}_0^{(p)}$ induces a Frobenius-linear endomorphism F of $H^2_{crys}(\mathfrak{X}_0/A)$ with respect to the chosen Frobenius endomorphism on A. Let $\mathbb{B}$ the Bogomolov-Beauville form on $H^2_{crys}(\mathfrak{X}_0/A) = H^2_{DR}(\mathfrak{X}/A)$. Then we have

$$\mathbb{B}(Fx, Fy) = p^2 \mathbb{B}(x,y)^{\sigma}, \quad x, y \in H^2_{crys}(\mathfrak{X}_0/A). \tag{68}$$

This formula is a consequence of Lemma 25 which we consider as a formula for crystalline cohomology classes. We are then allowed to apply the Frobenius. The proof is like that of the horizontality of the BB-form.

We can now prove a refinement of Proposition 18.

Proposition 30. *Let k be an algebraically closed field and let X_0 be a projective scheme of $K3$ type which lift to a projective smooth scheme over some discrete valuation ring O of mixed characteristics with residue class field k.*

Let $f : X \to \operatorname{Spec} R$ be a deformation of X_0 over an artinian local ring R with residue class field k. Then the crystalline cohomology $H^2_{crys}(X/\hat{W}(R))$ has a unique structure of a selfdual $\hat{\mathcal{W}}_R$-display which is functorial in R.

Proof. We can use the frames $\mathcal{A}_n$ introduced before Proposition 18. We consider the $\mathcal{A}_n$-display $\mathcal{P}$ we introduced on $H^2_{crys}(\mathfrak{X}_{R_n}/A_n)$. We have to show that the Beauville-Bogomolov form induces a bilinear form of $\mathcal{A}_n$-displays

$$\mathcal{P} \times \mathcal{P} \to \mathcal{U}(2). \tag{69}$$

But because we are in the torsion free case it suffices to show that this pairing is compatible with F_0 which follows from (68). We already know that the Beauville-Bogomolov form induces a selfdual pairing on

$$H^2_{crys}(\mathfrak{X}_{R_n}/A_n) = H^2_{DR}(\mathfrak{X}_{A_n}/A_n).$$

with respect to the Hodge filtration on the right hand side. This shows that (69) is perfect. The assertion of the Proposition is obtained by base change. $\square$

Remark: In the same way one can generalize the Corollary of Proposition 18) and obtain duality on the displays there.

We denote by the selfdual $\hat{\mathcal{W}}_k$-2-display $(\mathcal{P}_0, \mathbb{B}_0)$ associated to X_0. We assume that this 2-display is F_0-étale (Definition 11). By Corollary 17 the deformation functor of $(\mathcal{P}_0, \mathbb{B}_0)$ is pro-representable by $S_{disp} = \operatorname{Spf} A_{disp}$, where A_{disp} is a power series ring over $W(k)$. The universal object is a $\hat{\mathcal{W}}_{A_{disp}}$-display. Let $\mathfrak{X} \to S$ be the universal deformation of X_0. By Proposition 30 we have a general morphism

$$S \to S_{disp}. \tag{70}$$

Let $f : X \to \operatorname{Spec} R$ as in Proposition 30 and let $(\mathcal{P}, \lambda)$ be the correspoding $\hat{\mathcal{W}}_R$ display. Let $R' \to R$ be a surjection of artinian local rings with residue class field k and kernel $\mathfrak{a}'$. We assume that $(\mathfrak{a}')^2 = 0$. Then the liftings of X to R' and of $(\mathcal{P}, \lambda)$ to R' are by Proposition 16 and Corollary 28 in natural bijection. In particular (70) is an isomorphism. We obtain:

Theorem 31. *Let k be an algebraically closed field. Let X_0 be a scheme of $K3$-type over k and assume that the associated selfdual $\hat{\mathcal{W}}_k$-2-display $(\mathcal{P}_0, \lambda_0)$ is F_0-étale.*

Let R be a local artinian ring with residue class field k. The map which associates to a deformation of X/R of X_0 its $\hat{\mathcal{W}}_R$-2-display $(\mathcal{P}, \lambda)$ is a bijection to the deformations of $(\mathcal{P}_0, \lambda_0)$ to R.

Moreover an automorphism of X_0 lifts to an automorphism of X (necessarily unique) iff the induced automorphism of $(\mathcal{P}_0, \lambda_0)$ lifts to $(\mathcal{P}, \lambda)$

Proof. The last statement is a consequence of Corollary 29. $\square$

5 The relative de Rham-Witt complex of an ordinary K3-surface

In this section we relate our results to the results of [N] and prove the degeneration of the integral de Rham-Witt spectral sequence for ordinary $K3$-surfaces.

Let R be a ring such that p is nilpotent on R, let $X/\operatorname{Spec} R$ be a smooth projective scheme.

We assume that there exists a formal lifting $\mathfrak{X}$ of X over $\operatorname{Spf} W(R)$ and let $\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ be its de Rham complex. We recall the following complex from [LZ2], paragraph 4, denoted by $\mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$:

$$I_R \otimes_{W(R)} \Omega^{\circ}_{\mathfrak{X}/W(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} I_R \otimes_{W(R)} \Omega^{m-1}_{\mathfrak{X}/W(R)} \xrightarrow{d} \Omega^{m}_{W(R)} \xrightarrow{d} \cdots$$

where $I_R = VW(R)$.

Let $W\Omega_{X/R}$ denote the relative de Rham-Witt complex and $N^m W\Omega^{\cdot}_{X/R}$ the Nygaard complex (compare [LZ2], introduction):

$$(W\mathcal{O}_{X/R})_{[F]} \xrightarrow{d} \cdots \xrightarrow{d} (W\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W\Omega^m_{X/R} \xrightarrow{d} W\Omega^{m+1}_{X/R} \xrightarrow{d} \cdots$$

Here F means the restriction of scalars via $F : W(R) \to W(R)$.

Then we recall the following.

Conjecture 32. *There is a canonical isomorphism in the derived category $D^+(X_{\mathrm{zar}}, W(R))$ between the Nygaard complex and the complex $\mathcal{F}^m\Omega_{\mathfrak{X}/W(R)}$:*

$$N^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$$

Remark 33.

1. *The Conjecture holds for $m < p$, if R is a reduced ring ([LZ2], Corollary 4.7).*
2. *Under the additional assumptions about the de Rham spectral sequence associated to $\Omega^{\cdot}_{X/R}$, resp. $\Omega^{\cdot}_{\mathfrak{X}/W(R)}$, the conjecture is related to ([LZ2], Conjecture 5.8) which predicts that for varying m the hypercohomology group $\mathcal{H}^n(X, N^m W\Omega_{X/R})$ defines a display structure on the crystalline cohomology $H^n_{\mathrm{crys}}(X/W(R))$.*

We prove the following

Theorem 34. *Let X/R be a smooth projective scheme, such that R is artinian with perfect residue field k and such that the closed fibre X_k is an ordinary K3-surface. Let $\mathfrak{X}$ be a formal lifting of X to $\operatorname{Spf} W(R)$. Assume that Conjecture 32 holds for $m = 2$.*

Then the de Rham-Witt spectral sequence associated to the relative de Rham-Witt complex

$$E_1^{i,j} = H^j(X, W\Omega^i_{X/R}) \longrightarrow \mathcal{H}^{i+j}(W\Omega^{\cdot}_{X/R})$$

degenerates. Moreover, one has the following properties:

- $H^0_{\mathrm{crys}}(X/W(R)) = H^0(X, W\mathcal{O}_{X/R})$
- $H^1_{\mathrm{crys}}(X/W(R)) = H^3_{\mathrm{crys}}(X/W(R)) = 0$
- $H^i(X, W\Omega^j_{X/R}) = 0$ *for $i + j$ odd, or $i + j > 4$ or $i + j = 4,\ i \neq j$*

- $H^2(X, W\Omega^2_{X/R}) = H^4_{\text{crys}}(X/W(R)) = W(R)$
- $H^2_{\text{crys}}(X/W(R)) \cong H^0(X, W\Omega^2_{X/R}) \oplus H^1(X, W\Omega^1_{X/R}) \oplus H^2(X, W\mathcal{O}_X)$

which is a Hodge-de Rham-Witt decomposition (slope decomposition) in degree 2, lifting the slope decomposition over $W(k)$.

Moreover, as in the case $R = k$, $H^2(X, W\mathcal{O}_X)$ is the Cartier-Dieudonné module of $\hat{\mathrm{Br}}_{X/R} = \hat{\mathbb{G}}_m/R$, the formal Brauer group of X, $H^1(W\Omega^1_{X/R})$ is the Dieudonné-module of $\Psi^{\text{et}}_{X/R}$, which is the étale part of the extended Brauer group $\Psi_{X/R}$ and $H^0(X, W\Omega^2_{X/R})$ is the (shifted by -1) Dieudonné-module of the Cartier dual $(\hat{\mathrm{Br}}_{X/R})^*$.

Remark 35. *This is the first non-trivial example where the spectral sequence of the relative de Rham-Witt complex degenerates.*

Proof. It is known that, as we are in the ordinary case, $\hat{\mathrm{Br}}_{X/R} \cong \hat{\mathbb{G}}_{m,R}$ and $H^2(X, W\mathcal{O}_X)$ is the Cartier module of $\hat{\mathrm{Br}}_{X/R}$, hence $H^2(X, W\mathcal{O}_X) = W(R)$.

Using Nygaard-Ogus, Theorem 3.20 ([N-O]) we can consider the composite map

$$D((\hat{\mathrm{Br}}_{X/R})^*) \longrightarrow D(\Psi^*_{X/R}) \longrightarrow H^2_{\text{crys}}(X/W(R)) \longrightarrow H^2(X, W\mathcal{O}_X) \qquad (71)$$

where the last map is an edge morphism in the spectral sequence and conclude that it is in isomorphism; the argument is the same as in the case $R = k$ (compare [N-O] page 490), hence $H^2(X, W\mathcal{O}_X)$ is a direct summand of rank 1 in $H^2_{\text{crys}}(X/W(R))$. As $H^1_{\text{crys}}(X/W(R)) = 0$ and there is an exact sequence

$$0 \longrightarrow H^0(X, W\Omega^1_{X/R}) \longrightarrow H^1_{\text{crys}}(X/W(R)) \longrightarrow H^1(X, W\mathcal{O}_X)$$

we see that $H^0(X, W\Omega^1_{X/R})$ vanishes and $H^1(X, W\mathcal{O}_X)$, being the Cartier-Dieudonné-module of the connected part of the p-divisible group of the Picard scheme of X vanishes too because the Picard scheme is zero.

From the spectral sequence we get a direct sum decomposition

$$\mathcal{H}^2_{\text{crys}}(X/W(R)) = H^2(X, W\mathcal{O}_X) \oplus \mathcal{H}^2(W\Omega^{\geq 1}_{X/R}[-1]) \qquad (72)$$

Let $\mathfrak{X}$ be the formal lifting of X over $\operatorname{Spf} W(R)$. It is known that the Hodge-de Rham spectral sequence of $\mathfrak{X}$ degenerates; moreover the Hodge-de Rham spectral sequence associated to the complex $\mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ degenerates too (see

[LZ2], Prop. 3.1 and Prop. 3.2). We have a commutative diagram of exact rows

$$\begin{array}{ccccc} \mathcal{H}^2(\Omega^{\geq 1}_{\mathfrak{X}/W(R)}[-1]) & \hookrightarrow & H^2_{dR}(\mathfrak{X}/W(R)) & \twoheadrightarrow & H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \\ \uparrow & & \uparrow & & \uparrow \cdot p \\ \mathcal{H}^2(0 \to I_R \otimes \Omega^1 \to \Omega^2 \to 0) & \hookrightarrow & \mathcal{H}^2(\mathcal{F}^2\Omega_{\mathfrak{X}/W(R)}) & \twoheadrightarrow & I_R H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \end{array} \tag{73}$$

The vertical map on the right hand side may be identified with

$$H^1(\Omega^1_{\mathfrak{X}/W(R)}) \oplus H^0(\mathfrak{X},\Omega^2_{\mathfrak{X}/W(R)}) \to I_R H^1(\Omega^1_{\mathfrak{X}/W(R)}) \oplus H^0(\mathfrak{X},\Omega^2_{\mathfrak{X}/W(R)}).$$

The vertical map in the middle is the map described in [LZ2], the map on the left is the natural inclusion map.

Now we look at the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{H}^1(W\Omega^1 \overset{d}{\to} W\Omega^2) & \to & H^2_{\text{crys}}(X/W(R)) & \to & H^2(X,W\mathcal{O}_{\mathfrak{X}}) & \to & 0 \\ & \uparrow (V,\text{id}) & & \uparrow & & \uparrow pV & & \\ 0 \to & \mathcal{H}^1(W\Omega^1_{X(R)} \overset{dV}{\to} W\Omega^2_{X/R}) & \to & \mathcal{H}^2(N^2W\Omega^{\cdot}_{X/R}) & \to & H^2(X,W\mathcal{O}_X) & \to & 0 \end{array} \tag{74}$$

Here the last vertical arrow factors through $H^2(X,W\mathcal{O}_X) \overset{\sim}{\to} I_R H^2(X,W\mathcal{O}_X) \overset{\cdot p}{\to} H^2(X,W\mathcal{O}_X)$ where the first isomorphism is given by $\xi\omega \mapsto V\xi\omega$ where ω is a generator of the rank 1-module $H^2(X,W\mathcal{O}_X)$ and $N^2W\Omega^{\cdot}_{X/R}$ is the Nygaard complex and the middle vertical arrow comes from the diagram

$$\begin{array}{ccccccc} W\mathcal{O}_X & \overset{d}{\longrightarrow} & W\Omega^1_{X/R} & \overset{dV}{\longrightarrow} & W\Omega^2_{X/R} & : & N^2W\Omega^{\cdot}_{X/R} \\ \downarrow pV & & \downarrow V & & \downarrow = & & \\ W\mathcal{O}_X & \overset{d}{\longrightarrow} & W\Omega^1_{X/R} & \overset{d}{\longrightarrow} & W\Omega^2_{X/R} & : & W\Omega^{\cdot}_{X/R} \end{array} \tag{75}$$

Using Conjecture 32 we can identify the middle vertical arrows in diagrams (73) and (74). Moreover, since $H^2(X,W\mathcal{O}_X)$ is isomorphic to $H^2(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) \cong W(R)$, we see that the whole diagram (74) is isomorphic to the diagram (73).

Now consider the commutative diagrams with exact rows

$$\begin{array}{ccccc} H^0(X,W\Omega^2_{X/R}) & \hookrightarrow & \mathbb{H}^1(X,[W\Omega^1_{X/R} \overset{d}{\to} W\Omega^2_{X/R}]) & \to & H^1(X,W\Omega^1_{X/R}) - \\ \uparrow = & & \uparrow \hat{\alpha} & & \uparrow V \\ H^0(X,W\Omega^2_{X/R}) & \hookrightarrow & \mathbb{H}^1(X,[W\Omega^1_{X/R} \overset{dV}{\to} W\Omega^2_{X/R}]) & \to & H^1(X,W\Omega^1_{X/R}) - \end{array}$$

$$\begin{array}{c} \longrightarrow H^1(X,W\Omega^2_{X/R}) \\ \uparrow = \\ \longrightarrow H^1(X,W\Omega^2_{X/R}) \end{array} \tag{76}$$

where $\hat{\alpha}$ is the map defined by (V, id) in (74); via the identification of 74 and (73) we see that $\hat{\alpha}$ is injective and the cokernel of $\hat{\alpha}$ is $H^1(X, \Omega^1_{X/R})$.

The above diagram shows that

$$V : H^1(X, W\Omega^1_{X/R}) \longrightarrow H^1(X, W\Omega^1_{X/R})$$

is injective too and $\operatorname{coker} V = H^1(X, \Omega^1_{X/R})$.

As $H^1(X, W\Omega^1_{X/R})$ is V-adically complete and separated we see that $H^1(X, W\Omega^1_{X/R})$ is a free $W(R)$-module of rank $= \operatorname{rank}_R H^1(X, \Omega^1_{X/R})$.

The composite map

$$0 \longrightarrow D(\Psi^*_{X/R}) \longrightarrow H^2_{\mathrm{crys}}(X/W(R)) \longrightarrow H^1(X, W\Omega^1_{X/R})$$

induces a map $D({\Psi^{\mathrm{et}}_{X/R}}^*) \to H^1(X, W\Omega^1_{X/R})$ and fits into a commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ D({\Psi^{\mathrm{et}}_{X/R}}^*) & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \\ \downarrow V & & \downarrow V \\ D({\Psi^{\mathrm{et}}_{X/R}}^*) & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \\ \downarrow & & \downarrow \\ H^1(X, \Omega^1_{X/R}) & \underset{=}{\longrightarrow} & H^1(X, \Omega^1_{X/R}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{77}$$

As both $D({\Psi^{\mathrm{et}}_{X/R}}^*)$ and $H^1(X, W\Omega^1_{X/R})$ are V-adically complete and separated and of the same rank over $W(R)$ they must be isomorphic. This implies that

$$\mathcal{H}^2(W\Omega^{\geq 1}_{X/R}[-1]) = H^0(X, W\Omega^2) \oplus H^1(X, W\Omega^1_{X/R})$$

splits and the composite map

$$H^0(X, W\Omega^2_{X/R}) \longrightarrow H^2_{\mathrm{crys}}(X/W(R)) \longrightarrow D(\hat{\mathrm{Br}}_{X/R})[-1]$$

being an injective map between rank $1 - W(R)$-modules must be an isomorphism because the right map is surjective with kernel $= D(\Psi^*_{X/R})$ ([N-O], Theorem 3.20).

The above discussion implies that the map $H^1(X, W\Omega^2_{X/R}) \to H^3_{\text{crys}}(X/W(R))$ is injective and as $H^3_{\text{crys}}(X/W(R))$ vanishes because the Albanese scheme is zero, $H^1(X, W\Omega^2_{X/R})$ vanishes too.

Consider now a diagram similar to 76 but in one degree higher:

$$\begin{array}{ccccc} H^1(X, W\Omega^2_{X/R}) & \hookrightarrow & \mathbb{H}^2(X, [W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}]) & \to & H^2(X, W\Omega^1_{X/R}) - \\ \uparrow = & & \uparrow \hat{\alpha} & & \uparrow V \\ H^1(X, W\Omega^2_{X/R}) & \hookrightarrow & \mathbb{H}^2(X, [W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}]) & \to & H^2(X, W\Omega^1_{X/R}) - \end{array}$$

$$\begin{array}{c} \longrightarrow H^2(X, W\Omega^2_{X/R}) \\ \uparrow = \\ \longrightarrow H^2(X, W\Omega^2_{X/R}) \end{array} \tag{78}$$

By the same argument as in (76) using Conjecture 32 we see that $\hat{\alpha}$ is injective, but $\mathcal{H}^2(W\Omega^1 \xrightarrow{d} W\Omega^2)$ can be identified with $H^2(\Omega^1_{\mathfrak{X}/W(R)}) \oplus H^1(\Omega^2_{\mathfrak{X}/W(R)})$ which is zero. Hence $\hat{\alpha}$ is the zero map and $\mathcal{H}^2(W\Omega^2 \xrightarrow{dV} W\Omega^2)$ vanishes too. This implies that $V : H^2(W\Omega^1) \to H^2(W\Omega^1)$ is injective with $\operatorname{coker} V = H^2(X, \Omega^1_{X/R})$ which is zero.

This implies that $H^2(W\Omega^1) = 0$. We got an exact sequence

$$0 \longrightarrow H^2(W\Omega^2_{X/R}) \longrightarrow \begin{array}{c} H^4_{\text{crys}}(X/W(R)) \\ = W(R) \end{array} \longrightarrow H^4\left(W\mathcal{O}_X \xrightarrow{d} W\Omega^1_{X/R}\right)$$

The exact sequence

$$0 \longrightarrow W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

induces, since $H^3(X, \mathcal{O}_X) = 0$, an exact sequence

$$0 \longrightarrow H^4(W\mathcal{O}_X) \xrightarrow{V} H^4(W\mathcal{O}_X) \longrightarrow H^4(X, \mathcal{O}_X) = 0$$

hence $H^4(W\mathcal{O}_X) = 0$.

By the same arguments one shows that $V : H^3(X, W\Omega^1_{X/R}) \to H^3(X, W\Omega^1_{X/R})$ is injective with vanishing cokernel $= H^3(\Omega^1_{X/R})$.

So $H^3(W\Omega^1_{X/R}) = 0$, this means

$$H^2(X, W\Omega^2_{X/R}) \cong H^4_{\text{crys}}(X/W(R))$$

and this finishes the proof of the theorem. □

Proposition 36. *Under the assumptions of Theorem 34, the Hodge-de Rham Witt decomposition of $H^2_{\text{crys}}(X/W(R))$ extends to a direct sum decomposition of displays (over the usual Witt ring $W(R)$) associated to the formal Brauer group, the étale part of the extended Brauer group and its Cartier dual, shifted by -1 and where $H^2_{\text{crys}}(X/W(R))$ is equipped with the Display-structure arising from the Nygaard complex (see [LZ2]).*

Proof. This is clear. $\square$

As we have seen Theorem 34 and Proposition 36 are only unconditional if R is reduced. If R is not reduced we can still derive a Hodge-Witt decomposition for H^2_{crys}, unconditionally, as follows.

Let as before B be the universal deformation ring of X_k, X_B be the universal familiy of X_k over Spf B, define $X_n = X_B \times_{\text{Spf}\, B} \text{Spec}\, B/m^n$. Let $\tilde{\mathcal{Y}}$ be a formal p-adic lifting to $W(B)$ with induced liftings $\mathcal{Y}^k_n$ over $\text{Spec}\, W_k(B/m^n)$, compatible with the liftings X_n. We assume that n is big enough so that $B \to R$ factors through $B/m^n \to R$.

By [LZ2], Theorem 4.6, we have for $r < p$ a quasiisomorphism

$$Ru_{n*}J^{[r]}_{X_n/W_k(B/m^n)} \longrightarrow I^r W_k \Omega^{\cdot}_{X_n/(B/m^n)}$$

where $u_n : \text{Crys}(X_n/W_k(B/m^n)) \to X_n$ is the canonical morphism of sites and $I^r W_k \Omega^{\cdot}_{X_n/(B/m^n)}$ denotes the complex

$$p^{r-1}VW_{k-1}(\mathcal{O}_{X_n}) \longrightarrow \cdots \longrightarrow VW_{k-1}\Omega^{r-1}_{X_n/(B/m^n)} \longrightarrow W_k\Omega^r_{X_n/(B/m^n)} \cdots$$

By [B-O], Theorem 7.2, $Ru_{n*}J^{[r]}_{X_n/W_k(B/m^n)}$ is represented by the complex ($I^k_n := VW_{k-1}(B/m^n)$):

$$p^{r-1}I^k_n\Omega^0_{\mathcal{Y}^k_n/W_k(B/m^n)} \longrightarrow \cdots \longrightarrow I^k_n\Omega^{r-1}_{\mathcal{Y}^k_n/W_k(B/m^n)} \longrightarrow \Omega^r_{\mathcal{Y}^k_n/W_k(B/m^n)} \longrightarrow \cdots .$$

As we pass to the projective limit with respect to k, n and note that all inverse systems of sheaves in the above complexes are Mittag-Leffler-systems, we get an isomorphism of complexes in the derived category of $W(B)$-modules between

$$p^{r-1}VW(b)\Omega^0_{\tilde{\mathcal{Y}}/W(B)} \longrightarrow \cdots \longrightarrow VW(B)\Omega^{r-1}_{\tilde{\mathcal{Y}}/W(B)} \longrightarrow \Omega^r_{\tilde{\mathcal{Y}}/W(B)} \longrightarrow \cdots$$

and

$$p^{r-1}VW\mathcal{O}_{X_B} \longrightarrow \cdots \longrightarrow VW\Omega^{r-1}_{X_B/B} \longrightarrow W\Omega^r_{X_B/B} \longrightarrow \cdots$$

which is the inverse limit of the complexes $I^r W_k \Omega^{\cdot}_{X_n/(B/m^n)}$ with respect to k, n.

As multiplication by p is injective on $\Omega^{\cdot}_{\tilde{\mathcal{Y}}/W(B)}$ resp. $W\Omega^{\cdot}_{X_B/B}$ (this can be reduced to a local argument, and can be made explicit for polynomial algebras) the first complex is isomorphic to $\mathcal{F}^r\Omega_{\tilde{\mathcal{Y}}/W(B)}$ (notation as in Conjecture 32) and the second complex is isomorphic to the Nygaard complex $N^r W\Omega^{\cdot}_{X_B/B}$. The above considerations hold for any smooth proper X/R with a smooth deformation ring B and $r < p$. For K3-surfaces we take $r = 2$ and see that Conjecture 32 holds for the universal family X_B over B. Thus the statement of Theorem 34 holds for X_B over $\operatorname{Spf} B$. In particular the de Rham-Witt spectral sequence

$$H^j(X_B, W\Omega^i_{X_B/B}) \longrightarrow \mathcal{H}^{i+j}(X_B, W\Omega^{\cdot}_{X_B/B})$$

degenerates and we have a Hodge-Witt decomposition

$$H^2_{\text{crys}}(X_B/W(B)) = \bigoplus_{i+j=2} H^i(X_B, W\Omega^j_{X_B/B}). \tag{79}$$

By base change we get a decomposition over $W(R)$ as follows:

$$H^2_{\text{crys}}(X/W(R)) = \bigoplus_{i+j=2} H^i(X_B, W\Omega^j_{X_B/B}) \otimes_{W(B)} W(R). \tag{80}$$

Moreover we have the following evident properties of the direct summands:

- $H^2(X_B, W\mathcal{O}_{X_B}) \otimes_{W(B)} W(R) = H^2(X, W\mathcal{O}_X)$ is the Cartier-Dieudonné-module of $\hat{\text{Br}}_{X_R} = \hat{\mathbb{G}}_m/R$;
- $H^1(X_B, W\Omega^1_{X_B/B}) \otimes_{W(B)} W(R)$ is the Dieudonné-module of $\Psi^{\text{et}}_{X/R}$;
- $H^0(X_B, W\Omega^2_{X_B/B}) \otimes_{W(B)} W(R)$ is the (shifted by -1) Dieudonné-module of the Cartier dual $\hat{\text{Br}}^*_{X/R}$.

As in the reduced case, the decomposition 80 which is a direct sum decomposition of Dieudonné-modules of p-divisible groups, extends to a direct sum decomposition of the corresponding displays, where $H^2_{\text{crys}}(X/W(R))$ carries the display structure obtained by base change via $B \to R$ from the display structure on $H^2_{\text{crys}}(X_B/W(B))$.

If $pR = 0$, there exists a reduced ring R' with $pR' = 0$ and a surjection $R' \to R$. One can then simplify the previous argument by working with a lifting X' of X to R', applying Theorem 34 to X'/R' and getting a Hodge-Witt decomposition by base change via $R' \to R$.

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