Windows for Displays of p-Divisible Groups

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Introduction

The starting point of this work was the classification of p-divisible groups over a discrete valuation ring of characteristic 0 with perfect residue field of characteristic $p \geq 3$ obtained by C.Breuil in his note [B]. We will show that such a classification holds under quite general circumstances. We prove this by showing that the category used by Breuil to classify p-divisible groups is equivalent to the category of Dieudonné displays, which we defined in [Z-DD]. Breuil obtains his result by a very useful classification of finite flat group schemes over a discrete valuation ring as above. We have no generalization of such a classification.

Our results are much more general and easier to prove, if we exclude p-divisible groups which have an étale part. Therefore this case is of independent interest. We will describe it first.

We fix a prime number p. Let R be a commutative ring with 1, such that p is nilpotent in R.

Definition 1 *A frame for R consists of the following data:*

1) A p-adic ring A which is torsion free as an abelian group.

2) An endomorphism σ of A.

3) A surjective ring homomorphism $A \to R$, such that the kernel J is an ideal with divided powers.

We require that σ induces the Frobenius endomorphism on A/pA.

The next objects we define give the possibility to operate a display.

Definition 2 *Let (A, J, σ) be a frame over R. A Dieudonné A-window over R consists of the following data:*

1) A finitely generated projective A-module M.

2) A submodule $M_1 \subset M$ which contains JM.

3) A σ-linear map $\Phi : M \to M$.

The following conditions are satisfied:

(i) M/M_1 *is a projective R-module.*

(ii)

$$\Phi M_1 \subset pM,$$

and M *is generated by the union of* ΦM *and* $\frac{1}{p}\Phi M_1$ *as an* A*-module.*

Let (M, M_1, Φ) be a Dieudonné A-window. One can show that there is a unique A-linear map $\Psi : M \to A \otimes_{\sigma,A} M$ such that $\Psi(\Phi(m)) = p \otimes m$ for all $m \in M$.

Definition 3 *We will say that* (M, M_1, Φ) *satisfies the nilpotence condition, if there is a number* N*, such that* $\Psi^N(M) \subset J \otimes_{\sigma^N,A} M$*. A Dieudonné* A*-window which satisfies the nilpotence condition is called an* A*-window.*

Theorem 4 *Let* R *be an excellent ring. Then the category of* A*-windows over* R *is equivalent to the category of formal* p*-divisible groups over* R*.*

The proof of the theorem uses the classification of formal p-divisible groups by displays given in [Z-DFG]. We will assume that the reader is acquainted with the introduction of this paper. Notations and results in this introduction will be freely used.

Let us again consider the case where R is an arbitrary ring with p nilpotent. There is a Cartier morphism (compare [BAC] IX §1, 2 prop.2) $\delta : A \to W(A)$ which is characterized by the property:

$$\mathbf{w}_n(\delta(a)) = \sigma^n(a).$$

Here $\mathbf{w}_n$ denotes the $n-th$ Witt polynomial. Let us denote the composite of δ with the canonical map $W(A) \to W(R)$ by $\varkappa$. Then we may associate to a window (M, M_1, Φ) a display $\mathcal{P} = (P, Q, F, V^{-1})$ over R as follows. We set:

$$P = W(R) \otimes_{\varkappa,A} M.$$

We define Q as the kernel of the canonical map $P \to M/M_1$. Moreover F is the ^{F}-linear extension of Φ to P. Finally V^{-1} is uniquely determined by the following equations:

$$\begin{array}{rcll} V^{-1}(\xi \otimes m_1) & = & {}^F\xi \otimes \Phi_1 m_1, & \text{for } \xi \in W(R),\ m_1 \in M_1 \\ V^{-1}({}^V\xi \otimes m) & = & \xi \otimes \Phi m, & \text{for } m \in M \end{array}$$

In this way we obtain a functor Dsp from the category of A-windows over R to the category of displays over R. The quasi inverse of this functor is defined by the Dieudonné crystal $\mathcal{D}_\mathcal{P}$ which is associated to a display $\mathcal{P}$. Indeed we find the window associated to $\mathcal{P}$ if we set:

$$M = \mathcal{D}_\mathcal{P}(A)$$

This shows that the functor Dsp is an equivalence of categories. Therefore the theorem above follows from [Z-DFG].

As a typical example of a frame we can take for R the polynomial ring $k[T_1, \ldots, T_d]$ over a field of characteristic p and for A the ring of restricted power series $C_k\{T_1, \ldots, T_d\}$ over a Cohen ring for k. For σ we take an endomorphism which induces on C_k any lifting of the Frobenius on k, and which acts on the indeterminates by $\sigma(T_i) = T_i^p$.

We consider now the case of an arbitrary p-divisible group. In this case we restrict ourself to the category $\mathcal{Z}$ of rings R with the following property: R is a local ring with maximal ideal $\mathfrak{m}$ and perfect residue field k of characteristic $p \geq 3$. Moreover we assume that there is a number N such that $x^N = 0$ for each $x \in \mathfrak{m}$. The exact sequence:

$$0 \to W(\mathfrak{m}) \to W(R) \to W(k) \to 0,$$

has a unique ^{F}-equivariant section $W(k) \to W(R)$. We set $\hat{W}(R) = \hat{W}(\mathfrak{m}) \oplus W(k)$, where $\hat{W}(\mathfrak{m}) \subset W(\mathfrak{m})$ is the subring of Witt vectors whose components are almost all zero. In the sense of the splitting of the sequence above $\hat{W}(R)$ is a subring of $W(R)$, which is stable by the Frobenius ^{F} and the Verschiebung ^{V}. We note that the stability by ^{V} doesn't hold in the case $p = 2$ (compare [Z-DD] Lemma 2).

Consider a pd-thickening $S \to R$ in the category $\mathcal{Z}$. This is a surjection of rings, such that the kernel $\mathfrak{a}$ is equipped with divided powers. The Witt polynomials $\mathbf{w}_n : W(\mathfrak{a}) \to \mathfrak{a}$ may be divided by p^n in the sense of these divided powers. These divided Witt polynomials provide an isomorphism of additive groups:

$$\log : W(\mathfrak{a}) \to \mathfrak{a}^{\mathbb{N}}.$$

The inverse image of the infinite direct sum $\mathfrak{a}^{(\mathbb{N})} \subset \mathfrak{a}^{\mathbb{N}}$ by this homomorphism is denoted by $\widetilde{W}(\mathfrak{a})$. We have $\hat{W}(\mathfrak{a}) \subset \widetilde{W}(\mathfrak{a})$. This last inclusion is an equality if the divided powers on $\mathfrak{a}$ are pointwise nilpotent. Let us denote by $\widetilde{W}(S)$ the subring of $W(S)$ generated by $\hat{W}(S)$ and $\widetilde{W}(\mathfrak{a})$. Of course the ring $\widetilde{W}(S)$ depends of the divided powers, but we omit this in the notation.

We call a p-adic ring A a $\hat{\mathcal{Z}}$-ring if for each number n the ring A/p^nA is a $\mathcal{Z}$-ring. We set $\hat{W}(A) = \varprojlim \hat{W}(A/p^nA)$. If we have a pd-thickening $A \to R$ we set $\widetilde{W}(A) = \varprojlim \widetilde{W}(A/p^nA)$, where the last projective limit goes over numbers n, such that p^n is zero in R. The rings $\hat{W}(A)$ and $\widetilde{W}(A)$ may be identified with subrings of $W(A)$.

Definition 5 *A Dieudonné frame (A, J, σ) for a $\mathcal{Z}$-ring R consists of the following data:*

1) A $\hat{\mathcal{Z}}$-ring A which is torsion free as an abelian group.

2) An endomorphism σ of A.

3) A surjective ring homomorphism $A \to R$, such that the kernel J is an ideal with divided powers.

We require that σ induces the Frobenius endomorphism on A/pA, and that the Cartier morphism $A \to W(A)$ factors through $\widetilde{W}(A)$.

Let us denote by $\varkappa$ the composite of the morphisms $A \xrightarrow{\delta} \widetilde{W}(A) \to \hat{W}(R)$. If we tensorize a Dieudonné A-window by $\varkappa$ we obtain a Dieudonné display. Again this functor is an equivalence from the category of Dieudonné A-windows over the $\mathcal{Z}$-ring R to the category of Dieudonné displays over R. As a consequence of [Z-DD] we obtain:

Theorem 6 *Let R be a $\mathcal{Z}$-ring such that its maximal ideal is nilpotent. Let (A, J, σ) be a Dieudonné frame for R. Then the category of Dieudonné A-windows over R is equivalent to the category of p-divisible groups over R.*

We will now give some examples of Dieudonné frames.

Example 1: Let k be a perfect field of characteristic $p \geq 3$. Let d and s be arbitrary numbers. Then we consider the $\mathcal{Z}$-ring:

$$R = k[T_1, \dots, T_d]/(T_1^s, \dots, T_d^s)$$

We set $A = W(k)[T_1, \dots, T_d]/(T_1^s, \dots, T_d^s)$. The obvious surjection $A \to R$ has as kernel the pd-ideal pA. We define σ as the ^{F}-linear endomorphism of A such that $\sigma(T_i) = T_i^p$. Then we obtain a Dieudonné frame (A, J, σ).

We let s vary, and denote the objects just defined by R_s respectively A_s. We set $R = k[[T_1, \dots, T_d]]$ and $A = W(k)[[T_1, \dots, T_d]]$. Let us define the endomorphism $\sigma : A \to A$ in the same way as before. Then we obtain a frame $A \to R$, which is not a Dieudonné frame. But it is a limit of Dieudonné frames. Indeed, to give a Dieudonné A-window over R is the same thing as to give a Dieudonné A_s-window $\mathcal{M}_s$ over the ring R_s for each number s together with isomorphisms:

$$A_s \otimes_{A_{s+1}} \mathcal{M}_{s+1} \cong \mathcal{M}_s$$

As a result we obtain that the category of p-divisible groups over the ring $R = k[[T_1, \dots, T_d]]$ is equivalent to the category of Dieudonné A-windows over R.

Example 2: Let k be as in the previous example. Let S be a flat and finite local $W(k)$-algebra with residue field k. We set $R = S/p^u S$, where s is some fixed number. We choose a presentation:

$$0 \to \mathfrak{c} \to W(k)[T_1, \ldots, T_d] \to S \to 0,$$

such that each T_i for $i = 1, \ldots d$ is mapped to the maximal ideal of S. Let $A_0 \subset W(k)[T_1, \ldots, T_d] \otimes \mathbb{Q}$ be the subring generated over $W(k)[T_1, \ldots, T_d]$ by all elements of the form $c^n/n!$ for $c \in \mathfrak{c}$, $n \in \mathbb{N}$. We obtain an obvious surjection $A_0 \to S$, whose kernel is a pd-ideal. Let us denote by A the p-adic completion of A_0. Then $A \to S$ is a pd-thickening. Consider the ^{F}-linear endomorphism of σ of $W(k)[T_1, \ldots, T_d] \otimes \mathbb{Q}$ such that $\sigma(T_i) = T_i^p$ for $i = 1, \ldots, d$. It induces an endomorphism of A_0 and A. The pd-thickening $A \to R$ together with the endomorphism σ of A is a Dieudonné frame. As in example 1 this may be used to classify p-divisible groups by Dieudonné A-windows over S.

Example 3: Let $k[[T_1, \ldots, T_d]]$ be as in the last example. Let $f_1, \ldots, f_r \in k[[T_1, \ldots, T_d]]$ be elements such that $f_1, \ldots, f_r, T_{r+1} \ldots, T_d$ is a system of parameters of this local ring. We set

$$R = k[[T_1, \ldots, T_d]]/(f_1, \ldots, f_r, T_{r+1}^s \ldots, T_d^s),$$

where s is some fixed number.

We choose arbitrary liftings $\tilde{f}_1, \ldots, \tilde{f}_r \in W(k)[[T_1, \ldots, T_d]]$. Consider the following morphism of flat $W(k)[[T_{r+1}, \ldots, T_d]]$-algebras:

$$W(k)[[T_1, \ldots, T_d]] \to W(k)[[T_1, \ldots, T_d]]/(\tilde{f}_1, \ldots, \tilde{f}_r).$$

We denote by B the p-adic completion of the pd-hull of this morphism (compare BBM lemme 2.3.3). We set $A = B/(T_{r+1}^s, \ldots, T_d^s)$. This is a p-adic ring which is a pd-thickening of R. The endomorphism σ of $W(k)[[T_1, \ldots, T_d]]$ used in the last example extends to A. This gives a Dieudonné frame for R.

We may use Dieudonné windows over $S = k[[T_1, \ldots, T_d]]/(f_1, \ldots, f_r)$ with frame $\hat{B} = \varprojlim_s B/(T_{r+1}^s, \ldots, T_d^s) \to S$ to classify p-divisible groups over S as in the end of example 1.

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1 The Case of a Formal p-Divisible Group

Definition 1.1 *Let R be a ring such that p is nilpotent in R. A quasiframe for R consists of the following data:*

1) A p-adic ring A, and a surjective ring homomorphism $\alpha : A \to R$ whose kernel will be denoted by J.

2) A ring endomorphism $\sigma : A \to A$ and a σ-linear map $\sigma_1 : J \to A$, such that $\sigma \big|_J = p \cdot \sigma_1$.

3) A ring homomorphism $\varkappa : A \to W(R)$

We require that the following conditions are satisfied:

(i) The image of the ideal J in A/pA consists of nilpotent elements.

(ii) The following diagram is commutative:

$$\begin{tikzcd} A \arrow[rr, "\varkappa"] \arrow[dr, "\alpha"'] & & W(R) \arrow[dl, "\mathbf{w}_0"] \\ & R & \end{tikzcd}$$

(iii) The following relations are satisfied:

$$\varkappa(\sigma(a)) = {}^F\varkappa(a) \quad \text{for } a \in A$$
$${}^V\varkappa(\sigma_1(u)) = \varkappa(u) \quad \text{for } u \in J$$

We are going to make some comments on this definition.

The ideal J is contained in the radical of A. This is clear.

The first relation of (iii) implies:

$$\alpha(\sigma(a)) = \alpha(a)^p \mod pR$$

Any finitely generated projective R-module L_0 may be lifted to a finitely generated projective A-module L.

Indeed, for big m the morphism α factors $A \to A/p^m \to R$. Then (i) implies that any element in the kernel of the second arrow is nilpotent. Therefore finitely generated projective R-modules lift with respect to that surjection (Compare e.g.H.Bass: Algebraic K-Theory, Benjamin 1968, Chapt.III, 2.10). For the first arrow we obtain liftings by a limit argument.

Example: Let R be a complete discrete valuation ring of characteristic 0 with residue field k. Let C_k be a Cohen ring for k. We choose an endomorphism σ of C_k which lifts the Frobenius endomorphism on k. By the structure theorem for complete local rings R is an Eisenstein extension of C_k, i.e. we have an exact sequence:

$$0 \to E(u)C_k[u] \to C_k[u] \to R \to 0,$$

where $E(u) \in C_k[u]$ is an Eisenstein polynomial of degree e. We denote the image of u in R by π.

We consider the ring $A = C_k[u, \frac{u^{ep}}{p}]$. We extend σ to an endomorphism of A by the rule $\sigma(u) = u^p$. Because $E(u) = u^e + pF(u)$ for some polynomial $F(u)$ the morphism $C_k[u] \to R$ extends to a homomorphism $A \to R$. One checks that σ maps the kernel of this homomorphism to pA. This allows to define σ_1 as in 1.1. Finally we define a morphism

$$\varkappa : A \to W(R),$$

as follows. On the subring C_k this is the composite $C_k \to W(C_k) \to W(R)$, where the first arrow is the Cartier morphism associated to σ. To complete the definition we set:

$$\varkappa(u) = [\pi], \quad \varkappa(\frac{u^{ep}}{p}) = \frac{[\pi^{ep}]}{p}.$$

Here $[\pi]$ denotes the Teichmüller representative in $W(R)$. It is easy to check that $(A \to R, \sigma, \varkappa)$ is a quasiframe.

If we use the fact that the Witt ring $W(R)$ is p-adic ([Z-DFG] Prop.1.3), we obtain that $\varkappa$ extends to the completion $\hat{A} = C_k[[u, \frac{u^{ep}}{p}]]$. Hence we have also a quasiframe if we replace A by $\hat{A}$. Breuil used this quasiframe to prove Fontaines conjecture for p-divisible groups in the case, where k is a finite field of characteristic $p > 2$.

Let $\tilde{P}$ be a finitely generated projective A-module, and set $P = R \otimes_A \tilde{P}$. Then any direct summand L of P may be lifted to a direct summand $\tilde{L}$ of $\tilde{P}$, and hence any direct decomposition $P = L \oplus T$ may be lifted to a direct decomposition $\tilde{P} = \tilde{L} \oplus \tilde{T}$.

Definition 1.2 *Let A be a quasiframe over R. An A-window over R consists of the following data:*

1) A finitely generated projective A-module M.

2) A submodule $M_1 \subset M$ which contains JM.

3) Two σ-linear maps $\Phi : M \to M$ and $\Phi_1 : M_1 \to M$.

The following conditions are satisfied:

(i) M/M_1 is a projective R-module.

(ii)

$$\Phi(m_1) = p\Phi_1(m_1), \ \textit{for } m_1 \in M_1$$
$$\Phi_1(um) = \sigma_1(u)\Phi(m), \ \textit{for } m \in M, u \in J.$$

(iii) The union of $\Phi_1(M_1)$ and $\Phi(M)$ generates M as an A-module.

(iv) Let $\Psi : M \to A \otimes_{\sigma,A} M$ be the unique A-linear map with the property that $\Psi(\Phi_1(m_1)) = 1 \otimes m_1$, for $m_1 \in M_1$, and that $\Psi(\Phi m) = p \otimes m$ for $m \in M$. Then there is an integer N, such that $\Psi^N(M) \subset J \otimes_{\sigma^N,A} M$.

We will refer to (iv) as the nilpotence condition. We have to show the existence of Ψ in order to justify this definition. This is done exactly as in [Z-DFG] Lemma 1.10:

We may lift the direct summand $M_1/JM \subset M/JM$ to a direct summmand L of M, by the remark after the definition 1.1. Let T be a complement to L. Then we obtain decompositions of A-modules:

$$M = L \oplus T \qquad M_1 = L \oplus JT.$$

A decomposition of M which satisfies the last equation is called normal. We consider the following σ-linear map:

$$\Phi_1 \oplus \Phi : L \oplus T \to M.$$

It follows from (iii) and (ii) of definition 1.2 that this is a σ-linear isomorphism, i.e. the linearization $(\Phi_1 \oplus \Phi)^\sharp$ is an isomorphism. Then we set:

$$\Psi = (id_L \oplus p\, id_T)((\Phi_1 \oplus \Phi)^{\#})^{-1}.$$

One verifies that with this definition Ψ has the property required by (iv) above. Therefore this Ψ is independent of the chosen normal decomposition.

The obvious example of a quasiframe is $A = W(R), \alpha = \mathbf{w}_0, \varkappa = id$. The pd-ideal J is $I_R = {}^VW(R)$ with the natural divided powers ([Z-DFG] §2.3). The operators Φ, Φ_1, resp. Ψ are in this case denoted by F, V^{-1}, resp $V^\sharp$.

Next we define a functor from the category of A-windows to the category of displays over R. Let us start with an A-window (M, M_1, Φ, Φ_1). Then we obtain a display (P, Q, F, V^{-1}) as follows:

We set $P = W(R) \otimes_{\varkappa,A} M$. The kernel of the map

$$W(R) \otimes_{\varkappa,A} M \xrightarrow{\mathbf{w}_0 \otimes id} R \otimes_{\alpha,A} M \to M/M_1$$

is by definition Q. The map $F : W(R) \otimes_{\varkappa,A} M \to W(R) \otimes_{\varkappa,A} M$ is defined by

$$F(\xi \otimes m) = {}^F\xi \otimes \Phi m.$$

Finally we claim that there is a unique ^{F}-linear map $V^{-1} : Q \to P$, which satisfies the following relations:

$$\begin{array}{rcll} V^{-1}(\xi \otimes m_1) & = & {}^F\xi \otimes \Phi_1 m_1, & \text{for } \xi \in W(R), m_1 \in M_1 \\ V^{-1}({}^V\xi \otimes m) & = & \xi \otimes \Phi m, & \text{for } m \in M \end{array} \tag{1}$$

If we show that V^{-1} exists it is obvious that (P, Q, F, V^{-1}) is a display. For the existence of V^{-1} we choose a normal decomposition $M = L \oplus T$ and $M_1 = L \oplus JT$. Then we obtain $P = W(R) \otimes_{\varkappa,A} L \oplus W(R) \otimes_{\varkappa,A} T$ and $Q = W(R) \otimes_{\varkappa,A} L \oplus I_R \otimes_{\varkappa,A} T$. We define V^{-1} on the first direct summand of Q by the first equation of (1) and on the second direct summand by second equation of (1).

The first equation of (ii) shows that this definition is possible. We have to verify that with this definition the properties (1) are satisfied. For this it is enough to show that the first equation of (1) holds for $m_1 \in JT$ and the second equation of (1) holds for $m \in L$. Indeed, let $u \in J$ and $t \in T$. Then we obtain using (ii):

$$\begin{aligned} & V^{-1}(\xi \otimes ut) = V^{-1}(\xi\varkappa(u) \otimes t) = V^{-1}(\xi^V\varkappa(\sigma_1(u)) \otimes t) \\ =& V^{-1}({}^V({}^F\xi\varkappa(\sigma_1(u))) \otimes t) = {}^F\xi\varkappa(\sigma_1(u)) \otimes \Phi t \\ =& {}^F\xi \otimes \sigma_1(u)\Phi t = {}^F\xi \otimes \Phi_1(ut). \end{aligned}$$

Finally we check the second equation (1) for $m = l \in L$:

$$V^{-1}({}^V\xi \otimes l) = {}^{FV}\xi \otimes \Phi_1(l) = p\xi \otimes \Phi_1(l) = \xi \otimes \Phi(l).$$

Proposition 1.3 *The construction above provides a functor from the category of A-windows over R to the category of displays over R.*

Hence we have also a functor from the category of A-windows to the category of formal p-divisible groups.

We will now give conditions for the quasiframe A, which assure that the functor from A-windows to displays is an equivalence of categories.

Definition 1.4 *A frame over R consists of the following data:*

1) A torsion free (as an abelian group) p-adic ring A.

2) A surjective homomorphism $A \xrightarrow{\alpha} R$, whose kernel will be denoted by J.

3) An endomorphism $\sigma : A \to A$.

We require the following properties

(i) σ lifts the Frobenius on A/pA.

(ii) The ideal J has divided powers.

We associate to any A-frame a A-quasiframe as follows. According to Cartier there is a morphism:

$$\delta : A \to W(A),$$

which is uniquely determined by the property that

$$\mathbf{w}_n(\delta(a)) = \sigma^n(a).n$$

The divided powers on J define an inclusion (see (17) below) $J \to W(A)$, whose image will be denoted by $\breve{J} \subset W(A)$. Clearly $\breve{J}$ is a pd-ideal and so is $\breve{J} \oplus I_A \subset W(A)$. The ideal $\mathfrak{c} = \breve{J} \oplus I_A$ is the kernel of $W(A) \to R$. It follows that δ is a morphism of pd-extensions (since we have no p-torsion):

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & W(A) \arrow[dl] \\ & R & \end{tikzcd}$$

We extend the ^{F}-linear homomorphism ${}^{V^{-1}} : I_A \to W(A)$ to an ^{F}-linear homomorphism ${}^{V^{-1}} : \mathfrak{c} \to W(A)$, by setting ${}^{V^{-1}}\breve{J} = 0$.

We also introduce the σ-linear map:

$$\sigma_1 = \frac{1}{p}\sigma : J \to A.$$

Then we have the relation:

$$\delta(\sigma_1(a)) = {}^{V^{-1}}\delta(a) \quad a \in J \tag{2}$$

Since $W(A)$ has no p-torsion we may multiply the equation (2) by p to verify it. Since $p\ {}^{V^{-1}} = {}^F$ holds on the ideal $\mathfrak{c}$, we obtain:

$$\delta(\sigma(a)) = {}^F\delta(a) \tag{3}$$

But this follows directly from the definition of the Cartier morphism.

We define $\varkappa$ as the composition

$$\varkappa : A \xrightarrow{\delta} W(A) \xrightarrow{W(\alpha)} W(R)$$

Since $W(\alpha)(\breve{J}) = 0$ it follows that $W(\alpha)$ is a morphism of pd-thickenings of R.

$$\begin{tikzcd} W(\alpha) : W(A) \arrow[rr] \arrow[dr] & & W(R) \arrow[dl] \\ & R & \end{tikzcd}$$

Proposition 1.5 *The data $\alpha : A \to R$, σ, σ_1, $\varkappa$ define a quasiframe over R.*

Proof: Since J has divided powers we have $u^p = p\alpha_p(u) \equiv 0 \mod pA$ for $u \in J$ and therefore (i) is satisfied. The next condition (ii) is trivially satisfied. The first relation of (iii) is satisfied because of (3). From (2) we conclude:

$$\varkappa(\sigma_1(u)) = {}^{V^{-1}}\varkappa(u) \text{ for } u \in J$$

We note here that $VV^{-1} \neq id$ on the ideal $\mathfrak{c}$, so that the desired relation doesn't follow immediately. But since $\varkappa(u) \in W(\alpha)(\mathfrak{c}) \subset I_R$ we have indeed ${}^{VV^{-1}}\varkappa(u) = \varkappa(u)$. This proves the proposition. $Q.E.D.$

We note that the definition of a window greatly simplifies if A is a frame. Indeed, since A has no p-torsion the operator Φ_1 is uniquely determined by $p\Phi_1 = \Phi$. The second relation (ii) of definition 1.2 follows automatically from the fact that Φ is σ-linear. Hence the definition 3 of an A-window over R is equivalent to the definition 1.2.

In practice we have frames which satisfy the additional condition that $\sigma(J)$ generates pA as an A-module. With this assumption the condition (ii) of definition 2 is equivalent to the requirement that $\Phi(M_1)$ generates pM as an A-module.

We can also define a frame for a p-adic ring R. If (A, J, σ) is such a frame then $(A, J + p^m A, \sigma)$ is a frame for $R/p^m R$ for each number m. Then we define a window exactly as above, but we require the nilpotence condition (iii) only modulo pR or equivalently modulo $p^m R$ for any m. With this definition a window over R is the same thing a compatible system of windows $\{\mathcal{M}_m\}$, where $\mathcal{M}_m$ is a window over $R/p^m R$, such that $\mathcal{M}_m$ is obtained from $\mathcal{M}_{m+1}$ by base change.

Let us consider an A-window $\mathcal{M} = (M, M_1, \Phi)$. On M_1 we will set $\Phi_1 = \frac{1}{p}\Phi$. We have associated a display to $\mathcal{M}$ by setting:

$$\begin{array}{ll} P & = W(R) \otimes_{\varkappa, A} M \\ Q & = \text{Ker } W(R) \otimes_{\varkappa, A} M \to M/M_1 \\ F(\xi \otimes x) & = {}^F\xi \otimes \Phi(x), \quad \text{for } \xi \in W(R), x \in M \end{array} \tag{4}$$

Finally the operator $V^{-1} : Q \to P$ is uniquely determined by the relations:

$$\begin{array}{lcll} V^{-1}(\xi \otimes m_1) & = & {}^F\xi \otimes \Phi_1 m_1, & \text{for } m_1 \in M_1 \\ V^{-1}({}^V\xi \otimes m) & = & \xi \otimes \Phi m, & \text{for } m \in M \end{array} \tag{5}$$

Hence we have functors:

$$(A\text{-windows}) \xrightarrow{\text{Dsp}} (\text{displays}/R) \xrightarrow{\text{BT}} (\text{formal } p\text{-divisible groups } /R)$$

Here BT is the functor from [Z-DFG] chapt. 3. The composition with Dsp will be also denoted by BT.

Theorem 1.6 *If A is a frame the functor* Dsp *is an equivalence of categories.*

Proof: We construct a quasiinverse functor. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R. We set $\overline{R} = R/pR$ and we denote by $\overline{\mathcal{P}} = \mathcal{P}_{\overline{R}}$ the display obtained by base change. Then we will associate an A-window as follows:

In [Z-DFG] Definition 2.6 we have associated to the display $\mathcal{P}$ a crystal $\mathcal{D}_{\mathcal{P}}$ which was called the Dieudonné crystal. It is defined for pd-thickenings $S \to R$, such that p is nilpotent in S. For the pd-thickening $A \to R$ we set:

$$\mathcal{D}_{\mathcal{P}}(A) = \varprojlim \mathcal{D}_{\mathcal{P}}(A/p^n A)$$

This makes sense because for big n we have a pd-thickening $A/p^n A \to R$.

We define the finitely generated projective A-module M:

$$M = \mathcal{D}_{\mathcal{P}}(A) = \mathcal{D}_{\overline{\mathcal{P}}}(A)$$

Then the maps $F^{\#} : \overline{\mathcal{P}}^{(p)} \to \overline{\mathcal{P}}$ respectively $V^{\#} : \overline{\mathcal{P}} \to \overline{\mathcal{P}}^{(p)}$ induce A-linear maps by evaluating the Dieudonné crystal:

$$\begin{array}{ll} \Phi^{\#} = \mathcal{D}(F^{\#}) : & A \otimes_{\sigma,A} M \to M \\ \Psi^{\#} = \mathcal{D}(V^{\#}) : & M \to A \otimes_{\sigma,A} M, \end{array} \tag{6}$$

such that $\Phi^{\#} \circ \Psi^{\#} = p$ and $\Psi^{\#} \circ \Phi^{\#} = p$.

Since $\varkappa$ is a morphism of pd-extensions we obtain a canonical isomorphism:

$$W(R) \otimes_{\varkappa,A} M \cong P \cong \mathcal{D}_{\mathcal{P}}(W(R)). \tag{7}$$

The last isomorphism follows from [Z-DFG] Proposition 2.12. Here again we set:

$$\mathcal{D}_{\mathcal{P}}(W(R)) = \varprojlim \mathcal{D}_{\mathcal{P}}(W(R)/p^n W(R)).$$

The isomorphism (7) takes the maps $\Phi^{\#}$ respectively $\Psi^{\#}$ to the maps $F^{\#} : W(R) \otimes_{F,W(R)} P \to P$ respectively $V^{\#} : P \to W(R) \otimes_{F,W(R)} P$, which are associated to the display $\mathcal{P}$ by [Z-DFG] Corollary 2.17.

We define Φ to be the σ-linear map associated to $\Phi^{\#}$. We define M_1 as the kernel of the natural map:

$$M \to P/Q,$$

which is induced from the map $\mathcal{D}_{\mathcal{P}}(A) \to \mathcal{D}_{\mathcal{P}}(R) \cong P/I_R P$.

Our next aim is to show that $\Phi(M_1) \subset pM$. We note that for any display $\mathcal{P}$ there is an exact sequence:

$$P \xrightarrow{V^{\#}} W(R) \otimes_{F,W(R)} P \to W(R) \otimes_{F,W(R)} P/Q \to 0 \tag{8}$$

Indeed, P is generated by elements of the form $\xi \cdot V^{-1}y$, where $y \in Q$. Hence the image of $V^{\#}$ is generated by $V^{\#}(\xi V^{-1}y) = \xi \otimes y$, for $y \in Q$. This shows the exactness of (8).

We consider the $\overline{R}$-module:

$$H = \overline{R} \otimes_A M = \overline{R} \otimes_{W(R)} P$$

The image of Q or equivalently M_1 is a direct summand $H_1 \subset H$. If we tensor the exact sequence (8) with $\overline{R} \otimes_{W(R)}$ we obtain an exact sequence:

$$H \xrightarrow{\overline{V^{\#}}} \overline{R} \otimes_{Frob,\overline{R}} H \to \overline{R} \otimes_{Frob,\overline{R}} H/H_1 \to 0$$

The map $\overline{V}^{\#}$ is also obtained if we tensorize $\Psi^{\#} : M \to A \otimes_{\sigma,A} M$ with $\overline{R} \otimes_A$. Since the kernel of the map $A \to \overline{R}$ is $J + pA$ it follows that

$$\text{Image } (A \otimes_{\sigma,A} M_1) \subset \Psi^{\#}M + J \otimes_{\sigma,A} M + pA \otimes_{\sigma,A} M, \tag{9}$$

where the inclusion takes place in $A \otimes_{\sigma,A} M$. Since $\Phi^{\#}\Psi^{\#} = p\, id_M$, and since $\sigma(J) \subset pA$, we obtain

$$\Phi(M_1) \subset pM,$$

if we apply $\Phi^{\#}$ to (9). Hence we have defined a map $\Phi_1 : M_1 \to M$. We note that

$$\Phi_1(um) = \sigma_1(u)\Phi m, \quad \text{for } u \in J.$$

This equation is verified by multiplying it with p.

Next we describe the $\mathcal{P}$-triple over the pd-extension $A \to R$ in terms of $\mathcal{M} = (M, M_1, \Phi, \Phi_1)$. Let us denote this triple by $(\widetilde{P}, F, V^{-1})$. Because $\widetilde{P}$ is the value of the Dieudonné crystal $\mathcal{D}_{\mathcal{P}}$ evaluated at the pd-thickening $W(A) \to R$, and since more over δ is a morphism of pd-thickenings of R, we find a canonical isomorphism:

$$W(A) \otimes_{\delta,A} M \simeq \widetilde{P} \tag{10}$$

The maps Φ on M and F on $\widetilde{P}$ are induced by applying the Dieudonné crystal to $F^{\#} : \overline{\mathcal{P}}^{(p)} \to \overline{\mathcal{P}}$. Since (10) is canonical this shows that

$$^{F}\xi \otimes \Phi m = F(\xi \otimes m), \quad \text{for } m \in M, \xi \in W(A). \tag{11}$$

The map $F : \widetilde{P} \to \widetilde{P}$ already determines V^{-1} uniquely because $W(A)$ has no p-torsion. The domain of definition $\hat{Q}$ of V^{-1} is the kernel of the map:

$$W(A) \otimes_{\delta,A} M \to M/M_1,$$

induced by $W(A) \xrightarrow{\mathbf{w}_0} A \xrightarrow{\alpha} R$ on the first factor. The operator V^{-1} is determined on this kernel by the following equations:

$$\begin{array}{ll} V^{-1}(u \otimes m) = {}^{V^{-1}}u \otimes \Phi m, & \text{for } u \in \mathfrak{c},\ m \in M \\ V^{-1}(\xi \otimes m_1) = {}^{F}\xi \otimes \Phi_1 m_1, & \text{for } \xi \in W(A),\ m_1 \in M_1. \end{array} \tag{12}$$

These equations are verified by multiplying them with p. Hence we have obtained a full description of the triple $(\widetilde{P}, F, V^{-1})$ in terms of (M, M_1, Φ, Φ_1).

We are now able to show the commutativity of the diagram:

$$\begin{tikzcd} M_1 \arrow[r, "\Phi_1"] \arrow[d] & M \arrow[d] \\ Q \arrow[r, "V^{-1}"] & P, \end{tikzcd} \tag{13}$$

where $M \to P$ is induced by (7) and $M_1 \to Q$ is its restriction to M_1. Indeed, this diagram is the composition of two commutative diagrams

$$\begin{tikzcd} M_1 \arrow[r, "\Phi_1"] \arrow[d] & M \arrow[d] \\ \hat{Q} \arrow[r, "V^{-1}"] \arrow[d] & \widetilde{P} \arrow[d] \\ Q \arrow[r, "V^{-1}"] & P. \end{tikzcd}$$

The commutativity of the upper square was just proved. The point was that there is no p-torsion in this diagram, and therefore we could multiply the upper and lower horizontal arrow by p to verify it. The lower square is commutative by the definition of a triple.

Now we will show that (M, M_1, Φ, Φ_1) is a window. The only condition we have to verify is that the union of $\Phi_1 M_1$ and ΦM generates the A-module M. For this we choose a normal decomposition of M:

$$M = L_0 \oplus T_0 \quad M_1 = L_0 \oplus J T_0.$$

We consider the A-linear map:

$$U_0^{\#} = \Phi_1^{\#} \oplus \Phi^{\#} : A \otimes_{\sigma, A} L_0 \oplus A \otimes_{\sigma, A} T_0 \to M. \tag{14}$$

It suffices to show that this is an isomorphism. This is in fact equivalent to the condition (ii) of definition 2.

We set $L = W(R) \otimes_{\varkappa, A} L_0$ and $T = W(R) \otimes_{\varkappa, A} T$. By the commutative diagram (13) we see that $U^{\#} = W(R) \otimes_{\varkappa, A} U_0^{\#}$ is the map:

$$(V^{-1})^{\#} \oplus F^{\#} : W(R) \otimes_{F, W(R)} L \oplus W(R) \otimes_{F, W(R)} T \to P. \tag{15}$$

Since $P = L \oplus T$ is a normal decomposition the map (15) is an isomorphism by the definition of a display. The equality of determinants of endomorphisms of $R \otimes_{\mathbf{w}_0, W(R)} P = R \otimes_A M$:

$$\det(R \otimes_{\mathbf{w}_0, W(R)} U^{\#}) = \det(R \otimes_{\alpha, A} U_0^{\#}),$$

shows that the right hand side is a unit in R. From the assumption that J is in the radical of A it follows that det $U_0^{\#}$ is a unit too. Hence $\mathcal{M} = (M, M_1, \Phi, \Phi_1)$ is an A-window.

Therefore we have defined a functor:

$$\text{Win} : (\text{displays}/R) \to (A\text{-windows}).$$

It is obvious from our considerations that this functor is quasiinverse to Dsp. Indeed assume that either $\mathcal{P} = \text{Dsp}\,\mathcal{M}$ or that $\mathcal{M} = \text{Win}\,\mathcal{P}$. Then we have defined a canonical isomorphism $P \simeq W(R) \otimes_{\varkappa, A} M$, which is compatible with F and Φ resp. V^{-1} and Φ_1. *Q.E.D.*

Example: Let R be a ring, such that $pR = 0$, and assume that R admits a p-basis in the sense of [BM]. This means that R regarded as an R-module via the Frobenius $Frob : R \to R$ is a free R-module. It is shown in [BM] that there is a torsion free p-adic ring A, such that A/pA is isomorphic to R. Moreover the Frobenius endomorphism of R lifts to an endomorphism σ of A. If we set $J = pA$ we obtain a frame (A, J, σ) over R. Clearly $(A, p^m A, \sigma)$ is also a frame over $A/p^m A$ for any number m. By [Z-DFG] we obtain:

Corollary 1.7 *Let R be an excellent ring. Then the category of formal p-divisible groups over R is equivalent to the category of A-windows over R.*

Example: Let S be a torsion free p-adic ring. We set $R = S/p^m S$ for some number m. Then we construct a frame for R as follows:

Let I be an index set. We denote by $C = \mathbb{Z}_p\{X_i\}_{i \in I}$ the ring of restricted power series, i.e. the p-adic completion of the polynomial ring $\mathbb{Z}_p[X_i]_{i \in I}$. We consider the endomorphism σ of C given by $\sigma(X_i) = X_i^p$. Assume we are given an epimorphism of rings:

$$C \to S,$$

whose kernel will be denoted by $\mathfrak{c}$. We define B_0 as the subring of $C \otimes \mathbb{Q}$ which is generated by C and by the elements $\frac{c^n}{n!}$ for $c \in \mathfrak{c}$. We obtain a surjection $B_0 \to S$ whose kernel J_0 is generated by the elements $\frac{c^n}{n!}$ for $c \in \mathfrak{c}$. Then J_0 is a pd-ideal and consequently $J_0 + pB_0$ is a pd-ideal too. Since $\sigma(\mathfrak{c}) \subset J_0 + pB_0$, we obtain $\sigma(J_0) \subset J_0 + pB_0$. Therefore σ induces an endomorphism of B_0. We claim that $\sigma(b) = b^p \mod pB_0$ for $b \in B_0$. It is enough to verify this for generators of B_0. For $b \in C$ the congruence is trivial. For $c \in \mathfrak{c}$ we have to verify that:

$$\sigma(\frac{c^n}{n!}) = \left(\frac{c^n}{n!}\right)^p = 0 \mod pB_0,$$

where the last congruence follows since we have divided powers on J_0. We set $\sigma(c) = c^p + pu$ for $u \in C$. Then we obtain:

$$\sigma(\frac{c^n}{n!}) = \sum_{i+j=n} \frac{c^{pi}}{i!}\frac{p^j}{j!}u^j = 0 \mod pB_0.$$

We denote by B the p-adic completion of B_0. By continuity σ extends to an endomorphism of B, and the morphism $B_0 \to S$ gives a pd-thickening of $B \to S$, whose kernel will be denoted by J. Then $(B, p^m + J, \sigma)$ is a frame for S/p^mS, and (B, J, σ) is a frame for the p-adic ring S (Compare the remarks after proposition 1.5). Then we obtain:

Proposition 1.8 *Let S/pS be an excellent ring. The category of p-divisible groups over S whose reduction modulo p has no étale part is equivalent to the category of B-windows over S.*

2 Variants of the Witt Ring

As a preparation to the next section we will make some comments on the Witt ring and in particular on the Cartier morphism.

In [Z-DFG] §1.4 we applied the Witt ring functor W to a pd-thickening $S \to R$ of $\mathbb{Z}_{(p)}$-algebras, whose kernel will be denoted by $\mathfrak{a}$. The divided powers are uniquely determined by a function $\alpha_p : \mathfrak{a} \to \mathfrak{a}$, which defines a^p divided by p. The divided Witt polynomials $\mathbf{w}_n/p^n$ make sense on $W(\mathfrak{a})$ and they define an isomorphism of additive groups:

$$\log : W(\mathfrak{a}) \to \prod_{\mathbb{N}} \mathfrak{a}. \tag{16}$$

An element of the right hand side is denoted by $[a_0, a_1, \dots, a_n, \dots]$. We call this the logarithmic coordinates on $W(\mathfrak{a})$. For $a \in \mathfrak{a}$ we set:

$$\exp a = \log^{-1}[a, 0, \dots, 0, \dots]. \tag{17}$$

Then $\exp : \mathfrak{a} \to W(a)$ is an injective ring homomorphism. If we regard $\mathfrak{a}$ as a $W(S)$-module via restriction of scalars $\mathbf{w}_0 : W(S) \to S$ the morphism exp becomes a $W(S)$-module homomorphism. We often say that the divided powers define $\mathfrak{a}$ as an ideal of $W(S)$. We write $\mathfrak{a} \subset W(S)$, without mentioninig that this inclusion is defined by exp. A basic property is that ${}^F\mathfrak{a} = 0$.

We will consider commutative rings without unit. These are often denoted by $\mathcal{N}, \mathcal{M}$ e.t.c..

Definition 2.1 *We call $\mathcal{N}$ pointwise nilpotent, if each element $n \in \mathcal{N}$ is nilpotent. We call $\mathcal{N}$ bounded nilpotent, if there is a number $k \in \mathbb{N}$, such that $n^k = 0$ for all $n \in \mathcal{N}$. If there is a number k such that $\mathcal{N}^k = 0$, we call $\mathcal{N}$ nilpotent.*

Let $\hat{W}(\mathcal{N}) \subset W(\mathcal{N})$ be the subset of Witt vectors with only finitely many non-zero components. If $\mathcal{N}$ is pointwise nilpotent $\hat{W}(\mathcal{N})$ is an ideal in $W(\mathcal{N})$. Moreover $\hat{W}(\mathcal{N})$ is again pointwise nilpotent.

Lemma 2.2 *Assume that $\mathcal{N}$ is bounded nilpotent and annihilated by some power of p. Then $W(\mathcal{N})$ is bounded nilpotent and annihilated by some power of p.*

Proof: By considering the ideals $p^k\mathcal{N}$ one reduces to the case where $p\mathcal{N} = 0$. Then one finds a number M, such that ${}^{F^M}W(\mathcal{N}) = 0$.

Let us consider the set S of all Witt vectors in $W(\mathcal{N})$, which have the form:

$$[n_0] + {}^V[n_1] + \ldots + {}^{V^{M-1}}[n_{M-1}],$$

where $n_i \in \mathcal{N}$ for $i = 1, \ldots M-1$ are arbitrary elements. It is clear that there is a number N, such that $\xi^N = 0$ for each $\xi \in S$.

Next we note that each $\xi \in W(\mathcal{N})$ has a unique expression:

$$\xi = \xi_0 + {}^{V^M}\xi_1 + \ldots + {}^{V^{kM}}\xi_k + \ldots ,$$

where $\xi_k \in S$. It follows from the definition of M that the summands in the last sum are pairwise orthogonal. Hence we obtain $\xi^N = 0$. *Q.E.D.*

We call $\mathcal{N}$ modulo p bounded nilpotent, if $\mathcal{N}/p\mathcal{N}$ is bounded nilpotent. In this case $\mathcal{N}/p^k\mathcal{N}$ is bounded nilpotent too.

Proposition 2.3 *Assume that $\mathcal{N}$ is modulo p bounded nilpotent. Then the p-adic topology on $W(\mathcal{N})$ coincides with the linear topology, which has as a fundamental system of neighbourhoods of zero the subgroups $W(p^k\mathcal{N})$, where k runs through the natural numbers.*

Proof: Fix a number n. Then by the last lemma there is a number r, such that p^r annihilates $W(\mathcal{N}/p^n\mathcal{N})$. This shows:

$$p^rW(\mathcal{N}) \subset W(p^n\mathcal{N}).$$

Hence the p-adic topology is finer. On the other hand the natural divided powers on $p^r\mathcal{N}$, for some number $r \geq 1$ induce an isomorphism of additive groups:

$$W(p^r\mathcal{N}) \xrightarrow[\sim]{\log} \prod p^r\mathcal{N}$$

This implies $p^mW(p\mathcal{N}) = W(p^{m+1}\mathcal{N})$. Therefore we obtain:

$$W(p^n\mathcal{N}) = p^{n-1}W(p\mathcal{N}) \subset p^{n-1}W(\mathcal{N}).$$

Q.E.D.

If R is a p-adic ring then the ring $W(R)$ is p-adic too (see: [Z-DFG] Prop.1.3). Hence $W(\mathcal{N})$ is a p-adic ring in the situation of the last proposition.

We extend the functor $\hat{W}$ to rings $\mathcal{N}$ which are p-adic and modulo p bounded nilpotent.

$$\hat{W}(\mathcal{N}) = \varprojlim_{r} \hat{W}(\mathcal{N}/p^r\mathcal{N}) \subset W(\mathcal{N})$$

Since this may contradict our old definition of $\hat{W}$, we say that we take $\hat{W}$ in the topological sense. Obviously $\hat{W}(\mathcal{N})$ is a closed subgroup for the topology on $W(\mathcal{N})$ given by the subgroups $W(p^r\mathcal{N})$ and hence also for the p-adic topology. By definition we have:

$$\hat{W}(\mathcal{N}) \cap W(p^r\mathcal{N}) = \hat{W}(p^r\mathcal{N})$$

Let $1 \leq u < r$ be number. Then the divided powers on $p^u\mathcal{N}/p^r\mathcal{N}$ provide a homomorphism:

$$\log : \hat{W}(p^u\mathcal{N}/p^r\mathcal{N}) \rightarrow \oplus p^u\mathcal{N}/p^r\mathcal{N}.$$

This is an isomorphism, if the divided powers are nilpotent in the sense of [Z-DFG] (3.4) (i.e. pointwise), which is the case if $p \geq 3$ or if $u \geq 2$. Hence with these assumptions we obtain an isomorphism:

$$\log : \hat{W}(p^u\mathcal{N}) \rightarrow \hat{\oplus} p^u\mathcal{N},$$

where the right hand side denotes the set of those elements of $\prod p^u\mathcal{N}$, whose components converge p-adically to zero. Using this we can modify the proof of the last proposition to obtain:

Proposition 2.4 *Assume that $\mathcal{N}$ is p-adic and modulo p bounded nilpotent. Let us consider $\hat{W}(\mathcal{N})$ in the topological sense. Then the p-adic topology on $\hat{W}(\mathcal{N})$ coincides with the linear topology which has as a fundamental system of neighbourhoods of zero the subgroups $\hat{W}(p^r\mathcal{N})$.*

Moreover $\hat{W}(\mathcal{N}) \subset W(\mathcal{N})$ is a closed subgroup for the p-adic topology on $W(\mathcal{N})$, and the p-adic topology on $\hat{W}(\mathcal{N})$ is induced by the p-adic topology on $W(\mathcal{N})$. In particular the group $\hat{W}(\mathcal{N})$ is p-adic.

Next we consider the Cartier morphism:

$$\Delta : W(\mathcal{N}) \rightarrow W(W(\mathcal{N})). \tag{18}$$

It is the unique functorial ring homomorphism, such that:

$$\mathbf{w}_n(\Delta\xi) = {}^{F^n}\xi, \quad \text{for } \xi \in W(\mathcal{N}).$$

Assume that $\mathcal{N}$ is pointwise nilpotent. Then $\hat{W}(\mathcal{N})$ is pointwise nilpotent too. Therefore $\hat{W}(\hat{W}(\mathcal{N}))$ makes sense. We have inclusions

$$\hat{W}(\hat{W}(\mathcal{N})) \subset W(\hat{W}(\mathcal{N})) \subset W(W(\mathcal{N})).$$

Lemma 2.5 *: Assume that $\mathcal{N}$ is pointwise nilpotent. Then the Cartier morphism (18) induces a morphism*

$$\Delta : \hat{W}(\mathcal{N}) \to \hat{W}(\hat{W}(\mathcal{N}))$$

Proof: We begin to verify, that $\Delta(\hat{W}(\mathcal{N})) \subset W(\hat{W}(\mathcal{N}))$. By functoriality it suffices to prove this if $\mathcal{N}$ and hence $W(\mathcal{N})$ has no p-torsion. Indeed consider any element $\underline{n} = (n_0, n_1, \dots) \in \hat{W}(\mathcal{N})$. Then we have for suitable numbers r and s a homomorhism of pointwise nilpotent algebras:

$$\mathfrak{a} = (\mathbb{Z}[X_0, \dots, X_s]/(X_0^r, \dots X_s^r))^+ \to \mathcal{N},$$

such that X_i is mapped to n_i and $n_i = 0$ for $i > s$. Here the index $+$ denotes the polynomials without constant term. This reduces our assertion to the case, where $\mathcal{N}$ is the torsion free algebra $\mathfrak{a}$.

The Frobenius endomorphism on $\hat{W}(\mathcal{N})$ satisfies the congruence:

$${}^F\xi = \xi^p \bmod p\hat{W}(\mathcal{N})$$

Hence there is a Cartier morphism:

$$\hat{W}(\mathcal{N}) \to W(\hat{W}(\mathcal{N})).$$

Since this coincides with Δ our claim above is shown.

Let $\hat{I}_{\mathcal{N}} = {}^V\hat{W}(\mathcal{N})$. This ideal is equipped with divided powers:

$$\alpha_p({}^V\xi) = p^{p-2}\, {}^V(\xi^p)$$

These divided powers are pointwise nilpotent in the sense of [Z-DFG] (3.4) because $\hat{W}(\mathcal{N})$ is pointwise nilpotent. Hence they define an isomorphism:

$$\log : \hat{W}(\hat{I}_{\mathcal{N}}) \to \bigoplus_{\mathbb{N}} \hat{I}_{\mathcal{N}}. \tag{19}$$

Let us consider an element $\xi \in \hat{W}(\mathcal{N})$, such that $\Delta(\xi) \in \hat{W}(\hat{W}(\mathcal{N}))$. We claim that this implies $\Delta({}^V\xi) \in \hat{W}(\hat{W}(\mathcal{N}))$.

Indeed by [DFG] Lemma 2.11 we have the formula in $\hat{W}(\hat{W}(\mathcal{N}))$:

$$\Delta({}^V\xi) = {}^V\Delta(\xi) + \exp\ {}^V\xi$$

in the notation of (16). This shows our claim. Finally we know that Δ acts on a Teichmüller representative $[n] \in W(\mathcal{N})$ as follows:

$$\Delta([n]) = [[n]] \in \hat{W}(\mathcal{N}).$$

Hence we have $\Delta({}^{V^r}[n]) \in \hat{W}(\mathcal{N})$ by our claim above. This finishes the proof. *Q.E.D.*

Let us consider a ring $\mathcal{N}$ which is p-adic, and modulo p bounded nilpotent. Then the ring $\hat{W}(\mathcal{N})$ taken in the topological sense has exactly the same properties. By proposition 2.4 we obtain:

$$\begin{aligned}\hat{W}(\hat{W}(\mathcal{N})) &= \varprojlim_m \hat{W}(\hat{W}(\mathcal{N})/p^m\hat{W}(\mathcal{N})) \\ &= \varprojlim_m \hat{W}(\hat{W}(\mathcal{N}/p^m\mathcal{N}))\end{aligned}$$

Again we find that $\hat{W}(\hat{W}(\mathcal{N})) \subset W(W(\mathcal{N}))$ and that the Cartier morphism induces a ring homomorphism:

$$\Delta : \hat{W}(\mathcal{N}) \to \hat{W}(\hat{W}(\mathcal{N})). \tag{20}$$

Let R be a local ring with maximal ideal $\mathfrak{m}$ and with perfect residue field of characteristic $p \geq 3$. We assume that there is a number M, such that $x^M = 0$ for any $x \in \mathfrak{m}$. Let us denote the category of these rings R by $\mathcal{Z}$.

Let k be the residue field of R. The natural map $W(R) \to W(k)$ has a Teichmüller section [Z-DD] §2. Therefore we may write canonically:

$$W(R) = W(\mathfrak{m}) \oplus W(k)$$

We note that $\hat{W}(\mathfrak{m})$ is a $W(R)$-submodule of $W(\mathfrak{m})$.

We define a subring $\hat{W}(R) \subset W(R)$:

$$\hat{W}(R) = W(k) \oplus \hat{W}(\mathfrak{m}). \tag{21}$$

We will also consider p-adic local rings A, such that A/p^nA is in $\mathcal{Z}$ for any number n. We call them $\hat{\mathcal{Z}}$-rings. The ring $\hat{W}(R)$ is an example of a $\hat{\mathcal{Z}}$-ring.

We define:

$$\hat{W}(A) = \varprojlim_n \hat{W}(A/p^nA)$$

The maximal ideal $\mathfrak{n}$ of A is modulo p bounded nilpotent. We have a decomposition which is similar to (21):

$$\hat{W}(A) = W(A/\mathfrak{n}) \oplus \hat{W}(\mathfrak{a}) \tag{22}$$

The ring $\hat{W}(A)$ is a closed subring of $W(A)$ for the p-adic topology, and moreover the p-adic topology on $\hat{W}(A)$ coincides with the topology induced by the p-adic topology on $W(A)$. The ring $\hat{W}(A)$ is again a $\hat{\mathcal{Z}}$-ring. From (20) we deduce that the Cartier morphism induces a map:

$$\Delta : \hat{W}(A) \to \hat{W}(\hat{W}(A)) \tag{23}$$

Consider a surjection $S \to R$ in $\mathcal{Z}$, and assume that its kernel $\mathfrak{a}$ is equipped with divided powers, which are compatible with the natural divided powers on pS. We will denote the inverse image of $\bigoplus_{\mathbb{N}} \mathfrak{a}$ by the homomorphism (16) by $\widetilde{W}(\mathfrak{a})$. By [Z-DFG] (3.4) we have $\hat{W}(\mathfrak{a}) \subset \widetilde{W}(\mathfrak{a})$, and this inclusion is an equality, if the divided powers are pointwise nilpotent. We will denote by $\widetilde{W}(S)$ the subring of $W(S)$ generated by $\widetilde{W}(\mathfrak{a})$ and $\hat{W}(S)$. We note that $\hat{W}(S) = \widetilde{W}(S)$ if the divided powers are pointwisw nilpotent. In general we have an exact sequence:

$$0 \to \widetilde{W}(\mathfrak{a}) \to \widetilde{W}(S) \to \hat{W}(R) \to 0$$

The composite $\widetilde{W}(S) \to \hat{W}(R) \xrightarrow{\mathbf{w}_0} R$ is again a pd-thickening, if we define the pd-structure on the kernel $\mathfrak{c}$ of $\widetilde{W}(S) \to R$ as follows:

If we embed $\mathfrak{a} \subset \widetilde{W}(S)$ by exp (16), we find a direct decomposition:

$$\mathfrak{c} = \mathfrak{a} \oplus {}^V\widetilde{W}(S) \tag{24}$$

Since there are divided powers defined on each direct summand of (24), we obtain divided powers on $\mathfrak{c}$. If the natural number n is big enough we obtain a pd-thickening in $\mathcal{Z}$:

$$\widetilde{W}(S)/p^n\widetilde{W}(S) \to R. \tag{25}$$

Remark: We verify that the divided powers on $\mathfrak{c}$ and on $p\widetilde{W}(S)$ are compatible. We will even show that the same is true if we replace $\widetilde{W}(S)$ by $W(S)$. Let us denote by $\exp a$ the image of an element $a \in \mathfrak{a}$ by the embedding $\mathfrak{a} \subset W(S)$. Assume we are given an equation:

$$p\xi = \exp a + {}^V\eta, \qquad a \in \mathfrak{a},\ \xi, \eta \in W(S) \tag{26}$$

Let $\alpha_p : \mathfrak{a} \to \mathfrak{a}$ be the function given by the divided powers (see above (16)). Since $\exp a\ {}^V\eta = 0$ our assertion says exactly that:

$$p^{p-1}\xi^p = \exp \alpha_p(a) + p^{p-2}\ {}^V(\eta^p) \tag{27}$$

We set $\xi_0 = \mathbf{w}_0(\xi)$. From (26) we obtain $p\xi_0 = a$ and therefore $\alpha_p(a) = \xi_0 a^{p-1}$, since the divided powers on $\mathfrak{a}$ are compatible with those on pS. The left hand side of (27) is:

$$\begin{aligned} \xi(\exp a + {}^V\eta)^{p-1} &= \xi \exp a^{p-1} + \xi p^{p-2}\ {}^V(\eta^{p-1}) \\ &= \exp \xi_0 a^{p-1} + p^{p-2}\ {}^V({}^F\xi\eta^{p-1}) \end{aligned}$$

Hence it is sufficient to show that ${}^F\xi = \eta$. This is a consequence of the following general assertion:

Let S be a ring, and let $\exp : pS \to W(S)$ be the map defined by the natural divided powers of pS. Then we have for $\xi \in W(S)$ the identity:

$$ {}^{FV}\xi - {}^{VF}\xi = \exp p\xi_0 \tag{28}$$

We obtain this by multiplying the equation $p - {}^{V}1 = \exp p$ with ξ. The last equation may be verified in $W(\mathbb{Z})$ by applying the Witt polynomials. Comparing the equation (28) to (26) we obtain ${}^{VF}\xi = {}^{V}\eta$ as desired.

Consider a $\hat{\mathcal{Z}}$-ring A. Then a pd-thickening $A \to R$ is supposed to be compatible with the natural divided powers on pA. It induces pd-thickenings in $\mathcal{Z}$:

$$A/p^m A \to R,$$

if m is large. In this case we define:

$$\widetilde{W}(A) = \varprojlim_m \widetilde{W}(A/p^m A). \tag{29}$$

This is a subring of $W(A)$ because A is p-adic.

Consider again the pd-thickening $S \to R$. $\widetilde{W}(S)$ is a $\hat{\mathcal{Z}}$-ring and $\widetilde{W}(S) \to R$ a pd-thickening. Therefore we may form $\widetilde{W}(\widetilde{W}(S))$. This is a subring of $W(W(S))$.

Lemma 2.6 *The Cartier morphism induces a map:*

$$\widetilde{W}(S) \to \widetilde{W}(\widetilde{W}(S))$$

Proof: By (23) it is enough to show that the Cartier morphism induces a map $\widetilde{W}(\mathfrak{a}) \to \widetilde{W}(\widetilde{W}(\mathfrak{a}))$. In [Z-DFG] Lemma 2.14 we expressed the map $W(\mathfrak{a}) \to W(W(\mathfrak{a}))$ in terms of logarithmic coordinates. From this the assertion about $\widetilde{W}$ is obvious. *Q.E.D.*

This lemma has also a version if we replace S by a $\hat{\mathcal{Z}}$-ring A. Then the ring $\widetilde{W}(A)$ defined by (29) is again a $\hat{\mathcal{Z}}$-ring and the kernel of the composition:

$$\widetilde{W}(A) \xrightarrow{\mathbf{w}_0} A \to R,$$

has a natural pd-structure (compare (24)).

Let us verify that $\widetilde{W}(A)$ is indeed a $\hat{\mathcal{Z}}$-ring. We denote by $\mathfrak{n} \subset A$ the maximal ideal and by $k = A/\mathfrak{n}$ the residue field. With the obvious notation we have a direct decomposition:

$$\widetilde{W}(A) = W(k) \oplus \widetilde{W}(\mathfrak{n}). \tag{30}$$

Let $u \geq 2$ be a number such that $p^u\mathfrak{n} \subset \mathfrak{a}$. Then we have $p^n\widetilde{W}(p^u\mathfrak{n}) = \widetilde{W}(p^{n+u}\mathfrak{n})$ for each number n. As in the proof of proposition 2.4 we conclude that the p-adic topology on $\widetilde{W}(\mathfrak{n})$ coincides with the topology of projective limit used to define $\widetilde{W}(\mathfrak{n})$:

$$\widetilde{W}(\mathfrak{n}) = \varprojlim_{m} \widetilde{W}(\mathfrak{n}/p^m\mathfrak{n}). \tag{31}$$

In particular $\widetilde{W}(A)$ is a p-adic ring. Then we may apply $\widetilde{W}$ to the pd-thickening $\widetilde{W}(A) \to R$ in $\hat{\mathcal{Z}}$.

From the fact that the p-adic topology on $\widetilde{W}(\mathfrak{n})$ is given by (31) and from the decomposition (30) we deduce:

$$\widetilde{W}(\widetilde{W}(A)) := \varprojlim \widetilde{W}(\widetilde{W}(A)/p^m\widetilde{W}(A)) = \\ \varprojlim \widetilde{W}(\widetilde{W}(A/p^mA))$$

For each m the lemma 2.6 gives a Cartier morphism:

$$\widetilde{W}(A/p^mA) \to \widetilde{W}(\widetilde{W}(A/p^mA))$$

If we pass to the projective limit we obtain a Cartier morphism for a pd-thickening $A \to R$ where A is a $\hat{\mathcal{Z}}$-ring:

$$\widetilde{W}(A) \to \widetilde{W}(\widetilde{W}(A)). \tag{32}$$

3 The Case of a General p-Divisible Group

Definition 3.1 *Let R be a $\mathcal{Z}$-ring. A Dieudonné frame over R is a frame $(A \xrightarrow{\alpha} R, J, \sigma)$, such that A is a $\hat{\mathcal{Z}}$-ring and such that the Cartier morphism given by σ:*

$$\delta : A \to W(A),$$

factors through the subring $\widetilde{W}(A) \subset W(A)$.

The map $\alpha : A \to R$ induces a map $\widetilde{W}(A) \to \hat{W}(R)$. We define $\varkappa$ to be the composite of the following maps:

$$\varkappa : A \xrightarrow{\delta} \widetilde{W}(A) \to \hat{W}(R) \tag{33}$$

Let us assume that $\mathcal{M} = (M, M_1, \Phi)$ is a Dieudonné A-window over R in the sense of definition 2. We may replace in the equations (4) the ring $W(R)$ by $\hat{W}(R)$ we obtain a Dieudonné display since for such objects no nilpotence condition is required. Therefore we obtain a functor:

$$\mathbb{D}\mathbb{D} : (\text{Dieudonné } A\text{-windows}) \to (\text{Dieudonné displays}) \tag{34}$$

Theorem 3.2 *Let R be a $\mathcal{Z}$-ring and let $A \to R$ be a Dieudonné frame. Then the functor* $\mathbb{DD}$ *is an equivalence of categories.*

From this theorem we may prove our main theorem 6 of the introduction: For an artinian ring R we may simply refer to theorem 20 of [Z-DD], which says that in this case the category of Dieudonné displays is equivalent to the category of p-divisible groups. Now let R be any $\mathcal{Z}$-ring with nilpotent maximal ideal and residue class field k. Then for any p-divisible group X over R, we find an artinian local subring $R' \subset R$, with the same residue class field k, such that X is defined over R'. Indeed, this follows from the existence of the universal deformation of X_k over a power series ring over $W(k)$ proved by Grothendieck and Messing. Moreover let X' and Y' be p-divisible groups over R'. We denote by X and Y the objects obtained by base change over R. Then the base change map is bijective

$$\mathrm{Hom}(X', Y') \to \mathrm{Hom}(X, Y). \tag{35}$$

Indeed, let $\mathfrak{m}$ be the maximal ideal of R and consider the commutative diagram:

$$\begin{array}{ccc} R'/R' \cap \mathfrak{m}^n & \longrightarrow & R/\mathfrak{m}^n \\ \downarrow & & \downarrow \\ R'/R' \cap \mathfrak{m}^{n-1} & \longrightarrow & R/\mathfrak{m}^{n-1} \end{array}$$

The vertical arrows are pd-extension. Hence to lift a homomorphism of p-divisible groups from downstairs is the same as to lift the Hodge filtration of the crystal ([M] Chapt. V Thm. 1.6). Using this it follows this one shows easily that (35) is a bijection for the upper horizontal monomorphism, if it is bijective for the lower one.

We note that Theorem 3 and Theorem 4 of [Z-DD] hold for any $\mathcal{Z}$-ring with nilpotent maximal ideal (see theorem 3.4 below). Then we may apply the arguments for p-divisible groups above to Dieudonné displays. This reduces the general case of theorem 6 to the case where R is artinian. Hence the proof of theorem 6 is finished.

The proof of theorem 3.2 is based on the construction of a crystal associated to a Dieudonné display. In [Z-DD] we have constructed such a crystal on the nilpotent crystalline site. We will now extend this crystal to the whole crystalline site, without assuming that the divided powers are nilpotent.

We consider a pd-thickening $S \to R$ in the category $\mathcal{Z}$. Let us denote by $\mathfrak{a} \subset S$ the kernel. Using exp we consider $\mathfrak{a} \subset W(S)$ also as an ideal of $W(S)$.

Definition 3.3 *Let $\mathcal{P}$ be a Dieudonné display over R. A $\mathcal{P}$-triple over S is a triple $(\widetilde{P}, F, V^{-1})$ such that:*

1) $\widetilde{P}$ is a finitely generated free $\widetilde{W}(S)$-module, equipped with an isomorphism

$$\hat{W}(R) \otimes_{\widetilde{W}(S)} \widetilde{P} \xrightarrow{\sim} P.$$

2) An ^{F}-linear operator $F : \widetilde{P} \to \widetilde{P}$ which lifts $F : P \to P$.

3) An ^{F}-linear operator $V^{-1} : \hat{Q} \to \widetilde{P}$, which is defined on the inverse image $\hat{Q}$ of Q by the map $\widetilde{P} \to P$, and which lifts $V^{-1} : Q \to P$.

The following properties hold:

(i) $V^{-1}(\,^V\tilde{\xi}\tilde{x}) = \tilde{\xi}F\tilde{x}$, for $\tilde{x} \in \widetilde{P}, \tilde{\xi} \in \widetilde{W}(S)$

(ii) $V^{-1}(\mathfrak{a}\widetilde{P}) = 0$

We could obviously define a "display" with coefficients in $\widetilde{W}(S)$. Then [Z-DD] theorem 3 and its proof holds for any pd-thickening $S \to R$ (and not just nilpotent ones), if we replace $\widetilde{W}(S)$ by $\hat{W}(S)$. The essential point is that an extension of V^{-1} to $\hat{Q}$ may be defined over $\widetilde{W}(S)$ but not over $\hat{W}(S)$. From this we obtain the following result:

Theorem 3.4 *Let $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ be a morphism of Dieudonné displays over R. Let $\mathcal{T}_i$ be a $\mathcal{P}_i$-triple over S for $i = 1, 2$. Then there exists a unique morphism of triples $\tilde{\alpha} : \mathcal{T}_1 \to \mathcal{T}_2$ which lifts α.*

We associate to a Dieudonné display $\mathcal{P}$ over R a crystal as follows: Let $S \to R$ be a pd-thickening in $\mathcal{Z}$. Then we find a $\mathcal{P}$-triple $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ over S. This is done by writing down structural equations for $\mathcal{P}$ (see [Z-DD] (3)) and lifting them to S. By the last theorem $\mathcal{T}$ is unique up to canonical isomorphism. Therefore the following definition makes sense:

$$\mathcal{D}_{\mathcal{P}}(S) = S \otimes_{\mathbf{w}_0, \widetilde{W}(S)} \widetilde{P}.$$

We call $\mathcal{D}_{\mathcal{P}}$ the Dieudonné crystal of $\mathcal{P}$. If $S \to R$ is a nilpotent pd-thickening, we have $\widetilde{W}(S) = \hat{W}(S)$, and therefore $\mathcal{D}_{\mathcal{P}}(S)$ coincides with the Dieudonné crystal defined in [Z-DD] in this case.

We note that $\mathcal{D}_{\mathcal{P}}$ commutes with base change in the following sense: Assume we are given a morphism of pd-thickenings in the category $\mathcal{Z}$:

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

The vertical arrows are pd-thickenings with kernels $\mathfrak{a} \subset S$ resp. $\mathfrak{a}' \subset S'$. Assume we are given a Dieudonné display $\mathcal{P}$ over R, and let $\mathcal{P}' = \mathcal{P}_{R'}$ be the Dieudonné display obtained by base change. Let $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ be a $\mathcal{P}$-triple over S. Then we define a $\mathcal{P}'$-triple $\mathcal{T}' = (\widetilde{P}', F, V^{-1})$ over S' with coefficients in $\widetilde{W}(S')$ as follows:

$$\widetilde{P}' = \widetilde{W}(S') \otimes_{\widetilde{W}(S)} \widetilde{P}$$

The operator $F : \widetilde{P}' \to \widetilde{P}'$ is the obvious ^{F}-linear extension of $F : P \to P$. Finally the ^{F}-linear operator $V^{-1} : \hat{Q}' \to \widetilde{P}'$ is uniquely determined by the formulas:

$$\begin{aligned} &V^{-1}(\xi \otimes y) = {}^F\xi \otimes V^{-1}y, \text{ for } \xi \in \widetilde{W}(S'),\ y \in \hat{Q} \\ &V^{-1}({}^V\xi \otimes x) = \xi \otimes Fx, \text{ for } x \in \widetilde{P} \\ &V^{-1}(a' \otimes x) = 0 \text{ for } a' \in \mathfrak{a}' \subset \widetilde{W}(S'). \end{aligned}$$

This construction of the triple $\mathcal{T}'$ provides a canonical isomorphism:

$$\mathcal{D}_{\mathcal{P}'}(S') \cong S' \otimes_S \mathcal{D}_{\mathcal{P}}(S). \tag{36}$$

If A is a $\hat{\mathcal{Z}}$-ring and $A \to R$ is a pd-thickening we set

$$\mathcal{D}_{\mathcal{P}}(A) = \varprojlim_n \mathcal{D}_{\mathcal{P}}(A/p^nA),$$

where n is large, such that $A/p^nA \to R$ is a pd-thickening.

Proof of theorem 3.2: We are going to repeat the arguments of the proof of theorem 1.6 in the new context.

Let us begin with the construction of a quasiinverse functor:

$$\text{Win} : (\text{Dieudonné displays}) \to (\text{Dieudonné } A\text{-windows}) \tag{37}$$

We consider a Dieudonné display $\mathcal{P}$ over R. Then we have to associate an A-window (M, M_1, Φ). We set:

$$M = \mathcal{D}_{\mathcal{P}}(A).$$

We define M_1 as the kernel of the canonical map:

$$M = A \otimes_{\widetilde{W}(A)} \widetilde{P} \to R \otimes_{\hat{W}(R)} P \to P/Q,$$

where $(\widetilde{P}, F, V^{-1})$ is the $\mathcal{P}$-triple over A.

Finally we define Φ. To do this we consider the ring $\overline{R} = R/pR$. Then $A \to \overline{R}$ is again a pd-thickening with kernel $\mathfrak{a} + pA \subset$A.

If we apply base change to the morphisms of pd-thickenings

$$\begin{CD} A @>{id}>> A \\ @VVV @VVV \\ R @>>> R', \end{CD}$$

we obtain a canonical isomorphism:

$$\mathcal{D}_{\mathcal{P}}(A) \simeq \mathcal{D}_{\widetilde{P}}(A)$$

We note that the last diagram involves a priori two different rings $\widetilde{W}(A)$, one with respect to $A \to R$ and the second with respect $A \to \overline{R}$. In fact these two rings coincide, but this fact is irrelevant for us.

The same considerations as in the proof of theorem 1.6 lead to homomorphisms:

$$\Phi^{\#} : A \otimes_{\sigma,A} M \to M$$
$$\Psi^{\#} : M \to A \otimes_{\sigma,A} M.$$

The argument following equation (8) shows:

$$\Phi M_1 \subset pM.$$

We set $\Phi_1 = \frac{1}{p}\Phi : M_1 \to M$. To show that (M, M_1, Φ) is a Dieudonné window, we still have to verify that the union of ΦM and $\Phi_1 M_1$ generate M as an A-module.

Before we prove this we will describe the $\mathcal{P}$-triple $(\widetilde{P}, F, V^{-1})$ over A in terms of (M, M_1, Φ). We will describe a canonical isomorphism:

$$\widetilde{P} \cong \mathcal{D}_{\mathcal{P}}(\widetilde{W}(A)), \tag{38}$$

in the manner of [Z-DFG] Proposition 2.12. Let us postpone the proof of (38).

From this we may finish the proof of the theorem as follows: The morphism of pd-thickenings

$$\begin{tikzcd} A \arrow[rr, "\delta"] \arrow[dr] & & \widetilde{W}(A), \arrow[dl] \\ & R & \end{tikzcd}$$

provides a canonical isomorphism:

$$\widetilde{W}(A) \otimes_{\delta,A} M \cong \mathcal{D}_{\mathcal{P}}(\widetilde{W}(A)) \cong \widetilde{P}. \tag{39}$$

Since the isomorphisms are functorial they commute with the Frobenius:

$$F(\xi \otimes m) = {}^{F}\xi \otimes \Phi m, \text{ for } m \in M, \ \xi \in \widetilde{W}(A).$$

Here F is from the triple $(\widetilde{P}, F, V^{-1})$. Since $\widetilde{W}(A)$ is p-torsion free the isomorphism (39) is also compatible with V^{-1}. From (39) we deduce an isomorphism:

$$\hat{W}(R) \otimes_{\varkappa,A} M \simeq P.$$

It follows that this isomorphism is compatible with F and V^{-1} too. From this isomorphism we deduce as in the last part of theorem 1.6 that the union of

ΦM and $\Phi_1 M_1$ generates M as an A-module. This shows that we have an isomorphism of functors:

$$\mathbb{D}\mathbb{D} \circ \mathrm{Win} \cong id.$$

On the other hand let us start with a Dieudonné window (M, M_1, Φ). We denote by $\mathcal{P}$ the Dieudonné display associated by the functor $\mathbb{D}\mathbb{D}$. Then it is clear that $(\widetilde{P}, F, V^{-1})$ described above in terms of (M, M_1, Φ) is a $\mathcal{P}$-triple over A. From this we deduce an isomorphism:

$$\mathcal{D}_{\mathcal{P}}(A) \simeq M,$$

which is compatible with the Frobenius. Hence the functors $\mathbb{D}\mathbb{D}$ and Win are indeed quasi inverse. *Q.E.D.*

Finally we give a precise definition of the isomorphism (38). Let us consider a pd-thickening $S \to R$ in $\mathcal{Z}$.

Proposition 3.5 *Let $\mathcal{P}$ be a Dieudonné display over R. Consider a $\mathcal{P}$-triple $\mathcal{T} = (\widetilde{P}, F, V^{-1})$ over S. We have defined a pd-thickening in the category $\hat{\mathcal{Z}}$ (compare (24)) :*

$$\widetilde{W}(S) \to R$$

We denote the associated Cartier morphism by Δ: $\widetilde{W}(S) \to \widetilde{W}(\widetilde{W}(S))$ which is defined by lemma 2.6. Then a $\mathcal{P}$-triple over $\widetilde{W}(S)$ is obtained as follows:

$$\widetilde{\mathcal{T}} = (\widetilde{W}(\widetilde{W}(S)) \otimes_{\Delta, \widetilde{W}(S)} \widetilde{P}, F, V^{-1})$$

Here $F : \widetilde{W}(\widetilde{W}(S)) \otimes_{\Delta, \widetilde{W}(S)} \widetilde{P} \to \widetilde{W}(\widetilde{W}(S)) \otimes_{\Delta, \widetilde{W}(S)} \widetilde{P}$ denotes the ^{F}-linear extension of $F : \widetilde{P} \to \widetilde{P}$.

The operator V^{-1}, which is defined on the inverse image of Q by the canonical map $\widetilde{W}(\widetilde{W}(S)) \otimes_{\Delta, \widetilde{W}(S)} \widetilde{P} \to P$ is uniquely determined by the following equations:

$$\begin{aligned} V^{-1}({}^V\hat{\xi} \otimes x) &= \hat{\xi} \otimes Fx, \quad \mathit{for}\, x \in (\widetilde{W}(\widetilde{W}(S)) \\ V^{-1}(\hat{\xi} \otimes y) &= {}^F\hat{\xi} \otimes V^{-1}y, \quad \mathit{for}\, y \in \hat{Q} \subset \widetilde{P}. \end{aligned}$$

Proof: The proof is the repetition of the proof of proposition 2.12 in [Z-DFG], which we omit. *Q.E.D.*

As a corollary we obtain the canonical isomorphism (38).

We will now discuss the examples of Dieudonné frames in the introduction in more detail.

Example : Let k be a perfect field of characteristic $p \geq 3$. We fix numbers d and s and consider the ring

$$R = k[T_1, \ldots, T_d]/(T_1^s \ldots T_d^s) \tag{40}$$

We set $A = W(k)[T_1, \ldots, T_d]/T_1^s, \ldots, T_d^s)$. Then the canonical map:

$$A \to R,$$

is a pd-thickening with nilpotent divided powers on pA. We define $\sigma : A \to A$ by the equations:

$$\sigma(\xi) = {}^F\xi, \text{ for } \xi \in W(k), \tag{41}$$

$$\tag{42}$$

$$\sigma(T_i) = T_i^p, \text{ for } i = 1, \ldots, d. \tag{43}$$

We claim that A is a Dieudonné frame. For this we have to show that the Cartier morphism associated to σ:

$$\delta : A \to W(A),$$

factors through $\widetilde{W}(A)$. We note that $\widetilde{W}(A) = \hat{W}(A)$ in this example because the divided powers are nilpotent. Clearly $\delta(T_i)$ is the Teichmüller representative of T_i and therefore

$$\delta(T_i) = [T_i] \in \hat{W}(A)$$

It remains to be shown that $\delta(\xi) \in \hat{W}(A)$ for $\xi \in W(k)$. But $\delta : W(k) \to W(A)$ is the Teichmüller section of the homomorphism $W(A) \to W(k)$ induced by the canonical map $A \to k$. Since this Teichmüller section takes by definition values in $\hat{W}(A)$, we have proved that A is a Dieudonné frame.

Let us change the notation: Let $R = k[[T_1 \ldots T_d]]$ be the power series ring and $A = W(k)[[T_1, \ldots, T_d]]$. We define $\sigma : A \to A$ as before. Then A becomes a frame over R. Then we obtain as explained in example 1 of the introduction:

Proposition 3.6 *The category of p-divisible groups over $k[[T_1 \ldots T_d]]$ is equivalent to the category of Dieudonné A-windows.*

Example : Let k be as above. Let S be a local finite flat $W(k)$-algebra, with residue field k. We set $R = S/p^uS$, where u is some fixed number. We choose a presentation:

$$0 \to \mathfrak{c} \to W(k)[T_1, \ldots, T_d] \to S \to 0,$$

such that each T_i is mapped to the maximal ideal of S. We consider the subring $A_0 \subset W(k)[T_1, \ldots, T_d] \otimes \mathbb{Q}$ which is generated over $W(k)[T_1, \ldots, T_d]$ by all elements of the form $\frac{c^n}{n!}$, for $c \in \mathfrak{c}$, $n \in \mathbb{N}$. The natural map $A_0 \otimes \mathbb{Q} \to S \otimes \mathbb{Q}$ induces a map $A_0 \to S$, whose kernel J_0 is generated by the element $\frac{c^n}{n!}$ as an ideal.

Let σ be the ^{F}-linear $W(k)$-algebra endomorphism of $W(k)[T_1 \ldots T_d]$, such that $\sigma(T_i) = T_i^p$. Then σ leaves the ideal $\mathfrak{c} + pW(k)[T_1, \ldots, T_d]$ stable. Since

$J_0 + pA_0 \subset A_0$ is a pd-ideal, it follows that σ extends to an endomorphism of A_0. As in the example of proposition 1.8 one verifies:

$$\sigma(a) \equiv a^p \mod pA_0 \quad a \in A_0.$$

Let A be the p-adic completion of A_0. Then we obtain a pd-thickening $A \to S$, and hence also a pd-thickening $A \to R$. The endomorphism σ extends to A.

We claim that (A, σ) is a frame for R. Again all we need to show is that the Cartier morphism

$$\delta : A \to W(A)$$

factors through $\widetilde{W}(A)$. Since $\delta(T_i) = [T_i] \in \hat{W}(A)$ it follows that

$$\delta(h) \in \hat{W}(A), \tag{44}$$

for $h \in W(k)[T_1 \ldots T_d]$.

Taking into account that $W(A)$ is p-adic, it suffices to show that $\delta(J_0) \subset \widetilde{W}(A)$. By (44) we know that $\delta(\mathfrak{c}) \subset \widetilde{W}(A)$. Clearly $\delta(\mathfrak{c})$ is in the kernel of $\widetilde{W}(A) \to A \to S$. The divided power structure on $J \subset A$ defines an embedding $J \subset \widetilde{W}(A)$. Hence the kernel of the map $\widetilde{W}(A) \to S$ is $J \oplus {}^V\widetilde{W}(A)$. This kernel is a pd-ideal because each direct summand is. Hence $\delta(c) \in J \oplus {}^V\widetilde{W}(A)$ for $c \in \mathfrak{c}$ implies $\delta\left(\frac{c^n}{n!}\right) \in J \oplus {}^V\widetilde{W}(A)$. Since J_0 is generated over $W(k)[T_1, \ldots, T_d]$ by products of elements of the form $\frac{c^n}{n!}$, for $c \in \mathfrak{c}$ we conclude that $\delta(J_0) \subset \widetilde{W}(A)$. This concludes the proof that (A, σ) is a Dieudonné frame for $R = S/p^uS$. Hence A-windows for the ring R classify p-divisible groups over R.

As in the last example we obtain the same result for the frame $A \to S$ which is not a Dieudonné frame.

Proposition 3.7 *The category of Dieudonné A-windows is equivalent to the category of p-divisible groups over S.*

This result was proved by Breuil in the case where S is a discrete valuation ring by using the crystalline Dieudonné theory of Berthelot, Breen, and Messing [BBM].

Example : Consider the example 3 of the introduction. We are given elements $f_1, \ldots f_r \in k[[T_1, \ldots T_d]]$, such that $f_1, \ldots f_r, T_{r+1}, \ldots T_d$ is a system of parameters in $k[[T_1, \ldots T_d]]$. We set $R = k[[T_1, \ldots T_d]]/(f_1, \ldots, f_r)$. Then we have defined in the introduction a pd-thickening $\hat{B} \to R$. Moreover we have defined an endomorphism σ of $\hat{B}$. With this structure $\hat{B}$ becomes a frame over R. This is a frame but not a Dieudonné frame. Nevertheless we obtain by the limit argument given in the introduction:

Proposition 3.8 *The category of Dieudonné $\hat{B}$-windows over R is equivalent to the category of formal p-divisible groups over R.*

References

[B] Breuil, Ch.: Schémas en groupe et module filtré, C.R. Acad. Sci. Paris Sér.I Math. **328** (1999) no.2, 93-97.

[BAC] Bourbaki,N.: Algébre Commutative, Chapt. 8 et 9, Masson 1983.

[BBM] Berthelot, P., Breen, L., Messing, W.,: Théorie de Dieudonné Cristalline II, Springer Lecture Notes No. 930, 1982.

[BM] Berthelot, P., Messing, W.,: Théorie Dieudonné Cristalline III: Théorèmes d'Equivalence et de Pleine Fidelité, in: The Grothendieck Festschrift Volume I, Birkhäuser 1990.

[M] Messing, W.: The crystals associated to Barsotti-Tate groups, LNM **264**, Springer 1972.

[Z-DFG] Zink,Th.: The display of a formal p-divisible group, to appear in: Semestre p-Adique, Astérisque SFM.

[Z-DD] Zink,Th.: A Dieudonné Theory for p-Divisible Groups, http://www.mathematik.uni-bielefeld.de/˜zink