Simple Groups, Fixed Point Sets and Involutions

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Introduction

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group on a finite set Ω.

The **fixed point set** of $x \in G$ is

$$C_{\Omega}(x) = \{\omega \in \Omega : \omega \cdot x = \omega\}$$

Problem 1. *How large is* $C_{\Omega}(x)$*?*

Obtain bounds on

$$\mathrm{fix}(x) = |C_{\Omega}(x)|$$

Problem 2. *When is* $C_{\Omega}(x)$ *empty?*

Investigate the elements $x \in G$ with $C_{\Omega}(x) = \emptyset$: these are the **derangements** in G.

We focus on the case where G is **almost simple** and **primitive**, with $x \in G$ an **involution**.

Background

- Fixed points have been studied since the 19th century, especially for simple groups.
- ***G* exceptional** (type $G_2(q)$, $E_8(q)$, etc.)

  [LLS, 2002]: Upper bounds on $\mathrm{fix}(x)$.
- ***G* classical** ($\mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$, etc.)

  [B, 2007]: Upper bounds on $\mathrm{fix}(x)$, Ω primitive and non-subspace.
- It has been studied more recently by Liebeck and Shalev:

Theorem (LSh, 2015)**.** *Let $G \leq \mathrm{Sym}(\Omega)$ be an almost simple primitive group of degree n and socle T. Then with some known exceptions, there is an involution $x \in T$ such that*

$$\mathrm{fix}(x) > n^{\frac{1}{6}}$$

- **Aim:** Improve the above constant $\frac{1}{6}$, limiting the number of possible exceptions.

Main Results

Let G be an almost simple primitive group of degree n, with point stabiliser H.

Let $T = \mathrm{Soc}(G)$ be an **alternating**, **sporadic** or **classical** group.

Theorem (Covato, 2016)**.** *One of the following holds:*

(a) $H \cap T$ has odd order.

(b) There is an involution $x \in T$ such that

$$\mathrm{fix}(x) > n^{\frac{4}{9}}$$

(c) (G, H) belongs to a list of known exceptions.

- The cases (G, H) in (a) are known. Here all involutions in T are **derangements**.
- Let $I(T) = \{x \in T \mid x \neq 1, x^2 = 1\}$ be the set of involutions in T. In (c), $\mathrm{fix}(x) < n^{\frac{4}{9}}$ for all $x \in I(T)$. Thus, we compute α such that

  $$\max_{x \in I(T)} \mathrm{fix}(x) = n^{\alpha}$$

  In most cases, α is very close to $\frac{4}{9}$.

Example: Let $G = \mathscr{A}_9$ and $H = \mathrm{PSL}_2(8){:}3$, thus $n = 120$. Here $\max_{x \in I(T)} \mathrm{fix}(x) = 8$. Therefore

$$\alpha = \log 8 / \log 120 \approx 0{,}4343\ldots$$

Example: Let $G = J_1$ and $H = 2^3.7.3$. Here T has *only* one class of involutions x^T. Using GAP, we compute $\mathrm{fix}(x) = 5$. Since $n = 1045$, we have

$$\alpha = \log 5 / \log 1045 \approx 0{,}2315\ldots$$

The following result can be deduced fairly quickly from the Theorem above.

Corollary. *One of the following holds:*

(i) All involutions in T are derangements.

(ii) There is an involution $x \in T$ with $\mathrm{fix}(x) > n^{\frac{1}{3}}$.

(iii) (G, H) is a known exception.

- The above example $(G, H) = (J_1, 2^3.7.3)$, is one of the exceptions in (iii).

Main Ingredients

- Use of **The Classification of Finite Simple Groups**
- Information on the conjugacy classes of involutions in finite simple groups
- The two following key lemmas:

Lemma 1. *Let $H_0 = T \cap H$. Then $n = |T|/|H_0|$, and*

$$\mathrm{fix}(x) = \frac{|x^T \cap H_0|}{|x^T|} n, \quad x \in T$$

Lemma 2. *Let $x \in T$ be an involution such that $|x^T| < n^{\frac{5}{9}}$. Then $\mathrm{fix}(x) > n^{\frac{4}{9}}$.*

The main challenge is to compute $|x^T \cap H_0|$ by studying the fusion of H_0-classes of involutions in T.

- ***T* sporadic:** GAP computation.
- ***T* alternating:** If H_0 is **intransitive** or **imprimitive**, we *count* the points fixed by $x = (1\,2)(3\,4) \in T$ on Ω.

  Let H_0 be **primitive**. In the *affine* and *product*-type case, we study the fusion of H_0-classes. In the *diagonal* type and *almost simple* case, $x = (1\,2)(3\,4) \in T$ satisfies Lemma 2 for large n. For smaller n, we construct the action of G on Ω using MAGMA.
- ***T* classical:** Let V be the natural module for T. The following theorem of Aschbacher's describes the possibilities for H:

Theorem (A, 1984)**.** *Either H preserves a natural geometric structure on V, or H is almost simple and acts irreducibly on V.*

If H is a geometric subgroup then the structure of $H_0 = T \cap H$ is known, and one can bound $|x^T \cap H_0|$ by studying the fusion of H_0-classes.

If H is almost simple and irreducible, we use Lemma 2 for large values of n. For the remaining cases, we study the irreducible representations of H to get bounds on $|x^T \cap H_0|$.

Further Developments

- Study the analogous problem for groups with socle of **exceptional** type.

  This is a current joint work with Tim Burness and Adam Thomas.
- Prove an analogous result for elements of odd prime order.
- Continue the study of **derangements** of order 2 for almost simple primitive groups.
- Investigate **2-elusive** actions (see [BG, 2016]).

References

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