$\frac{3}{2}$-Generation of Finite Groups

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Introduction

As a consequence of the Classification, every finite simple group can be generated by two elements.

Indeed, there is a vast literature establishing that finite simple groups have many generating pairs and it is natural to ask how these pairs are distributed across the group.

> A group is $\frac{3}{2}$**-generated** if every non-trivial element is contained in a generating pair.

Highlighting the strong 2-generation properties of finite simple groups, Guralnick and Kantor prove the following result [5].

Theorem

Every finite simple group is $\frac{3}{2}$-generated.

Our main question is the following.

? Which finite groups are $\frac{3}{2}$-generated?

Background

There is a straightforward necessary condition for $\frac{3}{2}$-generation.

Proposition

If G is $\frac{3}{2}$-generated then every proper quotient of G is cyclic.

In [2], Breuer, Guralnick and Kantor make the following remarkable conjecture.

Conjecture

A finite group is $\frac{3}{2}$-generated if and only if every proper quotient is cyclic.

Recent work of Guralnick reduces this conjecture to **almost simple** groups. That is, it suffices to show that if T is a finite non-abelian simple group and $g \in \mathrm{Aut}(T)$ then $\langle T, g \rangle$ is $\frac{3}{2}$-generated.

The conjecture has been verified when T is
- alternating [1]
- sporadic [2]
- linear [4]

Results

We have extended these results by establishing the conjecture for symplectic and odd-dimensional orthogonal groups [6].

Theorem (H, 2016)

Let $T = \mathrm{PSp}_{2m}(q)$ or $T = \Omega_{2m+1}(q)$ for $m \geq 2$ and let $g \in \mathrm{Aut}(T)$. Then $\langle T, g \rangle$ is $\frac{3}{2}$-generated.

Our ultimate goal is the following.

Aim

Prove the conjecture for all almost simple groups of Lie type.

Main Tools

We adopt the **probabilistic approach** introduced by Guralnick and Kantor in [5] which is encapsulated by the Key Lemma below.

Let G be a finite group, Ω be a G-set and $g \in G$. Write

$$\mathcal{M}(g) : \text{ the set of maximal overgroups of } g$$

$$\mathrm{fpr}(g, \Omega) : \text{ the fixed point ratio of } g \text{ on } \Omega$$

Key Lemma

The group G is $\frac{3}{2}$-generated if there exists $s \in G$ such that

$$\sum_{H \in \mathcal{M}(s)} \mathrm{fpr}(x, G/H) < 1$$

for all $x \in G$ of prime order.

Additional ingredients:
- Aschbacher's **subgroup structure** theorem for classical groups
- Bounds on **fixed point ratios** for almost simple groups
- **Shintani descent** from the theory of algebraic groups

References

[1] Brenner, Wiegold, *Two generated groups I*, Michigan Math. J., (1975)

[2] Breuer, Guralnick, Kantor, *Probabilistic generation of finite simple groups II*, J. Algebra, (2008)

[3] Breuer, Guralnick, Lucchini, Maroti, Nagy, *Hamiltonian cycles in the generating graphs of finite simple groups*, Bull. London. Math. Soc, (2010)

[4] Burness, Guest, *On the uniform spread of almost simple linear groups*, Nagoya Math J., (2013)

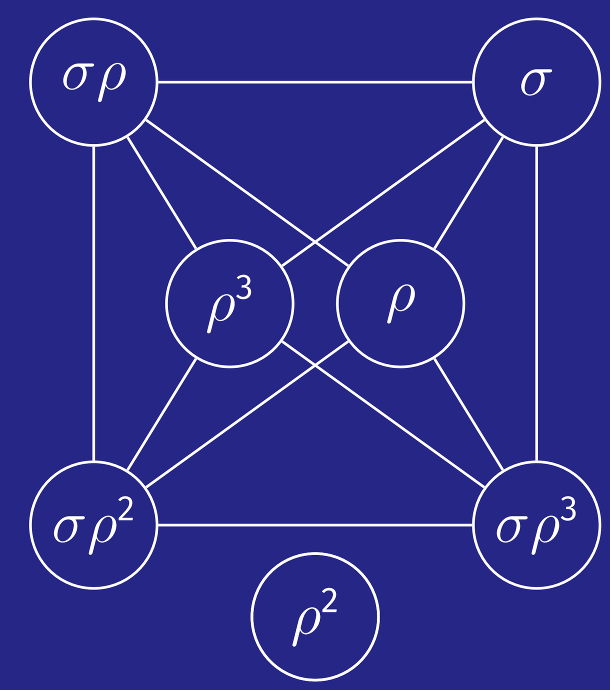
[5] Guralnick, Kantor, *Probabilistic generation of finite simple groups*, J. Algebra, (2000)

[6] Harper, *On the uniform spread of almost simple symplectic and orthogonal groups*, (In prep.)

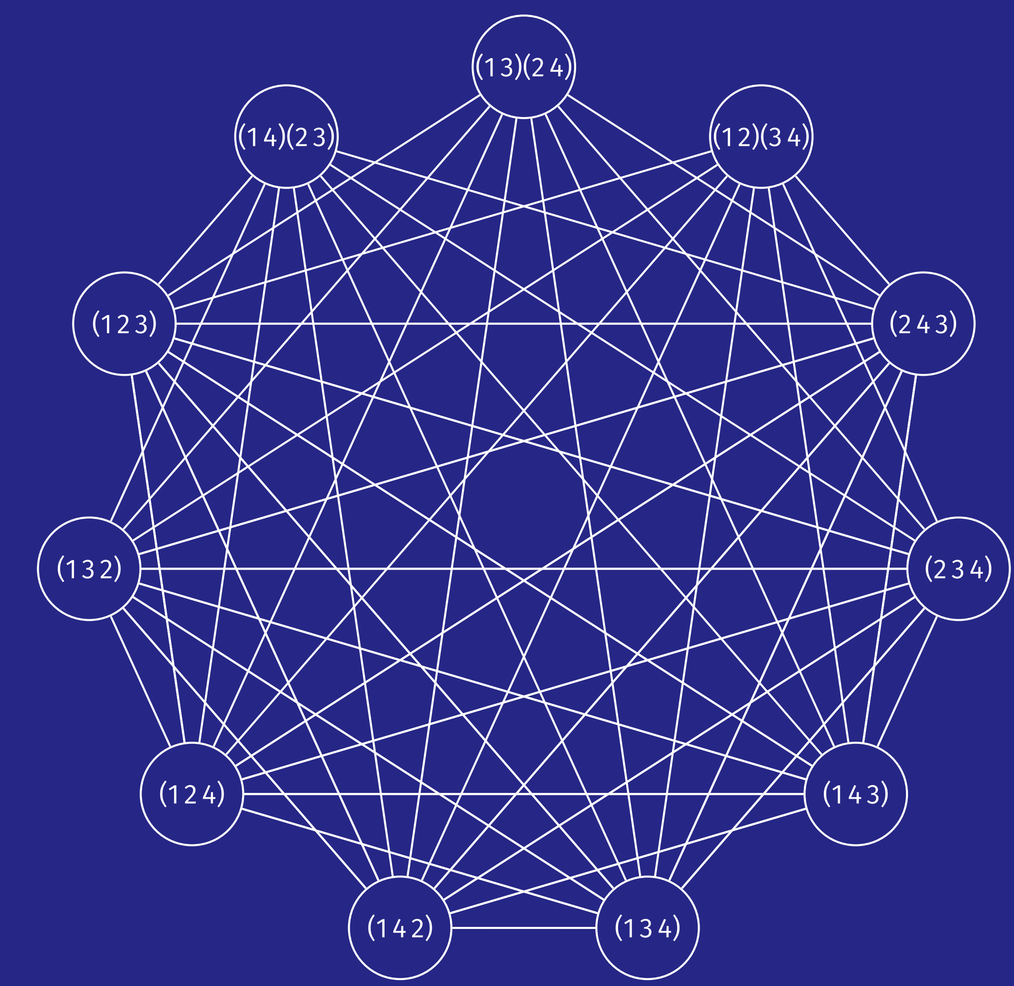
Generating Graphs

The **generating graph** of a group G is the graph $\Gamma(G)$ such that
- the vertices of $\Gamma(G)$ are the non-trivial elements of G
- two vertices g and h are adjacent if $\langle g, h \rangle = G$

Examples



$\Gamma(D_8)$



$\Gamma(A_4)$

Properties

A group G is $\frac{3}{2}$**-generated** if and only if $\Gamma(G)$ has **no isolated vertices**. In particular, $\Gamma(G)$ has no isolated vertices if G is finite and simple.

Let G be a finite simple group.
- $\Gamma(G)$ has diameter at most two [2]
- $\Gamma(G)$ has a Hamiltonian cycle if $|G|$ is sufficiently large [3]

We are interested in the following stronger conjecture.

Conjecture

Let G be a finite group. The following are equivalent.
- Every proper quotient of G is cyclic
- $\Gamma(G)$ has no isolated vertices
- $\Gamma(G)$ is connected with diameter at most two
- $\Gamma(G)$ has a Hamiltonian cycle

In [6], our main result extends in this direction.

Theorem (H, 2016)

Let $T = \mathrm{PSp}_{2m}(q)$ or $T = \Omega_{2m+1}(q)$ for $m \geq 2$ and let $g \in \mathrm{Aut}(T)$. Then $\Gamma(\langle T, g \rangle)$ is connected with diameter two.