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THE INVOLUTION WIDTH OF FINITE SIMPLE GROUPS

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WIDTH QUESTIONS

Let G be a finite group and let S be a normal subset of G (i.e. a subset closed under conjugation). Suppose that S generates the group, then for each $g \in G$, there exists an expression $g = t_1 \dots t_k$ where $t_i \in S$. It follows that we can write $G = S^d \cup S^{d-1} \cup \dots \cup S \cup \{1\}$ and we define the width of G with respect to S to be the minimal d for which this holds. We denote this by $width(G, S)$. Examples:

- $S = \{I(G)\}$, where $I(G)$ is the set of involutions. So width$(G, I(G))$ is the minimal n such that every element is the product of at most n involutions.
- $S = \{C(G)\}$, where $C(G) = \{[x, y] : x, y \in G\}$ - the commutator width
- $S = \{P_k(G)\}$, where $k \geq 2$ and $P_k(G) = \{x^k : x \in G\}$ is the set of k^{th} powers in G.

Width questions are not independent for different choices of S e.g. if k is odd then the k^{th} power map fixes every involution and it follows that width$(G, P_k(G)) \leq$width$(G, I(G))$.

Theorem: ([1]). There is an absolute constant $c > 0$ such that for any finite non-abelian simple group G and any non-identity normal subset $S \subseteq G$, we have $G = S^n$ for all $n \geq c \log|G|/\log|S|$.

Corollary: There is an absolute constant N such that every element of every finite non-abelian simple group is a product of N involutions.

AIM: Find the minimal value for N.

STRONGLY REAL SIMPLE GROUPS

Def: An element $x \in G$ is real if $x^g = x^{-1}$ for some $g \in G$. x is strongly real if g can be taken to be an involution. The group G is (strongly) real if all elements are (strongly) real.

Lemma: $x \in G$ is strongly real if and only if x is a product of 2 involutions.

Naturally if G is strongly real then we have a width as small as possible. The real FSGs were classified by Tiep and Zalesski [4] and it was later shown that G is real if and only if strongly real. This was completed by a series of different authors working through each family of the classification. Compiling the results we have

Theorem: A finite simple group is strongly real if and only if it is one of the following

- $PSp_{2n}(q)$ where $q \not\equiv 3 \pmod 4$ and $n \geq 1$;
- $P\Omega_{2n+1}(q)$ where $q \equiv 1 \pmod 4$ and $n \geq 3$;
- $P\Omega_9(q)$ where $q \equiv 3 \pmod 4$;
- $P\Omega^+_{4n}(q)$ where $q \not\equiv 3 \pmod 4$ and $n \geq 3$;
- $P\Omega^-_{4n}(q)$ $n \geq 2$;
- $P\Omega^+_8(q)$ or ${}^3D_4(q)$;
- $A_5, A_6, A_{10}, A_{14}, J_1, J_2$.

MAIN THEOREM

Theorem: (M. 2016, [3]) *Every non-abelian finite simple group has involution width at most 4.*

GROUPS OF LIE TYPE

The general procedure for finding the involution width of a simple group of lie type G, is as follows. We aim to pick regular semisimple elements x and y in particular classes of G, such that

- (1) The elements x and y are strongly real.
- (2) $G\backslash\{1\} \subset x^G \cdot y^G$.

From (1) and (2) it follows that width$(G) \leq$ width(x)+width$(y) \leq 4$. For (2) it suffices to show that

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(g^{-1})}{\chi(1)} \neq 0$$

for all $1 \neq g \in G$ ([1]).

- Complete character tables are currently unavailable for most groups of Lie type. We therefore rely on estimates of character values derived from the Deligne-Lusztig representation theory.
- Substantial difficulties are faced in the case of unitary groups, where we develop the theory of minimal degree characters using dual pairs.
- Similarly problematic are a number of exceptional groups of Lie type. Here we use an inductive approach, restricting to subgroups of G for which the involution width is known

ALTERNATING AND SPORADIC GROUPS

For these groups we can say more, and consider the width with regards to elements of any fixed prime order. We call this the p-width of the group.

Theorem: (M. 16). *Fix a prime p. The p width of A_n $(n \geq p)$ is at most 3.*

Theorem: (M. 16). Let (G, p) be a pair consisting of a sporadic finite simple group G and a prime p, dividing $|G|$. Then the p-width of G is 2, unless it is listed below (the p-width is 3 in these cases).

Prime	Width 3 Groups
2	All except J_1 and J_2
3	HS, Co_2, Co_3, Fi_{22}, Fi_{23}, BM
5	Fi_{22}, Fi_{23}

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