Finite generation of iterated wreath products of perfect groups

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Introduction

The wreath product construction is a classical, yet very prolific, source of examples in finite and infinite Group Theory. It provides an unparalleled way to combine two permutation groups and it also "contains" all possible extensions of said groups.

On the other hand when working with infinite groups we prefer to deal with "finitely generated" groups. It is then of considerable interest to construct finitely generated infinite groups with exotic properties.

Notation

For an integer $n \in \mathbb{N}$, we will denote the set $\{1, \dots, n\}$ by $[n]$.

⚠ Let d be an integer and let $\{m_k\}_{n\in\mathbb{N}}$ be a sequence of integers. Throughout this work we will denote by $\mathcal{S}$ a sequence $\{S_k\}_{k\in\mathbb{N}}$ where S_k is a transitive subgroup of $\mathrm{Sym}(m_k)$ that is perfect and at most d-generated.

For a topological group G, we denote by $\mathrm{d}(G)$ the minimal number of generators of a dense subgroup of G. In symbols, $\mathrm{d}(G) = \min\{|X| \mid X \subset G,\ \overline{\langle X\rangle} = G\}$.

Definitions

The abstract wreath product $A \mathrm{wr} B := A^n \rtimes B$ of two permutation groups $A \le \mathrm{Sym}([m])$ and $B \le \mathrm{Sym}([n])$ can be considered as a permutation group in two (in general inequivalent) ways. The first one is the **exponentiation** of A by B. This is the subgroup $A \textcircled{\wr} B$ of $\mathrm{Sym}([m]^n)$ together with the following action: if $(a_1, \dots, a_n)b \in A\mathrm{wr}B$ and $(x_1, \dots, x_n) \in [m]^n$ then the **product action** of $A\mathrm{wr}B$ is given by

$$(x_1, \dots, x_n)^{(a_1,\dots,a_n)} = (x_1^{a_1}, \dots, x_n^{a_n})$$

and

$$(x_1, \dots, x_n)^{b^{-1}} = (x_{1^b}, \dots, x_{n^b}).$$

Infinitely Iterated Exponentiation

The **iterated exponentiation** $\tilde{S}_n \le \mathrm{Sym}(\tilde{m}_n)$ of the groups in the sequence $\mathcal{S}$ is the permutation group inductively defined by: $\tilde{m}_1 = m_1$, $\tilde{S}_1 = S_1 \le \mathrm{Sym}(\tilde{m}_1)$ and $\tilde{m}_k = m_k^{\tilde{m}_{k-1}}$, $\tilde{S}_k = S_k \textcircled{\wr} \tilde{S}_{k-1} \le \mathrm{Sym}(\tilde{m}_k)$ for $k \ge 2$. The groups $\tilde{S}_k$, together with the projections $\tilde{S}_k \to \tilde{S}_{k-1}$, form an inverse system of finite groups. We call the profinite group $\tilde{W}(\mathcal{S}) = \varprojlim \tilde{S}_k$ the **infinitely iterated exponentiation** of the groups in $\mathcal{S}$.

We can prove that many iterated exponentiations are finitely generated.

Theorem 1

Let d be an integer. Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of transitive subgroups $S_k \le \mathrm{Sym}(m_k)$ such that each S_k is perfect and at most d-generated as an abstract group. Suppose that for every $k \in \mathbb{N}$ there exist elements i_k and j_k in $[m_k]$ with distinct stabilizers, i.e. such that $\mathrm{St}_{S_k}(i) \ne \mathrm{St}_{S_k}(j)$. Then the infinitely iterated exponentiation $\tilde{W}(\mathcal{S})$ of the groups in $\mathcal{S}$ is topologically finitely generated. In particular, $\mathrm{d}(\tilde{W}(\mathcal{S})) \le d + \mathrm{d}(S_1)$.

Moreover, the proof constructs explicitly the generators.

Using the same methods we can prove a slight improvement of Theorem 1 for sequences of 2-generated groups.

Corollary

Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of perfect 2-generated transitive subgroups $S_k \le \mathrm{Sym}(m_k)$. Suppose that for every $k \in \mathbb{N}$ and all $i, j \in [m_k]$ we have $\mathrm{St}_{S_k}(i) \ne \mathrm{St}_{S_k}(j)$. Then the infinitely iterated exponentiation $\tilde{W}(\mathcal{S})$ of the groups in $\mathcal{S}$ satisfies $\mathrm{d}(\tilde{W}(\mathcal{S})) \le 3$.

It is worth to note that the bound in Theorem 1 is "asymptotically sharp".

Lower bound lemma

Let N be a natural number. Let A be a finite simple group and let $B \le \mathrm{Sym}(n)$ be a finite permutation group. Then

$$\mathrm{d}(A^N \wr B) \ge \max\left\{\frac{1}{n}\left(\mathrm{d}\left(A^N\right) - \mathrm{d}(A) - 1\right), \mathrm{d}(B)\right\}.$$

The lemma relies on the fact that B swaps "rigidly" the n N-tuples of $(A^N)^n$, and one shows that $n \cdot \mathrm{d}(A^N \wr B) \ge \log_{|A|} N$. Also, $\log_{|A|} N$ grows as $\mathrm{d}(A^N)$.

A second way to consider the abstract wreath product $A\mathrm{wr}B$ of A by B as a permutation group is the **permutational wreath product** of A by B. This is the subgroup $A \wr B$ of $\mathrm{Sym}([m] \times [n])$ together with the following action: if $(a_1, \dots, a_n)b \in A\mathrm{wr}B$ and $(x, y) \in [m] \times [n]$ then the **permutational wreath action** of $A\mathrm{wr}B$ is given by

$$(x, y)^{(a_1,\dots,a_n)b} = (x^{a_y}, y^b).$$

Mixed iterated wreath product

Let $\{k_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive integers. Define the sequence $\{G_n\}_{n\in\mathbb{N}}$ of perfect transitive subgroups of $\mathrm{Sym}(r_n)$ starting from the groups in $\mathcal{S}$ in the following way: $G_0 = \{e\}$ and for $k \ge 1$

$$G_k = \begin{cases} S_k \textcircled{\wr} G_{k-1} & \text{if } k \in \{k_1, k_2, \dots\}, \\ S_k \wr G_{k-1} & \text{otherwise.} \end{cases}$$

The permutation groups G_n are called **iterated mixed wreath product of type** $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$.
Let m be an integer, if the sequence $\{k_n\}_{n\in\mathbb{N}}$ is such that $k_{n+1} - k_n \le m$ for every $n \in \mathbb{N}$, we say that the iterated mixed wreath product G_n of type $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$ has **stride at most** m.
The groups G_n, together with the projections $G_n \to G_{n-1}$, form an inverse system of finite groups. We say that the profinite group $\tilde{W}(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}}) = \varprojlim G_n$ is an **infinitely iterated mixed wreath product of type** $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$. If the groups G_n have stride at most m we say that $\tilde{W}(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$ has stride at most m.

Note that an infinitely iterated exponentiation is an infinitely iterated mixed wreath product of stride at most one. Thanks to an old theorem of Kalužnin, Klin and Suščans′kiĭ we can manipulate mixed iterated wreath products.

Theorem (Kalužnin, Klin, Suščans′kiĭ)

Let n_1, n_2 and n_3 be integers and let $A \le \mathrm{Sym}(n_1), B \le \mathrm{Sym}(n_2)$ and $C \le \mathrm{Sym}(n_3)$ be permutation groups. Then $A \textcircled{\wr} (B \wr C)$ and $(A \textcircled{\wr} B) \textcircled{\wr} C$ are isomorphic as permutation groups.

Theorem 2

Let d be an integer. Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of transitive subgroups $S_k \le \mathrm{Sym}(m_k)$ such that each S_k is perfect and at most d-generated as an abstract group. Suppose that for every $k \in \mathbb{N}$ there exist elements i_k and j_k in $[m_k]$ such that $\mathrm{St}_{S_k}(i) \ne \mathrm{St}_{S_k}(j)$. Let $G = \varprojlim G_n$ be an infinitely iterated mixed wreath product of type $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$ of stride at most m. Then G is topologically finitely generated.

An open question

It is known that the infinitely iterated wreath product of any sequence of *finite non-abelian simple groups* is 2-generated, by results of M. Quick. We believe that the estimate of the corollary is sharp for perfect non-simple groups, but we have been unable to confirm or disprove this.

An application

On the other hand, in some special cases we are able to improve the previous bounds further.

Proposition

Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of 2-generated perfect transitive subgroups $S_k \le \mathrm{Sym}(m_k)$. Suppose that for every $k \in \mathbb{N}$ there exist two generators a_k, b_k of S_k such that:

- $\mathrm{fix}(a_k)$ and $\mathrm{fix}(b_k)$ are non-empty,
- $(|a_1|, |b_j|) = 1$ and $(|b_1|, |a_j|) = 1$ for $j \ge 2$.

Then the infinitely iterated exponentiation $\tilde{W}(\mathcal{S})$ is topologically 2-generated and we produce explicitly two generators for the group.

References

[1] L. A. Kalužnin, M. H. Klin, and V. Ī. Suščans′kiĭ.
Exponentiation of permutation groups. I.
Izv. Vyssh. Uchebn. Zaved. Mat., 8:26–33, 1979.

[2] Martyn Quick.
Probabilistic generation of wreath products of non-abelian finite simple groups. II.
Internat. J. Algebra Comput. 16(3):493–503, 2006.

[3] M. Vannacci.
Finite generation of iterated wreath products in product action.
Arch. der Math., 105(3):205–214, 2015.

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