Majorana algebras and subgroups of the Monster

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1. The Monster group

The Monster group is denoted $\mathbb{M}$. It was first constructed as $Aut(V_{\mathbb{M}})$, where $V_{\mathbb{M}}$ is a 196 884 dimensional real, commutative, non-associative algebra known as the Griess algebra.

It contains 2 conjugacy classes of involutions; *2A* and *2B* and $\mathbb{M} = \langle 2A \rangle$. If $t,s \in 2A$ then ts lies in one of 9 conjugacy classes: *1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A* or *6A*.

There exists a bijection ψ between the *2A* involutions and certain idempotents in $V_{\mathbb{M}}$ called *2A* axes and $V_{\mathbb{M}} = \langle\langle \psi(2A) \rangle\rangle$.

If $t,s \in 2A$ then the algebra $\langle\langle \psi(t), \psi(s) \rangle\rangle$ is called a dihedral algebra and has one of nine isomorphism types, depending on the class of ts.

2. VOAs

In 1992, R.Borcherds famously proved Conway and Norton's monstrous moonshine conjectures.

The central object in his proof is the moonshine module, denoted $V^{\#}$. It belongs to a class of graded algebras known as vertex operator algebras, or VOAs.

In particular, we have $Aut(V^{\#}) \cong \mathbb{M}$.

If we take a vertex operator algebra $V = \bigoplus_{n=0}^{\infty} V_n$ such that $V_0 = \mathbb{R}$ and $V_1 = 0$ then V_2 is a real, commutative, non-associative algebra called a generalised Griess algebra.

M.Miyamoto showed that there exist involutions $\tau_a \in Aut(V)$ called Miyamoto involutions which are in bijection with generating involutions a in V_2 called Ising vectors.

In particular, if $V = V^{\#}$ then $V_2 \cong V_{\mathbb{M}}$, the Miyamoto involutions are the *2A* involutions and the Ising vectors are the *2A* axes.

We have shown that a certain class of Majorana algebras correspond exactly to an important class of subgroups in the Monster group. This forms the first step in a classification of these algebras.

Theorem 1: Let V be a Majorana algebra generated by three Majorana axes a_1, a_2 and a_3 such that a_1 and a_2 generate a *2A* dihedral algebra. Then the group $G = \langle \tau(a_1), \tau(a_2), \tau(a_3) \rangle$ must be a triangle-point subgroup of $\mathbb{M}$. Conversely, every triangle-point subgroup of $\mathbb{M}$ gives rise to such an algebra.

3. Majorana theory

Majorana theory is an axiomatisation of certain properties of generalised Griess algebras, providing a powerful framework in which to study the Griess algebra and related objects.

Definition: A Majorana algebra V is a real, commutative, non-associative algebra such that

- $V = \langle A \rangle$ where A is a set of idempotents called Majorana axes;
- For each $a \in A$, we can construct an involution $\tau(a) \in Aut(V)$ called a Majorana involution;
- The algebra obeys seven further axioms, which we omit here.

4. Triangle-point groups

The classification of Majorana algebras generated by two axes was completed by A.A.Ivanov *et al* in 2010. The question of algebras generated by three axes is a much larger problem.

However, a natural first step is to classify algebras generated by a *2A* algebra along with one further axis (as in Theorem 1). The group generated by the Majorana involutions of such an algebra must necessarily form a triangle-point group:

Definition: A group G is a triangle-point group if

- $G = \langle a,b,c \rangle$ for $a,b,c \in G$ of order dividing 2 such that $ab = ba$;
- $\forall t,s \in a^G \cup b^G \cup c^G \cup (ab)^G$, $o(ts) \leq 6$.

The triangle-point subgroups of the Monster were studied by S.P.Norton in 1978 who investigated the possibility of using them to give a new construction of the Griess algebra.

In 2012, S.Decelle showed that every triangle-point group occurs as the quotient of one of 11 finite groups. I recently showed that the triangle-point groups which do not embed into the Monster may not occur as groups generated by the Majorana involutions of one of the algebras in question, thus proving Theorem 1.

R.E.Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.* 109:405-44, 1992. • S.Decelle. *Majorana representations and the Coxeter groups.* PhD thesis, 2013. • A.A.Ivanov, D.V.Pasechnik, Á.Seress, S.Shpectorov. Majorana representations of the symmetric group of degree 4. *J. Algebra*, 324:2432-63, 2010. • M.Miyamoto. Griess algebras and conformal vectors in vertex operator algebras. *J. Algebra,* 179:523-48, 1996. • S.P.Norton. The uniqueness of the Fischer-Griess monster. *Contemp. Math.*, 45:271-85, 1985.