THE UNIVERSITY OF WESTERN AUSTRALIA

Three transpositions, Graphs and Groupoids

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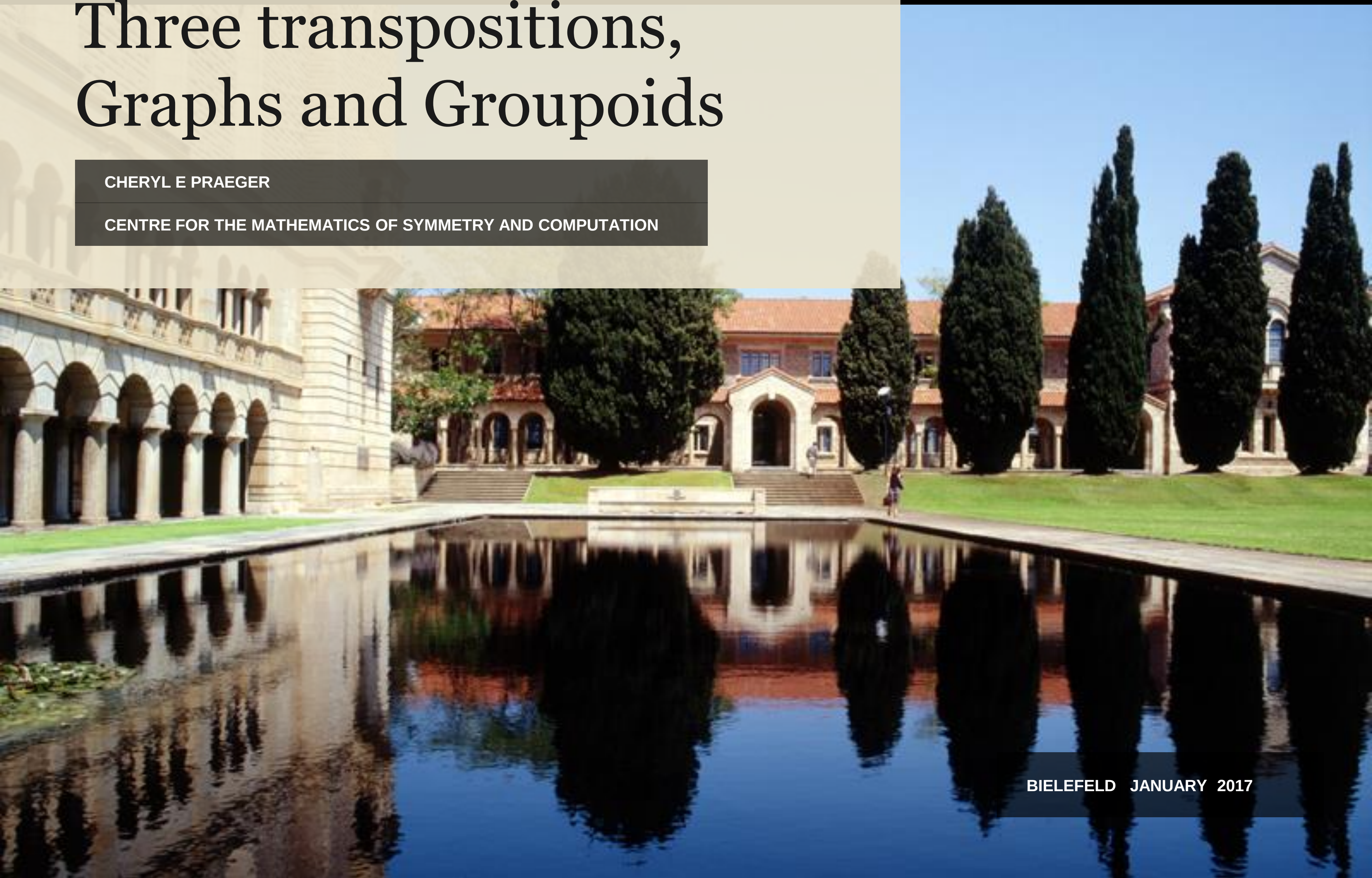


BIELEFELD JANUARY 2017

My congratulations to Bernd Fischer and thanks

- I appreciate the invitation to speak in honour of

Our colleague Bernd Fischer

Photographs: courtesy Ludwig Danzer

1969 Fischer theory of three transposition groups published

- In particular: wonderful constructions of the three Fischer sporadic finite simple groups

- “Three-transposition theory” caught the imagination of mathematicians world-wide and in many areas
  - In group theory, combinatorics, geometry

- My aim:
  - Trace several paths either influenced by “Three-transposition theory”
  - Or where Three-transposition groups appeared unexpectedly

  - And they keep on arising …..

1969 Lecture Note, University of Warwick
1971 Inventiones paper

- Definitions:
- **Group G**
- **family C of 3-transpositions** in G:
  1) C closed under conjugation,
  2) For all x, y in C, | xy | is 1 or 2 or 3

Distinct x, y
Either commute
Or generate Sym(3)

- **G called a 3-transposition group**
  - if G generated a family of 3-transpositions
  - Usually refer to (G, C) as a three transposition group

- Fischer classifies all finite almost simple 3-transposition groups – beautiful concept, beautiful proof

Fischer's classification:

- **Given** (G, C) a three transposition group
- **Assume** each normal {2,3}-subgroup central, and G' = G"
- **Then** G/Z(G) is known explicitly: one of
  1) Sym(n) , Sp(2n,2) , $O^{\varepsilon}(2n,2)$, PSU(n,2) $O^{\varepsilon}(2n,3)$ or
  2) One of the three Fischer sporadic groups Fi_{22} , Fi_{23} , Fi_{24}
- And the class C (modulo Z(G)) was specified in each case

- This result and the underlying theory was very influential

47 MathSciNet citations, 297 cites in Google Scholar

Huge impact in Group Theory: simple group classification

- **1973 Aschbacher:** extended theory to “odd transposition groups”

Fischer groups investigated:

- **1974 Hunt:** determined conjugacy classes of Fi_{23} & some character values
- **1981 Parrott:** characterised Fi_{22} , Fi_{23} , Fi_{24} by their centralisers of a central involution

Inspired and underpinned studies of subgroup structure of simple groups:

- **1979 Kantor:** Subgroups of finite classical groups generated by long root elements

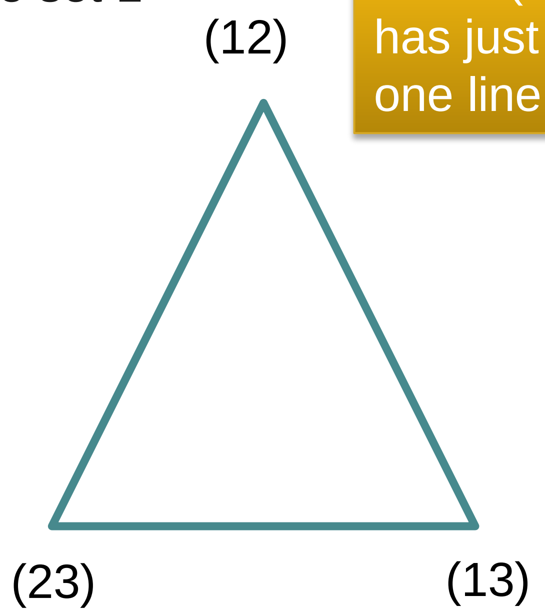
Even quite recently: for example

- **2006 (Chris) Parker:** 3-local characterisation of Fi_{22}

Geometrical and Combinatorial impact

- **1974 Buekenhout: Fischer spaces Π**
  - Partial linear space (P, L) with point set P, line set L
  - Each line incident with 3 points
  - Each intersecting line pair contained in a "Subspace" AG(2,3) or dual of AG(2,2)

- Each three transposition group (G, C)
  - Gives Fischer space Π(G,C) = (C, L)
  - lines are Sym(3) 's



Ex: Π(G,C) has just one line

- **Buekenhout:** 1-1 correspondence between connected Fischer spaces and three transposition groups with trivial centre

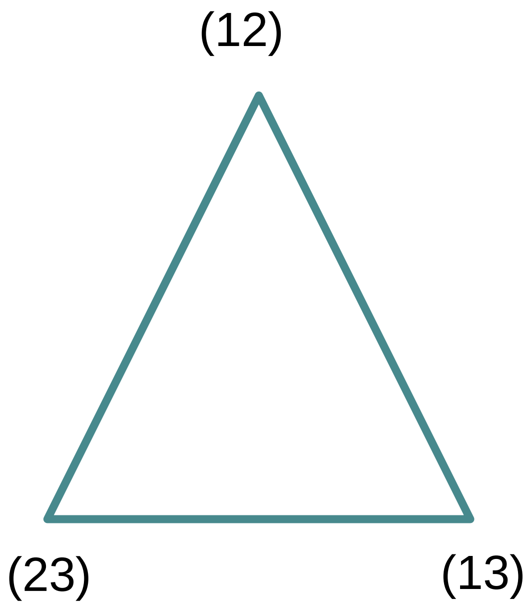
Connected: collinearity graph connected

Geometrical and Combinatorial impact

- **1971 Fischer: diagram** D of a three transposition group (G, C)
  - Graph with vertex set C
  - { x, y } an edge ⇔ | xy | = 3
  - [in analogy with Coxeter diagrams]

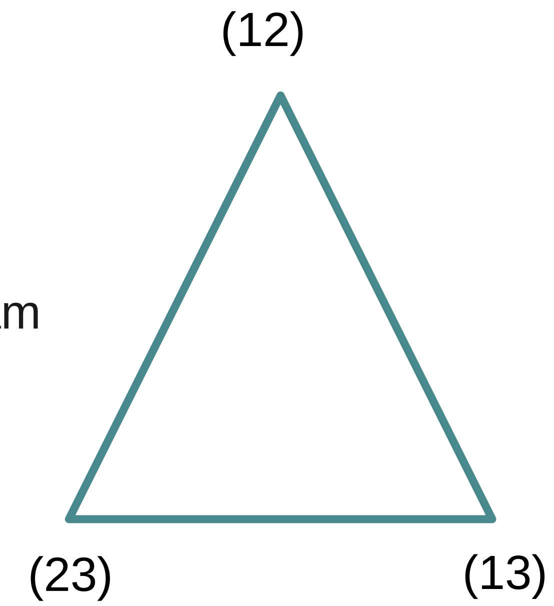
- Example G = Sym(3), C = { (12), (23), (13) }

- Paper contains diagrams like this
- So there was a combinatorial way of thinking



Geometrical and Combinatorial impact: Cuypers and (J I) Hall

- **1989 - 1997 [3 of JIH, 1 by HC, 4 joint] :** extend to infinite three transposition groups (G, C) – strong use of graph theoretic methodology
- As well as the **diagram D**, they study
- The **commuting graph A** of (G, C)
  - Graph with vertex set C
  - { x, y } an edge ⇔ | xy | = 2
  - Commuting graph is **complement** of diagram



- Example G = Sym(3), C = { (12), (23), (13) }

Commuting graph is the empty graph

- Note that G is a group of automorphisms of both D and A

Geometrical and Combinatorial impact: Cuypers and (J I) Hall

- **Two equivalence relations on C**
- **D-relation:**
  - $\mathbf{x \equiv_D y \Leftrightarrow x, y}$ have same neighbour set in D
- **A-relation:**
  - $\mathbf{x \equiv_A y \Leftrightarrow x, y}$ have same neighbour set in A

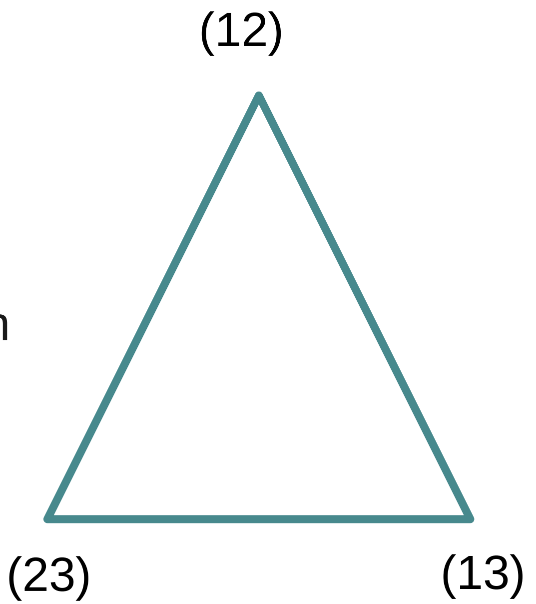
- Both relations are G-invariant – induced G-action

On the sets of equivalence classes

- G is **irreducible** if G faithful on the

Equivalence classes for each relation

Ex: both relations Trivial for Sym(3)



All finite three transposition groups with no nontrivial soluble normal subgroups are irreducible

Geometrical and Combinatorial impact: Cuypers and (J I) Hall

Ex: both relations Trivial for Sym(3)

- **Two equivalence relations on C**
- **D-relation:**
  - **x** $\equiv_D$ **y** ⇔ **x, y** have same neighbour set in D
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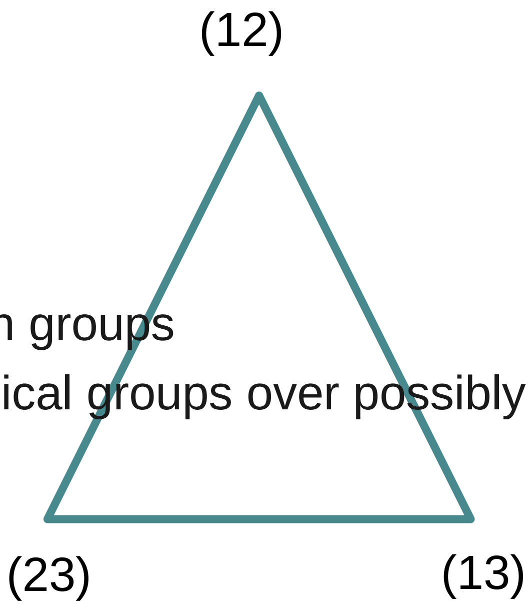


- **Classification:** all irreducible three transposition groups
  - Essentially same as finite case – same classical groups over possibly infinite dimensional spaces.

All finite three transposition groups with no nontrivial soluble normal subgroups are irreducible

Commuting graphs and diagrams

- **Broader context:** Group G and class C of involutions (union of conjugacy classes; often a single class)
  - Graph with vertex set C
  - { x, y } an edge ⇔ **CONDITION** holds

- **CONDITION:** "commuting" that is $| xy | = 2$
  - Motivating examples: all simply laced Weyl groups
  - Bates, Bundy, Perkins, Rowley [2003 + +]
  - Studied for all Coxeter groups: connectivity, diameters of components
  - Many generalisations in literature

Commuting graphs and diagrams

- **Broader context:** Group G and class C of involutions (union of conjugacy classes; often a single class)
  - Graph with vertex set C
  - { x, y } an edge ⇔ **CONDITION** holds

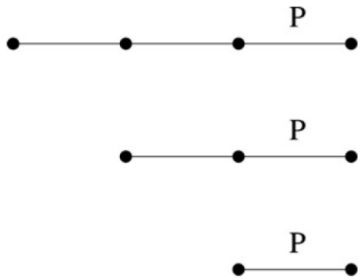
- **CONDITION:** $| xy | = 3$ equivalently $\langle x, y \rangle = \mathrm{Sym}(3)$
  - Called **Sym(3) - involution graph**
  - Devillers, Giudici [2008 - several papers]
  - General theory on connectivity, automorphisms, existence of triangles

- Motivated by ….

$$A_5 < \mathrm{PSL}(2, 11) < M_{11} < M_{12}.$$

Tower of graphs admitting interesting groups

- Arose from general study of decomposing edges of a Johnson graph J(v,k) “nicely” into isomorphic subgraphs [Devillers, Giudici, Li, CEP 2008]
  - Exceptional example J(12,4) [valency 32, 495 vertices]
    - admits M_{12} decomposing into 12 copies of Σ [valency 8, 165 vertices] admitting M_{11}
  - Exceptional example J(11,3)
    - admits M_{11} decomposing into 12 copies of Π [valency 6, 55 vertices] admitting PSL(2,11)
  - Use Witt designs to understand graphs J(12,4), Σ, Π
  - Or diagram geometry to understand $A_5 < \mathrm{PSL}(2,11) < M_{11}$



- Most uniform interpretation was as a set of four Involution graphs
- **CONDITION:** $< x, y > = \mathrm{Sym}(3)$ PLUS something extra
  - Devillers, Giudici, Li, CEP [2010]

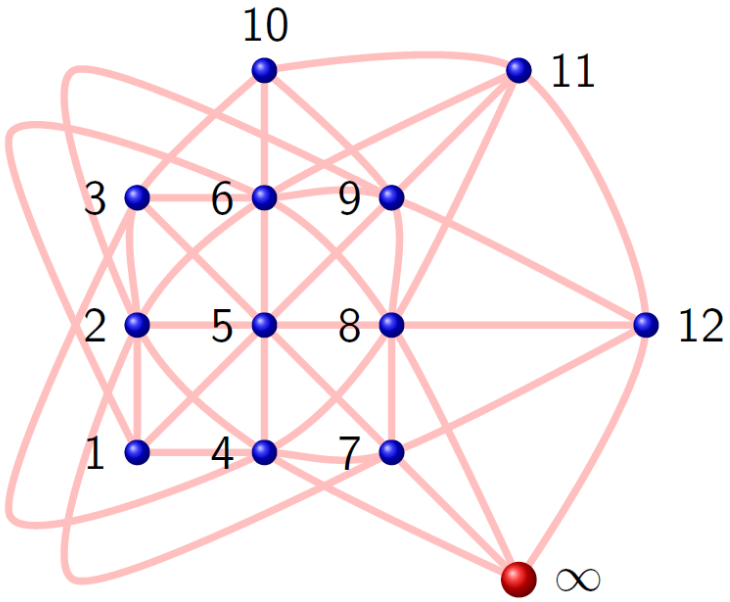
Commuting graphs and diagrams

- **Broader context:** Group G and class C of involutions (union of conjugacy classes; often a single class)
  - Graph with vertex set C
  - { x, y } an edge ⇔ **CONDITION** holds
- **CONDITION:** | xy | lies in given set π of positive integers
  - **π - Local fusion graph** or **Local fusion graph if π = {all odd integers}**
  - Ballantyne, Greer, Rowley [2013 - several papers]
  - For symmetric groups, sporadic simple groups: diameter at most 2
- **Theorem:** for all r, m exists G, C where local fusion graph has m components, each of diameter r

Now for something different: beginning with M_{12}

- Conway's Game on PG(3,3)

- Start from a specified point ∞
- Move to a second point



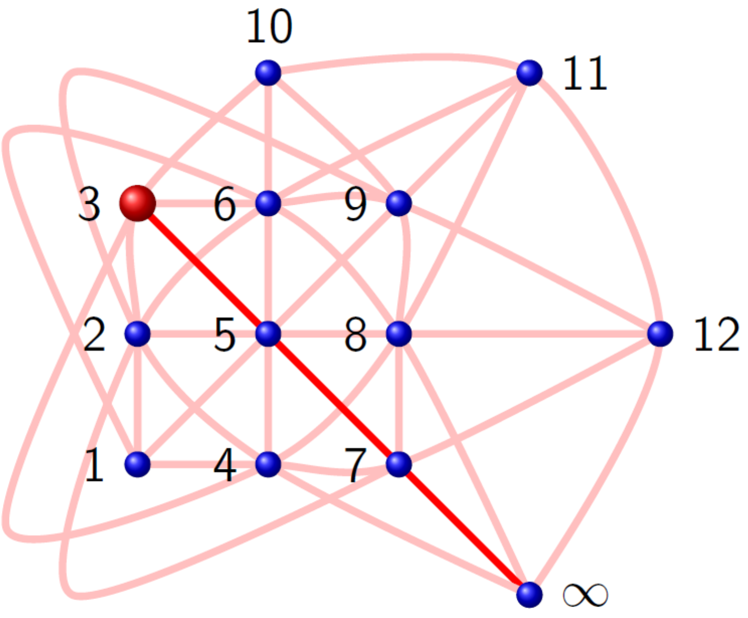
Now for something different: beginning with M_{12}

- Conway's Game on PG(3,3)

- Start from a specified point ∞
- Move to a second point, say 3

- Associate move with permutation

$$[\infty, 3] = (\infty,3)\ (5,7)$$



Now for something different: beginning with M_{12}

- Conway's Game on PG(3,3)

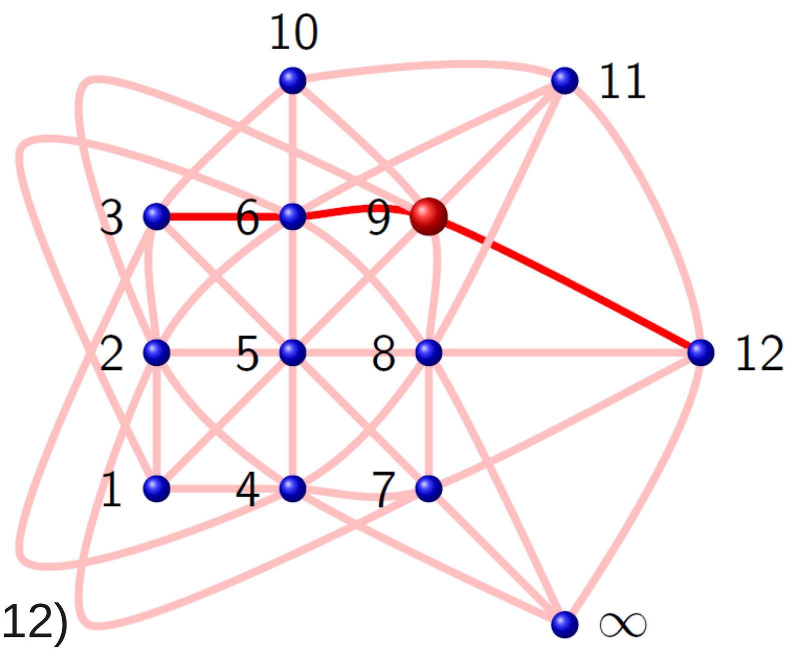
- Start from a specified point ∞
- Move to a second point, say 3

- Associate move with permutation

  $[\infty, 3] = (\infty,3)\ (5,7)$
- Repeat: $[3,9] = (3,9)\ (6,12)$
- Composite move sequence

$[\infty, 3, 9] = [\infty, 3]\ [3,9] = (\infty,3)\ (5,7)\ (3,9)\ (6,12)$

$= (\infty, 9, 3\ )\ (5, 7)\ (6,12)$



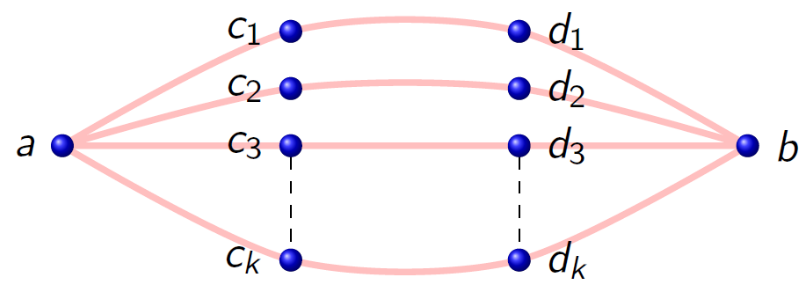
Now for something different: beginning with M_{12}

- Conway's Game on PG(3,3)

- L_∞(PG(3,3)) := SET of all move sequences starting with ∞
- "Conway's groupoid" – subset of Sym(13) – not a group

- Π_∞(PG(3,3)) := SET of all move sequences starting AND ENDING with ∞
- "hole stabiliser" – is a group
- Isomorphic to M_{12}

- Gill, Gillespie, Nixon, Semeraro: where else can we play this game?



Try a 2-(n,4,k) design D

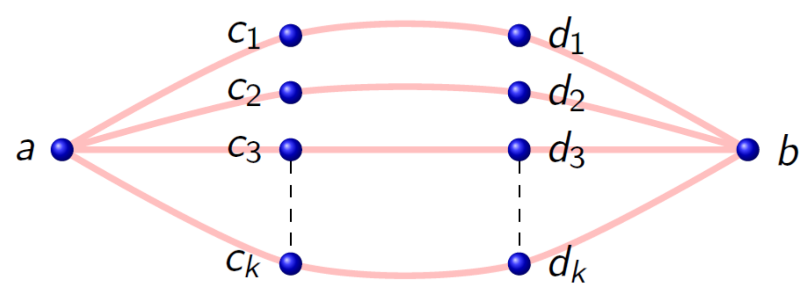
- n points, each point pair { a, b } lies on k lines [all of size 4]
- Try to define

$$[a, b] = (a, b) \prod_{i=1}^{k} (c_i, d_i)$$

- Well defined provided the points c_i and d_i are pairwise distinct
- So need D supersimple distinct lines have at most two common points

- $L_\infty(D)$:= SET of all move sequences starting with distinguished point ∞
- $\Pi_\infty(D)$:= SET of all move sequences starting AND ENDING with ∞

- Gill, Gillespie, Nixon, Semeraro: computer searches and some theory

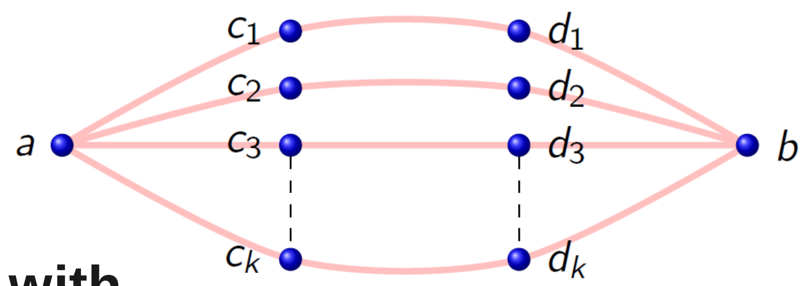


Supersimple 2-(n,4,k) design D

- Each point pair { a, b } has elementary move $$[a, b] = (a, b) \prod_{i=1}^{k} (c_i, d_i)$$
- For k=1 found: either Conway's groupoid or $\Pi_\infty(D) = Alt(n-1)$, $L_\infty(D) = Alt(n)$
- For k=2 found: INTERESTING CASE or $\Pi_\infty(D) = Sym(n-1)$, $L_\infty(D) = Sym(n)$
- INTERESTING CASE n=10, $\Pi_\infty(D) = O^+(4,2)$, $L_\infty(D) = Sp(4,2)$ [a group!]
- And D satisfies:
  - Symmetric difference of two intersecting lines is also a line
  - Each 4-subset of points contains 0, 2 or 4 collinear triples

Collinear triples forms regular two graph

$$[a, b] = (a, b) \prod_{i=1}^{k} (c_i, d_i)$$



Supersimple 2-(n,4,k) designs D with

- Symmetric difference of two intersecting lines is also a line
- Each 4-subset of points contains 0, 2 or 4 collinear triples

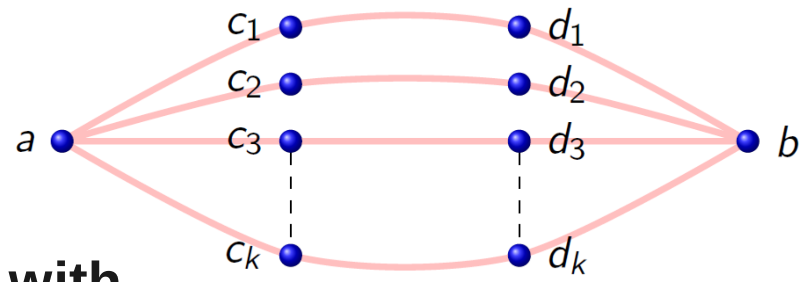
↘ 2017 Gill, Gillespie, CEP, Semeraro:

- L_∞(D) always a group

↘ For E := { [a,b] | distinct points a, b }

- E conjugacy class of L_∞(D)
- (L_∞(D), E) three transposition group

$$[a, b] = (a, b) \prod_{i=1}^{k} (c_i, d_i)$$



Supersimple 2-(n,4,k) designs D with

- Symmetric difference of intersection lines is also a line
- Each 4-subset of points contains 0, 2 or 4 collinear triples

- Using the Fischer classification of three transposition groups we find

- 2017 Gill, Gillespie, CEP, Semeraro: One of the following
  1) $\Pi_\infty(D) = 1$ and $L_\infty(D) = E(2^m)$
  2) $\Pi_\infty(D) = O^+(2m,2)$, $L_\infty(D) = Sp(2m,2)$
  3) $\Pi_\infty(D) = O^-(2m,2)$, $L_\infty(D) = Sp(2m,2)$
  4) $\Pi_\infty(D) = Sp(2m,2)$, $L_\infty(D) = 2^{2m}.Sp(2m,2)$
- D described explicitly e.g. in case 1) points and planes of AG(m,2)

Thank you

➢To Professor Bernd Fischer

➢For your beautiful mathematics

➢Congratulations on the milestone celebrated at this conference

Photo. Courtesy: Joan Costa joancostaphoto.com

Thank you

Photo. Courtesy: Joan Costa joancostaphoto.com