Generalized cycle decompositions and quasi-Coxeter elements in Coxeter groups

Thomas Gobet

Elie Cartan Institute, Nancy

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Cycle decompositions in symmetric groups

Coxeter groups

Cycle decompositions in arbitrary Coxeter groups

Parabolic quasi-Coxeter elements

Cycle decompositions in finite Coxeter groups

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$$W = \langle\ s_1, \dots, s_n \mid s_i^2 = e,\ \ \underbrace{s_i s_j \cdots}_{m_{ij}\ \text{copies}} = \underbrace{s_j s_i \cdots}_{m_{ji}\ \text{copies}}\ \text{ if } i \neq j\ \rangle,$$

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- Examples: dihedral groups, hyperoctahedral groups, Weyl groups of semisimple alg. groups or Kac-Moody groups, ...
- Coxeter groups can be realized as real reflection groups, that is, groups generated by reflections on finite-dimensional real spaces, preserving a symmetric bilinear form (which is non-degenerate iff W is finite).

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  3. Cycle decompositions in finite Coxeter groups.
- The main result, which is a characterization of (parabolic) quasi-Coxeter elements in finite Coxeter groups, is a joint work with B. Baumeister, K. Roberts, and P. Wegener.

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  Moreover, the set T is the set of all the conjugates to elements of S. In the Coxeter theoretic language: T is the set of *reflections* of W.

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Question 2

How should we define a “cycle” in W?

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- Counterexample for equivalence in general: dihedral group of order 10.

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Question 3

Is any element $w \in W$ a Coxeter element in a (finitely generated) reflection subgroup $W' \subseteq W$?

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- A *(parabolic) quasi-Coxeter element* (short (p)qc) $w \in W$ is an element admitting a T-reduced factorization $t_1 t_2 \cdots t_k$ such that $\langle t_1, t_2, \ldots, t_k \rangle = W$ (resp.$\langle t_1, t_2, \ldots, t_k \rangle$ is a parabolic subgroup of W).

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Conjecture 1

Let $w \in W$ be a pqc, then for **every** *$t_1 t_2 \cdots t_k \in \mathrm{Red}_T(w)$ we have $\langle t_1, t_2, \ldots, t_k \rangle = P(w)$.*

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Let $w \in W$ be a pqc, $t_1 t_2 \cdots t_k$, $q_1 q_2 \cdots q_k \in \mathrm{Red}_T(w)$ such that the subgroups $W' := \langle t_1, t_2, \ldots, t_k \rangle$ and $W'' := \langle q_1, q_2, \ldots, q_k \rangle$ are both parabolic. Then $W' = W''$. We denote this parabolic subgroup by $P(w)$ and call it the parabolic closure of w.

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Let $w \in W$ be a pqc, then for **every** $t_1 t_2 \cdots t_k \in \mathrm{Red}_T(w)$ *we have $\langle t_1, t_2, \ldots, t_k \rangle = P(w)$. (OK for finite W, BGRW).*

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pqc's in finite Coxeter groups

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- If the Hurwitz action is transitive on $\mathrm{Red}_T(w)$, then one can pass from any T-reduced expression of w to any other just by applying "dual braid relations", that is, relations of the form $ab = bc$ with $a, b, c \in T$. Hence a dual analogue of the Matsumoto property holds.

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- Unfortunately, the Hurwitz action on $\mathrm{Red}_T(w)$ is not transitive in general. It fails for example if w is the longest element in W of type B_2.

Main result

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- In type A_n every element is a pqc, and we recover the cycle decomposition.

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Proposition

There is a one-to-one correspondence

$$\{\mathcal{O}_1, \ldots, \mathcal{O}_\ell\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{Reflection subgroups of } W \\ \textit{in which } w \textit{ is a qc} \end{array} \right\},$$

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In particular, in each of these reflection subgroups, w has a unique generalized cycle decomposition.

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Merci !