Overgroups of Irreducible Quasisimple Subgroups in Finite Classical Groups

K. Magaard

University of Birmingham

January 14, 2017

Theorem [Aschbacher-Scott 1985]

To settle the maximal subgroup problem for general finite groups it suffices to

1. Determine the conjugacy classes of maximal subgroups of the almost simple groups.
2. For every quasisimple group G and all prime divisors ℓ of $|G|$ determine $H^1(G, V)$ for every irreducible $\mathbb{F}_\ell G$-module V.

The choices for G are *s*poradic (for example $\mathbb{M}$), alternating, of exceptional Lie type, classical (for example $\mathrm{PSL}_n(q)$).

The possibilities for V are not completely classified.We need all decomposition numbers and minimal fields of definition.

Aschbacher's Theorem

Let X be a classical group with natural module $V = k^m$, $\mathrm{Char}(\mathrm{k}) = \ell$. Aschbacher defines 8 families $\mathcal{C}_i(X)$ of "geometric" subgroups of X.

Theorem [Aschbacher 1984]

If $H \leq X$ is maximal, then either $H \in \mathcal{C}_i(X)$, or $H \in \mathcal{S}(X)$ meaning

1. The generalized Fitting subgroup $F^*(H)$ is quasisimple;
2. $F^*(H)$ acts absolutely irreducibly on V;
3. the action of $F^*(H)$ on V can not be defined over a smaller field;
4. any bilinear, quadratic or sesquilinear form on V that is stabilized by $F^*(H)$ is also stabilized by X.

Aschbacher calls the collection of subgroups of X which satisfy conditions 1 through 4 above $\mathcal{S}(X)$.

Some remarks

The geometric families

$\mathcal{C}_1(X)$ are the stabilizers of subspaces of V.

$\dim(V) \leq 12$

Recent work of Bray, Holt and Roney-Dougal achieves a full classification of conjugacy classes of maximal subgroups.

$\dim(V) \geq 13$

Kleidman and Liebeck classify those situations when a maximal member of $\mathcal{C}_i(X)$ is not maximal in G. Generally such occurrences are rare.

Open Question

When are members of $\mathcal{S}(X)$ maximal in X?

General Observation

If H is quasisimple and $\varphi \in \mathrm{IBr}_\ell$, then there exists a quasisimple classical group X with natural module V affording φ and a corresponding homomorphism $\Phi : H \mapsto X \leq GL(V)$ such that $H\Phi \in \mathcal{S}(X)$.

Possible obstructions to the maximality of $N_X(H\Phi)$ in X

The definition of $\mathcal{S}(X)$ implies that if $N_X(H\Phi) < G < X$, then

$$G \in \mathcal{C}_2(X) \cup \mathcal{C}_4(X) \cup \mathcal{C}_6(X) \cup \mathcal{C}_7(X) \cup \mathcal{S}(X).$$

$$\mathcal{S}(X) = \textbf{Spor} \cup \textbf{Alt} \cup \textbf{CL d} \cup \textbf{CL c} \cup \textbf{EX d} \cup \textbf{EX c}$$

Problem 1

For each $H\Phi$, name the possible types of G; i.e. identify the members of $\mathcal{C}_2(X) \cup \mathcal{C}_4(X) \cup \mathcal{C}_6(X) \cup \mathcal{C}_7(X) \cup \mathcal{S}(X)$ which may contain $H\Phi$.

Problem 2

Name X; i.e., determine the minimal field k of definition of Φ, and the H invariant form(s) on $V = k^m$. Choose X minimal subject to containing $H\Phi$ and also determine $N_{\mathrm{GL}(V)}(X)$.

Problem 3

For each $\tilde{X} \in N_{\mathrm{GL}(V)}(X)$ with $X \leq \tilde{X}$ determine $N_{\tilde{X}}(H\Phi)$ and $N_{\tilde{X}}(H\Phi) \cap G$. More generally determine the action of $N_{\tilde{X}}(H\Phi)$ on the set of G's lying over $H\Phi$.

Conclusion:

G obstructs the maximality of $N_{\tilde{X}}(H)$ iff $N_{\tilde{X}}(H) < \tilde{X} \cap G$.

Problem 1: Necessary conditions for the existence of obstructions G to the maximality of $N_X(H\Phi)$ in X.

- If $G \in \mathcal{C}_2(X)$, then Φ is an induced representation.
- If $G \in \mathcal{C}_4(X)$, then Φ is a tensor (Kronecker) product of two representations of unequal degrees > 1.
- If $G \in \mathcal{C}_7(X)$, then Φ is a tensor product of two or more representations of equal degrees; i.e., Φ is tensor induced.
- If $G \in \mathcal{C}_6(X)$,then $H \leq N_X(E)$ where E is extraspecial resp. symplectic type of order r^{1+2s} respectively $2 \times 2^{1+2s}$ and $\dim(V) = r^s$ resp. 2^s.
- If $G \in \mathcal{S}(X)$, then $(F^*(G), H, V)$ is an irreducible triple; i.e., the irreducible $F^*(G)$-module V restricts irreducibly to H.This is a special case of the branching problem.

Potential obstructions G to the maximality of $H \in \mathcal{S}(X)$

$\mathcal{C}_2$	HHM	DM/NN	Seitz	HHM	Seitz	HHM	induced
$\mathcal{C}_4$	MT	BK	Stei	MT	Stei	MT	tensor prod
$\mathcal{C}_6$	MT	and	Bray	inde	pend	ently	$r^{1+2n}\mathrm{Sp}_{2n}(r)$
$\mathcal{C}_7$	$\leftarrow$	—	MT	—	—	$\rightarrow$	tensor ind
Spor	LM-wip	Hu	LSS		LSS		$\uparrow$
Alt	S/KW BK/KS KTS	S/KW BK/KS KTS	S/KW JS/BK KS/KTS	S/KW BK/KS KTS	S/KW BK/KS KTS	S/KW BK/KS KTS	|
CL d	LM-wip	Hu	D/S/T	MRT	D/S/T	MRT	branching
CL c	LM-wip	Hu	LSS	Seitz	LSS	Se/S/N	rules
EX d	LM-wip	Hu	D/S/T	MRT	D/S/T	MRT	|
EX c	LM-wip	Hu	LSS	Seitz	LSS	Seitz	$\downarrow$
G / H	**Spor**	**Alt**	**CL d**	**CL c**	**EX d**	**EX c**	H rep'n

HHM = Hiss Husen Magaard, MT = Magaard Tiep, MRT = Magaard Röhrle Testerman, BK = Bessenrodt Klechshev, DM/NN = Djokovic Malzan / Nett Noeske, D/S/T = Dynkin/Seitz/Testerman, Hu = Husen, JS = Jantzen Seitz, S/KW= Saxl/Kleidman Wales, BK/KS/KST = Brundan Kleshchev/Kleshchev Sheth/Kleshchev Sin Tiep, LSS = Liebeck Saxl Seitz, Se/S/N = Seitz/ Schaefer-Frey / H.N. Nguyen, LM = Le Magaard

Irreducible Tensor Decomposable H-modules

Recall: If $H \in \mathcal{S}(X)$ is $\mathcal{C}_4$ obstructed, then V is tensor decomposable.

Theorem [Seitz, M.-Tiep]If $H \in \mathcal{S}(X)$ is of Lie type defined over $\mathbb{F}_q$, with $q = p^a$ and V is a tensor decomposable irreducible H-module, then one of the following is true:

- $p = \ell$ and the H-module V is **not** ℓ-restricted, (Steinberg's Thm.)
- V **is** ℓ-restricted, and either $p = \ell = 2$ and H is of Lie type B_n, C_n, F_4 or G_2, or $p = \ell = 3$ and H is of type G_2,
- $p \neq \ell$ and $q \leq 3$,
- $H \cong \mathrm{Sp}_{2n}(5)$, or $\mathrm{Sp}_{2n}(2^a)$
- $H/Z(H) \cong \mathrm{PSL}_3(4)$,
- $H \cong F_4(2^a)$, or ${}^2F_4(2^{2b+1})$.

Not all tensor product factorizations lead to $\mathcal{C}_4(X)$ or $\mathcal{C}_7(X)$ obstructions

- If $H = 4_1 \cdot \mathrm{PSL}_3(4)$, then $8_a \otimes 8_d = 64_a$. The Frobenius-Schur indicator of 8_a and 8_d is 0, whereas it is $+$ for 64_a. Thus $H \in \mathcal{S}(\Omega_{64}^+(\ell))$ for $\ell \neq 2, 3, 5, 7$, yet the maximal $\mathcal{C}_4(\Omega_{64}^+(\ell))$ subgroups are $N_X(\mathrm{Sp}_8(\ell) \circ \mathrm{Sp}_8(\ell))$ and $N_X(\Omega_8^\pm(\ell) \circ \Omega_8^\pm(\ell))$.
- Another example: $H = M_{24} \in \mathcal{S}(\Omega_{10395}(\ell))$, via it's largest irreducible character χ_{26}, whenever $(\ell, |M_{24}|) = 1$. Here $\chi_{26} = \chi_3 \otimes \chi_5 = \chi_4 \otimes \chi_5 = \chi_3 \otimes \chi_6 = \chi_4 \otimes \chi_6$ The indicator of χ_{26} is $+$, whereas those of $\chi_4, \chi_5, \chi_6 \chi_7$ are 0.
- If $H = \mathbb{M}$, then 11 of the 194 ordinary irreducible characters are tensor decomposable.Of these, 8 are $\mathcal{C}_4$-obstructed, and 3 are not. For example $\chi_{55} = \chi_2 \otimes \chi_{16}$ is $\mathcal{C}_4$ obstructed, whereas $\chi_{185} = \chi_6 \otimes \chi_{17}$ is not.

$\mathcal{S}(X)$ obstructions: The case $H \cong M_{11}$

How often does $H \in \mathcal{S}(X)$, imply H maximal in X?
Morally this should happen most of the time, but ....

Theorem

If $H \simeq M_{11}$ and $H \in \mathcal{S}(X)$, then $N_X(H)$ is maximal in X iff $F^*(X) = \mathrm{SL}_5(3)$.

φ	Type of X	$\mathcal{C}_2$	$\mathcal{C}_{4,7}$	$\mathcal{C}_6$	$\mathcal{S}_{Lie}$	$\mathcal{S}_{altspor}$
φ_2	$\Omega_9(11)$					A_{11}
φ_3	$\mathrm{SL}_{10}(11)$					$2.M_{12}$
φ_4	$\mathrm{SL}_{10}(11)$					$2.M_{12}$
φ_5	$\Omega_{11}(11)$	$2^{10} \rtimes M_{11} < 2^{10} \rtimes A_{11}$				$M_{12} < A_{12}$
φ_6	$\Omega^+_{16}(11)$				$\mathrm{Spin}^+_9(11)$	$2.A_{11}, M_{12}$
φ_7	$\Omega^+_{44}(11)$				$\Omega_9(11)$	A_{11}
φ_8	$\Omega_{55}(11)$	$2^{54} \rtimes M_{11} < 2^{54} \rtimes A_{55}$			$P\Omega_{11}(11)$	$M_{12} < A_{12}$

Table: Obstructions to the maximality of M_{11} embeddings; the case $\ell = 11$

Irreducible Imprimitive Representations

Theorem [Hiß, M. 2016] If H is quasisimple and ℓ is not a divisor of $|H|$, then all imprimitive elements of $\mathrm{IBr}_\ell(H)$ are known.

Theorem[Hiß, Husen, M. 2015]If $F^*(H)$ is quasisimple and V is irreducible and imprimitive, then one of the following is true:

1. H is of Lie type and characteristic $p \neq \ell$ and V is Harish-Chandra induced or H has an exceptional Schur multiplier,
2. $H \in \mathrm{Lie}(p)$, $p = \ell$, $F^*(H) \in \{\mathrm{SL}_2(5), \mathrm{SL}_2(7) \cong \mathrm{SL}_3(2), \mathrm{Sp}_4(3)\}$,
3. $H = A_n$ and V is from one of three families, or $\ell \leq n$, $H = 2 \cdot A_n$ and the block stabilizer is intransitive, or $n \leq 9$.
4. $\frac{F^*(H)}{Z(F^*(H))} \in \{M_{11}, M_{12}, M_{22}, M_{24}, HS, McL, ON, Co_2, Fi_{22}, Co_1, Fi_{24}'\}$

Theorem 7.3 [Hiß, Husen, M.] due to Lusztig

Let $s \in G^*$ be semisimple such that $C_{\mathbf{G}^*}(s)$ is contained in a proper split Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$. Let $\mathbf{L}$ be a split Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$.

- Then every ordinary irreducible character of G contained in $\mathcal{E}(G, [s])$ is Harish-Chandra induced from a character of $\mathcal{E}(L, [s])$.
- If s is ℓ-regular for some prime ℓ not dividing q, then every irreducible ℓ-modular character of G contained in $\mathcal{E}_\ell(G, [s])$ is Harish-Chandra induced from a Brauer character lying in $\mathcal{E}_\ell(L, [s])$.

Corollary: The proportion of primitive irreducible characters in $\mathrm{Irr}(\mathrm{SL}_n(q))$ approaches $1/n$ for large values of q.

V irreducible and imprimitve does not necessarily imply the existence of a $\mathcal{C}_2(X)$ obstruction.

- If $H = \mathrm{SL}_2(q)$, ℓ is odd, and V affords $R_{T,\Theta}$ with $|T| = q - 1$, then V is irreducible and imprimitive of dimension $q + 1 = |H : T|_{p'}$. In fact $V = \mathrm{Ind}_B^H(L)$ is irreducible and imprimitive. Here B is a Borel subgroup of H,and L affords $\mathrm{Infl}_T^B(\Theta) \in \mathrm{IBr}_\ell(B)$.
- Then $H \leq k^* \wr S_m \leq \mathrm{GL}_m(k)$, and H projects to a transitive subgroup of S_m.
- The module V is a self-dual H-module which implies that H must be in $\mathbb{Z}_2 \wr S_m \leq \mathrm{O}_m(k)$; which is impossible if $|\Theta| > 2$.
- If $|\Theta| = 2$, then q is odd and $R_{T,\Theta}$ and is not irreducible.
- Even though roughly half of the irreducible H representations are induced, none are $\mathcal{C}_2(X)$ obstructed.

Examples of $\mathcal{C}_2(X)$ type obstructions

- Let $H = \mathrm{SL}_3(5)$, $5 < \ell \neq 31$, and $V = \mathrm{Ind}_P^G(\Theta)$, where P is one of the two classes of the maximal parabolics and $1 \neq \Theta$ is a linear character of P.
- If $|\Theta| = 4$, then $H \leq \mathrm{SL}_{31}^{\epsilon}(\ell)$ where $\ell \equiv \epsilon \mod 4$.
- If $X = \mathrm{SL}_{31}^{\epsilon}(\ell)$, then $N_{\tilde{X}}(H)$ is $\mathcal{C}_2(\tilde{X})$ obstructed in $\tilde{X}$ iff no element of $\tilde{X}$ induces a graph automorphism on $F^*(H)$.
- If $|\Theta| = 2$, then $H \leq X = \Omega_{31}(\ell)$ and $\mathrm{Aut}(H) \leq \mathrm{SO}_{31}(\ell)$. Also $\mathrm{Aut}(H) \leq \Omega_{31}(\ell)$ iff $\ell \equiv 1 \mod 4$.
- So $N_{\Omega_{31}(\ell)}(H)$ is $\mathcal{C}_2(\Omega_{31}(\ell))$ obstructed iff $\ell \equiv 3 \mod 4$.
- If $\ell \equiv 3 \mod 4$, then $N_{\tilde{X}}(H)$ is not $\mathcal{C}_2(\tilde{X})$ obstructed if $\mathrm{SO}_{31}(\ell) \leq \tilde{X}$.

THANK YOU!