QEM “Optimization”, WS 2017/18

Part 1. Convergence in Metric Spaces
A Big List of Exercises to Start with

1. Prove that in any metric space (X, d):
(a) [**1 Point**] $|d(x,z) - d(y,z)| \leq d(x,y)$ (inverse triangle inequality).
(b) [**1 Point**] $|d(x,z) - d(y,u)| \leq d(x,y) + d(z,u)$ (quadrangle inequality).
2. **(a)** [**1 Point**] Let $a, b, c \geq 0$ and $a \leq b + c$. Check that

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

(b) [**1 Point**] Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$, $n \in \mathbb{N}$. Show that

$$d(x,y) := \frac{\|x-y\|}{1+\|x-y\|}, \quad x, y \in \mathbb{R}^n,$$

defines a metric on $\mathbb{R}^n$.
(c) [**1 Point**] Show that $0 \leq d(x,y) \leq 1$ for all $x, y \in \mathbb{R}^n$.
(d) [**1 Point**] The metric $d(x,y)$ cannot be generated by any norm $\|\cdot\|_d$ on $\mathbb{R}^n$, i.e., the presentation $d(x,y) = \|x-y\|_d$ is impossible.
3. [**2 Points**] Check that

$$d(x,y) := \frac{|x-y|}{\sqrt{1+x^2}\sqrt{1+y^2}}, \quad x, y \in \mathbb{R},$$

defines a metric on the real line $\mathbb{R}$. Is this metric induced by some norm on $\mathbb{R}$?
4. [**2 Points**] For $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ let us define

$$d(x,y) := \left(\sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|}\right).$$

Is it a metric on $\mathbb{R}^2$?
5. [**2 Points**] Two metrics d_1 and d_2 on X are called *equivalent* if from $d_1(x_n, x) \to 0$ it follows that $d_2(x_n, x) \to 0$, and vice versa. In other words,

d_1 and d_2 define the same system of open sets. Prove that the following two metrics on $X = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are equivalent:

$$d_1(x,y) := |x-y|, \quad d_2(x,y) := |\tan x - \tan y|.$$

6. Let (X,d) be a metric space.
(a) [**1 Point**] Show that

$$d_1(x,y) := \min\{1, d(x,y)\}, \quad x, y \in X,$$

defines a new metric on X.
(b) [**1 Point**] Show that these metrics are equivalent.
7. [**2 Points**] Let X be a set, (Y,ρ) a metric space, and $F\colon X \to Y$ a one-to-one function. Show that

$$d(x,y) := \rho(F(x), F(y)), \quad x, y \in X,$$

defines a metric on X.
8. Let

$$\mathbb{R}^\infty := \{x = (x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{R},\ \forall i \in \mathbb{N}\}$$

be the space of all real sequences.
(a) [**2 Points**] Using Exercise 2 check that both

$$d(x,y) := \sum_i \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|},$$
$$d_1(x,y) := \sup_i \frac{|x_i - y_i|}{1 + |x_i - y_i|},$$
$$x = (x_i)_{i\in\mathbb{N}},\ y = (y_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty,$$

define metrics on $\mathbb{R}^\infty$.
(b) [**1 Point**] Show that

$$0 \le d(x,y) \le d_1(x,y) \le 1, \quad \forall x, y \in \mathbb{R}^\infty.$$

(c) [**2 Points**] What does it mean for the sequences $x^{(n)} = (x_i^{(n)})_{i\in\mathbb{N}}$, $n \in \mathbb{N}$, to converge in these metrics? Show that $d(x^{(n)}, x) \to 0$ is equivalent to the *pointwise* convergence $|x_i^{(n)} - x_i| \to 0$ for each $i \in \mathbb{N}$, whereas $d_1(x^{(n)}, x) \to 0$ is equivalent to the *uniform* convergence $\sup_i |x_i^{(n)} \to x_i| \to 0$ as $n \to \infty$.

(d) [**2 Points**] Show that both spaces are complete.

(e*) [**4 Points**] Show that $(\mathbb{R}^\infty, d)$ is separable, but $(\mathbb{R}^\infty, d_1)$ is not.

9. Let

$$l_1 := \left\{ x = (x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \mid \sum_i |x_i| < \infty \right\}$$

be a set of all *summable* real sequences. As usual, define

$$\begin{aligned} (x_i)_{i\in\mathbb{N}} + (y_i)_{i\in\mathbb{N}} &:= (x_i + y_i)_{i\in\mathbb{N}}, \\ a(x_i)_{i\in\mathbb{N}} &:= (ax_i)_{i\in\mathbb{N}} \end{aligned}$$

for all $x = (x_i)_{i\in\mathbb{N}}$, $y = (y_i)_{i\in\mathbb{N}} \in l_1$ and $a \in \mathbb{R}$. Show that:

(a) [**1 Point**] l_1 is a vector space with respect to the above operations.

(b) [**1 Point**] Both

$$\|x\|_\infty := \sup_i |x_i| \text{ and } \|x\|_1 := \sum_i |x_i|$$

define norms on l_1. Furthermore, $\|x\|_\infty \leq \|x\|_1$.

(c) [**1 Point**] These norms are not equivalent, i.e., one cannot find some $C \in (0, \infty)$ such that $\|x\|_1 \leq C\|x\|_\infty$ for all $x \in l_1$. To this end, in l_1 construct a sequence $x^{(n)} = (x_i^{(n)})_{i\in\mathbb{N}}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|x^{(n)}\|_\infty = 0, \quad \lim_{n\to\infty} \|x^{(n)}\|_1 = \infty.$$

(d) [**2 Points**] Let $0 \leq a \leq \infty$. Show that in l_1 there exist sequences $x^{(n)} = (x_i^{(n)})_{i\in\mathbb{N}}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} x_i^{(n)} = 0 \text{ for all } i \in \mathbb{N},$$

but

$$\lim_{n\to\infty} \|x^{(n)}\|_\infty = \lim_{n\to\infty} \|x^{(n)}\|_1 = a.$$

10. For $p \in [1, \infty)$ define

$$l_p := \left\{ x = (x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \mid \|x\|_p := \left[\sum_i |x_i|^p \right]^{1/p} < \infty \right\}.$$

Show that:

(a) [**1 Points**] $\|\cdot\|_p$ is a norm on l_p [**Hint**: use Minkovski's inequality, see Lecture 4].

(b) [**2 Points**] $(l_p, \|\cdot\|_p)$ is a Banach space.

(c) [**2 Points**] $(l_p, \|\cdot\|_p)$ is separable [**Hint**: consider a countable subset of all finite sequences with $x_i \in \mathbb{Q}$].

(d) [**1 Point**] $l_p \subset l_{p'}$ whenever $p < p'$.

11. Consider the discrete metric space (X, d) with

$$d(x,y) := \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

(a) [**1 Point**] Show that any set in (X, d) is open and closed simultaneously.

(b) [**1 Point**] Describe all convergent sequences in (X, d).

(c) [**2 Points**] If (X, d) is separable or not?

12. **(a)** [**1 Point**] Prove that in any metric space (X, d), the *closed ball* defined by

$$\overline{B_r(x)} := \{y \in X \mid d(x,y) \leq r\}, \quad x \in X, r \in \mathbb{R}_+,$$

is a closed set.

(b) [**1 Point**] For each open ball

$$B_r(x) := \{y \in X \mid d(x,y) < r\},$$

its closure is contained in $\overline{B_r(x)}$.

(c) [**2 Points**] Show that in general there is no identity in (b). **Hint:** consider the discrete metric space from Exercise 7.

(d) [**2 Points**] Show that the closure of $B_r(x)$ coincides with $\overline{B_r(x)}$ in the space $C[a,b]$ with the uniform metric.

13. [**2 Points**] Show that a set $A \subseteq X$ is open iff $A \cap \partial A = \varnothing$ and A is closed if $\partial A \subseteq A$.

14. [**2 Points**] In $\mathbb{R}^2$ consider two sets

$$A := \{(x,y) \mid y = 0\} \text{ and } B := \{(x,y) \mid xy = 1\}.$$

Prove that $A \cap B = 0$ but $\text{dist}(A,B) := \inf\{d(x,y) \mid x \in A, y \in B\} = 0$.

15. **(a)** [**2 Points**] Prove that for any points $x, y \in X$ and any nonempty set $A \subseteq X$

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Here, $d(x, A) := \inf\{d(x, z) \mid z \in A\}$.

(b) [**1 Point**] Conclude from (a) that the mapping

$$(X, d) \ni x \to d(x, A) \in \mathbb{R}_+$$

is continuous.

16. [**2 Points**] Let $A \subseteq X$ be closed and $x \notin A$. Prove that $d(x, A) > 0$.

17. [**2 Points**] Prove that for any sets $A, B \subseteq X$

$$d(A, B) = d(A, \bar{B}) = d(\bar{A}, B) = d(\bar{A}, \bar{B}).$$

18. Prove the following properties of the interior:

(a) [**1 Point**] $A \subseteq B$ implies $A^\circ \subseteq B^\circ$.

(b) [**1 Point**] $(A \cap B)^\circ = A^\circ \cap B^\circ$ for any $A, B \subseteq X$.

(c) [**2 Points**] $(A \cup B)^\circ = A^\circ \cup B^\circ$ if $A \subseteq X$ is arbitrary and $B \subseteq X$ is closed.

19. [**2 Points**] Let g be some function from $C[a, b]$. Prove that the set

$$\{f \in C[a, b] \mid f(t) < g(t) \text{ for all } t \in [a, b]\}$$

is open in $C[a, b]$.

20. Check whether the following sequences are convergent or not:

(a) [**1 Point**] $X = l_1$, $x_n := \left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ times}}, 0, 0, \ldots\right)$.

(b) [**1 Point**] $X = l_2$, $x_n := \left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n^2 \text{ times}}, 0, 0, \ldots\right)$.

(c) [**1 Point**] $X = l_3$, $x_n := \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$.

21. [**2 Points**] Prove that the sequence

$$f_n(t) := t^n - t^{2n}, \quad t \in [0, 1],$$

is not convergent in $C[0,1]$.

22. [**2 Points**] Prove that the space $C[0,1]$ of all continuous functions $f\colon [0,1] \to \mathbb{R}$ is not complete with respect to the metric

$$d(f,g) := \left(\int_0^1 |f(t) - g(t)|^2 dt \right)^{1/2}.$$

Hint: consider the sequence

$$f_n(t) := \begin{cases} -1, & \text{if } -1 \le t \le -\frac{1}{n}, \\ nt, & \text{if } -\frac{1}{n} \le t \le \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} \le t \le 1. \end{cases}$$

23*. [**3 Points**] Let f be a twice continuously differentiable function on $\mathbb{R}_+ := \{t \in \mathbb{R} \mid t \ge 0\}$ obeying:

(a) $f(0) = 0$ and $f(t) > 0$ if $t > 0$; **(b)** $f(t)$ is strictly increasing for $t \ge 0$; **(c)** $f''(t) \le 0$ for $t > 0$.

Show that $\rho(x,y) := f(|x-y|)$ defines a metric on $\mathbb{R}$. In particular, this applies to $f(t) := \frac{t}{1+t}$.

24*. [**3 Points**] Let f be a continuously differentiable function on $\mathbb{R}_+ := \{t \in \mathbb{R} \mid t \ge 0\}$ obeying:

(a) $f(0) = 0$ and $f(t) > 0$ if $t > 0$; **(b)** $f(t)$ is increasing for $t \ge 0$; **(c)** $\frac{f(t)}{t}$ is decreasing for $t > 0$.

Show that $\rho(x,y) := f(|x-y|)$ defines a metric on $\mathbb{R}$. In particular, this applies to $f(t) := \arctan t$.

25*. [**3 Points**] Let $\mathbb{Q}$ be a metric space of all *rational* numbers with the metric $d(x,y) := |x-y|$. Prove that the set $M := \{p \in \mathbb{Q} \mid 2 < p^2 < 3\}$ is closed and bounded, but not compact in $\mathbb{Q}$.

26*. By definition, the space c consists of all convergent sequences $x = (x_i)_{i \in \mathbb{N}}$ from $\mathbb{R}^\infty$. Respectively, c_0 is a subset of c consisting of all sequences with $\lim_{i \to \infty} x_i = 0$. Define the distance

$$d(x,y) = \|x-y\|_\infty := \sup_i |x_i - y_i|.$$

Show that:

(a) [**3 Points**] c and c_0 are closed subsets in l_∞; hence c and c_0 are Banach spaces with respect to the metric $\|x-y\|_\infty$.

(b) [**3 Points**] The spaces c and c_0 are separable (in contrast to l_∞).

27*. [**3 Points**] Show that a normed space $(X, \|\cdot\|)$ is Banach if and only if any series $\sum_{n\geq 1} x_n$, for which $\sum_{n\geq 1} \|x_n\| < \infty$, is convergent in X.