QE "Optimization", WS 2017/18

Problem Set No. 11

Submit your solutions by **27.11.2017**.
The problems will be discussed in the tutorials.

Questions marked with a star (*) are slightly more challenging and can be skipped if you get too stuck.

1. [**2 points**] Prove that any intersection of convex sets is convex: if each U_i, $i \in I$, is convex and I is an arbitrary index set, then $\bigcap_{i\in I} U_i$ is convex.

2. [**5 points**] Determine whether the following sets are convex (if helpful, draw them). Briefly explain the answer.

(i) $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3\}$;

(ii) $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1 \text{ and } x_1^2 + x_2^2 \leq 1\}$;

(iii) $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1 \text{ or } x_1^2 + x_3^2 \leq 1\}$;

(iv) $\{x = (x_i)_{i=1}^n \in \mathbb{R}^n \mid -1 \leq \sum_{i=1}^n x_i \leq 1\}$;

(v) $\{x = (x_i)_{i=1}^n \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\}$.

3*. [**4 points**] Show that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex (resp. concave) if and only if for each fixed $x, y \in \mathbb{R}^n$ the function $\varphi\colon [0,1] \to \mathbb{R}$ defined by

$$\varphi(\lambda) := f(\lambda x + (1-\lambda)y)$$

is convex (resp. concave) on $[0,1]$.

4. [**3 points**] Show that the distance function

$$\mathbb{R}^{2n} \ni (x, y) \to \|x - y\|, \quad x, y \in \mathbb{R}^n,$$

is convex. Use the definition of convexity directly.

5. [**4 points**] Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a concave function satisfying $f(0) = 0$. Prove that for all $0 \leq \lambda \leq 1$ and $k \geq 1$ we have

$$f(\lambda x) \geq \lambda f(x), \quad kf(x) \geq f(kx).$$

6. [**4 points**] Show (by using the 2nd order sufficient condition) that the function

$$f(x,y) = \frac{x^2}{y}$$

is convex on the domain $U := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

7. [**4 Points**] Show that

$$f(x,y) := \frac{1}{2}e^{-(x+y)} - e^{-x} - e^{y}$$

is concave for $x, y > 0$.

8*. [**5 Points**] Let

$$f(x,y) = (\ln x)^a (\ln y)^b,$$

defined for $x > 1$, $y > 1$. Assume that $a > 0$, $b > 0$ and $a + b < 1$. Show that f is strictly concave.

9. [**5 Points**] Consider the CES production functions with $a, b > 0$

$$f(x,y) := (ax^p + by^p)^{1/p}$$

on the domain $U := \{(x,y) \mid x > 0, y > 0\} \subset \mathbb{R}^2$. For which $p > 0$ is this function convex / concave?