QE "Optimization", WS 2017/18

Problem Set No. 13

Solutions do not need to be submitted.
The problems will be discussed in the lecture

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1. [**7 Points**] Using the Karush-Kuhn-Tucker Theorem, solve the inequality constraint optimization problem in $\mathbb{R}^2$

$$\begin{array}{rl} \max & f(x,y) = xy \\ \text{subject to} & x^2 + y^2 \leq 1. \end{array}$$

(i) Explain why the max problem is solvable;
(ii) Show that the constrained qualification fails only at $x = y = 0$;
(iii) Form the Lagrangean and write the Karush-Kuhn-Tucker conditions [KKT-1], [KKT-2];
(iv) Find the next two candidates ($x = y = 1/\sqrt{2}$, $x = y = -1/\sqrt{2}$) for a solution of the constrained optimization problem;
(v) Compare the values of $f(x,y)$ at all these 3 points and choose the global max among them.

2. [**8 Points**] Find the maximum and minimum values of $f(x,y) = ax + by$ subject to $x^2 + y^2 \leq 1$.
(i) Consider separately the case $a = b = 0$;
(ii) Assume that either $a \neq 0$ or $b \neq 0$, apply the KKT method;
(iii) What sufficient conditions can you apply in this case?
Answer: $\pm\sqrt{a^2 + b^2}$, $\lambda = \pm\frac{1}{2}\sqrt{a^2 + b^2}$, $x = a/(2\lambda)$, $y = b/(2\lambda)$. There are no critical points in the interior, and there are exactly two critical points on the boundary.

3. [**10 Points**] Robinson Crusoe lives on an island where he produces two goods, x and y, according to the production possibility frontier

$$x^2 + y^2 \leq 400,$$

and he consumes all the goods himself. Robinson also faces an environmental constraint on his total output of both goods. Consider the problem

$$x + y \leq 28.$$

His utility function is

$$u(x, y) = x^{1/2}y^{1/2}.$$

(i) Write out the necessary KKT conditions for a point (x, y) to solve the max problem;
(ii) Find all points that satisfy these conditions. Identify which constraints are binding and check the Constraint Qualification;
(iii) Do any of those points solve the max problem? What sufficient conditions can you apply in this case?