OQE - PROBLEM SET 11 - SOLUTIONS

Exercise 1. Let $\{U_i\}_{i\in I}$ be a collection of convex sets. We claim that the set $U = \bigcap_{i\in I} U_i$ is convex. To prove so, let $x, y \in U$ and let $\lambda \in [0,1]$. For each $i \in I$, the elements x, y belong to U_i and so, by the convexity of U_i, also $\lambda x + (1-\lambda)y$ belongs to U_i. As a consequence, $\lambda x + (1-\lambda)y$ belongs to $\bigcap_{i\in I} U_i = U$ and, the choice of x, y, λ being arbitrary, it follows that U is convex.

Exercise 2. We determine whether the following sets are convex:

(a) $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \le x_2 \le x_3\}$;
(b) $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le 1\}$;
(c) $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le 1 \text{ or } x_1^2 + x_3^2 \le 1\}$;
(d) $D = \{(x_i)_{i=1}^n \in \mathbb{R}^n : |\sum_{i=1}^n x_i| \le 1\}$;
(e) $E = \{(x_i)_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 = 1\}$.

(a) We claim that A is convex. Let indeed $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in A and let moreover $\lambda \in [0,1]$. Since both λ and $1-\lambda$ are non-negative, we have

$$\lambda x_1 + (1-\lambda)y_1 \le \lambda x_2 + (1-\lambda)y_2 \le \lambda x_3 + (1-\lambda)y_3$$

and thus $\lambda x + (1-\lambda)y$ belongs to A.
(b) We claim that B is convex. To show this, we take the elements $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in B and $\lambda \in [0,1]$. We claim that $\lambda x + (1-\lambda)y$ lives in B. We recall that, for each $a, b \in \mathbb{R}$, one has $(a-b)^2 \ge 0$ and thus $ab \le (a^2+b^2)/2$. In view of this, we write $\mu = 1-\lambda$ and compute

$$\begin{aligned}
(\lambda x_1 + \mu y_1)^2 + (\lambda x_2 + \mu y_2)^2 &= \lambda^2(x_1^2 + x_2^2) + 2\lambda\mu(x_1y_1 + x_2y_2) + \mu^2(y_1^2 + y_2^2) \\
&\le \lambda^2 + 2\lambda\mu\frac{(x_1^2 + y_1^2 + x_2^2 + y_2^2)}{2} + \mu^2 \\
&\le \lambda^2 + 2\lambda\mu + \mu^2 \\
&= (\lambda + \mu)^2 = 1.
\end{aligned}$$

(c) We claim that C is not convex. We define $x = (0, -1, 5)$ and $y = (0, -2, -1)$, which are clearly elements of C. Set now $\lambda = 1/2$ and write $z = (x+y)/2$. Then we have

$$z_1^2 + z_2^2 = \Big(\frac{-1-2}{2}\Big)^2 = \frac{9}{4} > 1$$

and also

$$z_1^2 + z_3^2 = \Big(\frac{5-1}{2}\Big)^2 = 4 > 1$$

so C is not convex and the claim is proven.
(d) We claim that D is convex. Let indeed $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ be elemets of D and let moreover $\lambda \in [0,1]$. Set $\mu = 1 - \lambda$. Then one has

$$\Big|\sum_{i=1}^n (\lambda x_i + \mu y_i)\Big| \leq \Big|\sum_{i=1}^n \lambda x_i\Big| + \Big|\sum_{i=1}^n \mu y_i\Big| = \lambda\Big|\sum_{i=1}^n x_i\Big| + \mu\Big|\sum_{i=1}^n y_i\Big| \leq \lambda + \mu = 1$$

and therefore $\lambda x + \mu y$ belongs to D. The choice of x, y, λ being abitrary, D is convex.
(e) We claim that E is not convex. Indeed, for each $x \in E$ also $-x$ belongs to E, but $\frac{1}{2}x + \frac{1}{2}(-x) = 0$ does not belong to E.

Exercise 3. For each $f : \mathbb{R}^n \to \mathbb{R}$ and for each $x, y \in \mathbb{R}^n$, define $\phi_{f,x,y} : [0,1] \to \mathbb{R}$ by

$$\lambda \mapsto \phi_{f,x,y}(\lambda) = f(\lambda x + (1-\lambda)y).$$

Fix now $f : \mathbb{R}^n \to \mathbb{R}$. We claim that the following are equivalent:

(a) the function f is convex;
(b) for each $x, y \in \mathbb{R}^n$, the function $\phi_{f,x,y}$ is convex.

Let $\lambda_1, \lambda_2, \delta \in [0,1]$ and define $\mu_i = 1 - \lambda_i$ and $\epsilon = 1 - \delta$. Fix moreover $x, y \in \mathbb{R}^n$ and, to light the notation, write $\phi = \phi_{f,x,y}$. Then we have

(1) $\phi(\delta\lambda_1 + \epsilon\lambda_2) = f(\delta(\lambda_1 x + \mu_1 y) + \epsilon(\lambda_2 x + \mu_2 y))$; and
(2) $\delta\phi(\lambda_1) + \epsilon\phi(\lambda_2) = \delta f(\lambda_1 x + \mu_1 y) + \epsilon f(\lambda_2 x + \mu_2 y)$.

The implication $(a) \Rightarrow (b)$ is clear from (1) and (2). To prove $(b) \Rightarrow (a)$, let $\lambda \in [0,1]$. Setting $\delta = \lambda$ and $(\lambda_1, \lambda_2) = (1, 0)$, it follows from (1) and (2) that, if ϕ is convex, then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

All choices involved being arbitrary, the claim is proven. An analogue statement involving concavity can be proven in a similar way.

Exercise 4. We prove that the Euclidean distance $d : \mathbb{R}^{2n} \to \mathbb{R}$, defined by

$$x = (x_i)_{i=1}^{2n} \mapsto d(x) = \|(x_i)_{i=1}^n - (x_i)_{i=n+1}^{2n}\|$$

is a convex function. To this end, let $x = (x_i)_{i=1}^{2n}$ and $y = (y_i)_{i=1}^{2n}$ be elements of $\mathbb{R}^{2n}$ and let $\lambda \in [0,1]$. Set moreover $\mu = 1 - \lambda$. Then we have

$$\begin{aligned} d(\lambda x + \mu y) &= \|(\lambda x_i + \mu y_i)_{i=1}^n - (\lambda x_i + \mu y_i)_{i=n+1}^{2n}\| \\ &= \|\lambda(x_i - x_{i+n})_{i=1}^n + \mu(y_i - y_{i+n})_{i=1}^n\| \\ &\leq \lambda\|(x_i - x_{i+n})_{i=1}^n\| + \mu\|(y_i - y_{i+n})_{i=1}^n\| \\ &= \lambda d(x) + \mu d(y). \end{aligned}$$

The choices of x, y, λ being arbitrary, the function d is convex.

Exercise 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function satisfying $f(0) = 0$. We will show that, for each $x \in \mathbb{R}^n$, $\lambda \in [0, 1]$, and $k \in \mathbb{R}_{\geq 1}$, one has

(a) $f(\lambda x) \geq \lambda f(x)$; and
(b) $kf(x) \geq f(kx)$.

Fix $x \in \mathbb{R}^n$, $\lambda \in [0, 1]$, and $k \in \mathbb{R}_{\geq 1}$. Since $-f$ is convex, we have

$$-f(\lambda x) = -f(\lambda x + (1 - \lambda)0) \leq -\lambda f(x) - (1 - \lambda)f(0) = -\lambda f(x)$$

and so (a) is proven. To prove (b), we note that $1/k \in [0, 1]$ and so, applying (a), we get

$$kf(x) = kf\Big(\frac{1}{k}kx\Big) \geq k\frac{1}{k}f(kx) = f(kx).$$

Exercise 6. Let $U = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and let $f : U \to \mathbb{R}$ be defined by

$$(x, y) \mapsto f(x, y) = \frac{x^2}{y}.$$

We claim that f is convex. In order to apply Theorem 3.1.4, we compute

$$D^2 f(x, y) = \begin{bmatrix} 2/y & -2x/y^2 \\ -2x/y^2 & 2x^2/y^3 \end{bmatrix}$$

and therefore we have

$$\det D^2 f(x, y) = \frac{4x^2}{y^4} - \frac{4x^2}{y^4} = 0.$$

However, the elements $2/y$ and $2x^2/y^3$ are both non-negative and so $D^2 f$ is positive semi-definite. It follows that f is convex.

Exercise 7. Let $U = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ and define $f : U \to \mathbb{R}$ by

$$(x, y) \mapsto f(x, y) = \frac{1}{2}e^{-(x+y)} - e^{-x} - e^{y}.$$

We claim that f is concave. To show this, we compute

$$D^2 f(x, y) = \begin{bmatrix} e^{-(x+y)}/2 - e^{-x} & e^{-(x+y)}/2 \\ e^{-(x+y)}/2 & e^{-(x+y)}/2 - e^{y} \end{bmatrix}$$

and therefore

$$\begin{aligned} \det D^2 f(x, y) &= \Big(\frac{e^{-(x+y)}}{2} - e^{-x}\Big)\Big(\frac{e^{-(x+y)}}{2} - e^{y}\Big) - \frac{e^{-2(x+y)}}{4} \\ &= -\frac{e^{-x}}{2} - \frac{e^{-2x}e^{-y}}{2} + e^{-x}e^{y} \\ &= \frac{e^{-x}}{2}(2e^{y} - e^{-x}e^{-y} - 1). \end{aligned}$$

If we take $(x, y) \in U$, then both x and y are positive and therefore $e^y > 1$ and $e^{-x}e^{-y} < 1$ and thus $\det D^2 f(x, y) > 0$. Moreover, we have

$$\frac{e^{-(x+y)}}{2} - e^{-x} = \frac{e^{-x}}{2}(e^{-y} - 2) < 0$$

and therefore $D^2 f$ is negative definite. It follows that f is concave.

Exercise 8. Let $U = \{(x, y) \in \mathbb{R}^2 : x, y > 1\}$ and a, b be positive real numbers satisfying $a + b < 1$. Let moreover $f : U \to \mathbb{R}$ be defined by

$$(x, y) \mapsto f(x, y) = (\log x)^a (\log y)^b.$$

We claim that f is strictly concave. We will use the following fact (stronger version of Lemma 3.3.1): *If $g : U \to \mathbb{R}$ is strictly concave and $l : \mathbb{R} \to \mathbb{R}$ is increasing and concave, then $l \circ g$ is strictly concave.* We define

(1) $g : U \to \mathbb{R}$ by $(x, y) \mapsto g(x, y) = \log(f(x, y))$;
(2) $l_1 : \mathbb{R} \to \mathbb{R}$ by $x \mapsto l_1(x) = -e^{-x}$ (concave increasing);
(3) $l_2 : \mathbb{R}_{<0} \to \mathbb{R}$ by $-1/x$ (concave increasing).

We observe that, for each $(x, y) \in U$, one has

$$l_2 l_1 g(x, y) = -\frac{1}{-e^{-\log(f(x,y))}} = e^{\log(f(x,y))} = f(x, y)$$

and thus to prove that $f(x, y)$ is concave, it suffices to prove that g is strictly concave. For each $(x, y) \in U$, the element $f(x, y)$ is positive (so g is well-defined) and moreover we have

$$g(x, y) = \log(f(x, y)) = \log((\log x)^a (\log y)^b) = a \log \log x + b \log \log y.$$

For each $(x, y) \in U$, we compute

$$\nabla g(x, y) = \Big(\frac{a}{x \log x}, \frac{b}{x \log x}\Big)$$

and therefore we have

$$D^2 g(x, y) = \begin{bmatrix} -a(1 + \log x)/(x \log x)^2 & 0 \\ 0 & -b(1 + \log y)/(y \log y)^2 \end{bmatrix}.$$

The matrix $D^2 g(x, y)$ being negative definite for any choice of $(x, y) \in U$, it follows that g is strictly concave.

Exercise 9. Let $U = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$ and let $a, b, p \in \mathbb{R}_{>0}$. Define $f : U \to \mathbb{R}$ by

$$(x, y) \mapsto f(x, y) = (ax^p + by^p)^{1/p}.$$

We claim that

(a) f is convex for $p \geq 1$;
(b) f is concave for $p \leq 1$.

We separate the two cases.
(a) Assume that $p \geq 1$. We will use the following fact (which is very easy to prove):

Fact 1. *Let V be a convex subset of $\mathbb{R}^n$ and let $\phi : V \to \mathbb{R}$ be a function. Let moreover $\delta : V \to V$ be a linear bijection (as defined in Section 1.10). Then ϕ is convex if and only if $\phi \circ \delta$ is convex.*

Define the map $\delta : U \to U$ by $(x, y) \mapsto (a^{1/p}x, b^{1/p}y)$. Then $f = \| \cdot \|_p \circ \delta$. Since norms are convex functions and δ is a linear bijection, Fact 1 yields that f is convex.
(b) Assume now that $p \leq 1$ and let $g : U \to \mathbb{R}$ be defined, for each $(x, y) \in U$, by $g(x, y) = ax^p + by^p$. Then, for each $(x, y) \in U$, we have

$$D^2 g(x, y) = \begin{bmatrix} p(p-1)x^{p-2} & 0 \\ 0 & p(p-1)y^{p-2} \end{bmatrix}$$

and so $D^2 g(x, y)$ is negative semi-definite for any choice of $(x, y) \in U$. It follows that g is concave. Using the same trick from Exercise 8, we get, for any choice of $(x, y) \in U$, that

$$-\frac{1}{-e^{-\log(g(x,y))/p}} = \frac{1}{e^{-\log(g(x,y))/p}} = e^{\log(g(x,y))/p} = e^{\log(f(x,y))} = f(x, y)$$

and therefore f is concave.