OQE - PROBLEM SET 13 - SOLUTIONS

Exercise 1. We look at the inequality constraint optimization problem in $\mathbb{R}^2$

$$\max f(x,y) = xy$$
$$\text{subject to } x^2 + y^2 \leq 1.$$

(a) Define $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and observe that D is closed and bounded, hence compact. The function f being continuous, it achieves a global maximum on D.

(b) We define $g : \mathbb{R}^2 \to \mathbb{R}$ by $(x,y) \mapsto g(x,y) = x^2 + y^2 - 1$. We consequently compute, for each $(x,y) \in \mathbb{R}^2$, the gradient of g to be $\nabla g(x,y) = (2x, 2y)$ which is equal to $(0,0)$ if and only if $(x,y) = (0,0)$. However, since the constraint is not active at $(0,0)$, this gives us no problems.

(c) For $(x,y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}_{\geq 0}$, we define the Lagrangian function

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = xy - \lambda(x^2 + y^2 - 1)$$

and consequently we impose

(KKT-1)
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) = y - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) = x - 2\lambda y = 0 \end{cases}$$

and also

(KKT-2)
$$\begin{cases} x^2 + y^2 - 1 = 0 \text{ if } \lambda > 0 \\ x^2 + y^2 - 1 < 0 \text{ if } \lambda = 0 \end{cases} .$$

(d) Solving the systems in (c), one finds the three points

(1) $(0,0)$;
(2) $(\sqrt{2}/2, \sqrt{2}/2)$;
(3) $(-\sqrt{2}/2, -\sqrt{2}/2)$.

(e) We compute

(1) $f(0,0) = 0$;
(2) $f(\sqrt{2}/2, \sqrt{2}/2) = 1/2$;
(3) $f(-\sqrt{2}/2, -\sqrt{2}/2) = 1/2$.

The global maximum on D has thus value $1/2$ and is achieved in $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$.

Exercise 2. Let $a, b \in \mathbb{R}$ and consider the inequality constraint optimization problem on $\mathbb{R}^2$ given by

$$\max / \min \; f(x,y) = ax + by$$
$$\text{subject to } x^2 + y^2 \leq 1.$$

(a) Assume that $a = b = 0$. Then the function f is equal to the constant funciton 0 and each element of $\mathbb{R}^2$ is a maximum/minimum of f in $\mathbb{R}^2$ and therefore the same applies to every element of $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Suppose, from now on, that $a \neq 0$ or $b \neq 0$.

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x,y) \mapsto^2 + y^2 - 1$ and let D be as in (a). To find the maxima of f, we define, for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{R}_{\geq 0}$, the Lagrangian function

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = ax + by - \lambda(x^2 + y^2 - 1)$$

and we impose

$$\text{(KKT-1)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) = a - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) = b - 2\lambda y = 0 \end{cases}$$

and also

$$\text{(KKT-2)} \qquad \begin{cases} x^2 + y^2 - 1 = 0 \text{ if } \lambda > 0 \\ x^2 + y^2 - 1 < 0 \text{ if } \lambda = 0 \end{cases} .$$

Solving the systems, we get two points:

(1) $(0,0)$ – corresponds to $\lambda = 0$ and refers back to (a); and
(2) $P = (a/(\sqrt{a^2+b^2}), b/(\sqrt{a^2+b^2}))$ with $f(P) = \sqrt{a^2+b^2}$ and corresponding to $\lambda_P = \sqrt{a^2+b^2}/2$

so P is the point in D in which f achieves its global maximum. To compute the minima of f in D, one applies the same method to $-f$, finding the minimum point $Q = (-a/(\sqrt{a^2+b^2}), -b/(\sqrt{a^2+b^2}))$ with $f(Q) = -\sqrt{a^2+b^2}$ and corresponding to $\lambda_Q = -\sqrt{a^2+b^2}/2$.

(c) As $\lambda_P, \lambda_Q > 0$, one can reduce to an EC problem and look at the functions

i. $\mathcal{L}_P(x,y) = f(x,y) - \lambda_P g(x,y) = ax + by - \sqrt{a^2+b^2}(x^2+y^2-1)/2$;
ii. $\mathcal{L}_Q(x,y) = f(x,y) - \lambda_P g(x,y) = ax + by + \sqrt{a^2+b^2}(x^2+y^2-1)/2$.

The associated Hessians are then

$$D^2\mathcal{L}_P(x,y) = \begin{bmatrix} -2\lambda_P & 0 \\ 0 & -2\lambda_P \end{bmatrix} = \begin{bmatrix} -\sqrt{a^2+b^2} & 0 \\ 0 & -\sqrt{a^2+b^2} \end{bmatrix}$$

and

$$D^2\mathcal{L}_Q(x,y) = \begin{bmatrix} -2\lambda_Q & 0 \\ 0 & -2\lambda_Q \end{bmatrix} = \begin{bmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{bmatrix}.$$

Theorem 4.3.1 yields that P is a global maximum and Q is a global minimum of f in D.

Exercise 3. Let $U = \{(x,y) \in \mathbb{R}^2_{\geq 0}\}$ and consider the inequality constraint problem on U

$$\max\ u(x,y) = x^{1/2}y^{1/2}$$
$$\text{subject to } x^2+y^2 \leq 400,\ x+y \leq 28.$$

(a–b) Define $g_1 : \mathbb{R}^2 \to \mathbb{R}$ by $(x,y) \mapsto g_1(x,y) = x^2+y^2-400$ and let also $g_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x,y) \mapsto g_2(x,y) = x+y-28$. We compute the Jacobian associated to $g = (g_1, g_2)$ at any point $(x,y) \in \mathbb{R}^2$:

$$Dg(x,y) = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix}$$

and note that the system

$$\begin{cases} 2x - 2y = 0 \\ x^2+y^2 = 400 \\ x+y = 28 \end{cases}$$

has no solution, so CQ is satisfied whenever both g_1 and g_2 are active constraint at a point (x,y). In all other cases CQ is trivially satisfied. We now observe that, if $(x,0),(0,y) \in U$, then $f(x,0) = f(0,y) = 0$ and, since u achieves only non-negative values and we are concerned with finding the maximum value of f, we can restrict our investigation to the domain $U^* = \{(x,y) \in U : xy \neq 0\}$. We define, for $(x,y) \in U^*$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$, the Lagrangian function

$$\begin{aligned} \mathcal{L}(x,y,\lambda,\mu) &= u(x,y) - \lambda g_1(x,y) - \mu g_2(x,y) \\ &= x^{1/2}y^{1/2} - \lambda(x^2+y^2-400) - \mu(x+y-28) \end{aligned}$$

and we impose consequently

$$\text{(KKT-1)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda,\mu) = \frac{1}{2}x^{-1/2}y^{1/2} - 2\lambda x - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda,\mu) = \frac{1}{2}x^{1/2}y^{-1/2} - 2\lambda y - \mu = 0 \end{cases}$$

and also

$$\text{(KKT-2)} \qquad \begin{cases} x^2+y^2-400 = 0 \text{ if } \lambda > 0 \\ x^2+y^2-400 < 0 \text{ if } \lambda = 0 \\ x+y-28 = 0 \text{ if } \mu > 0 \\ x+y-28 < 0 \text{ if } \mu = 0 \end{cases}.$$

From (KKT-1) we derive

$$\frac{1}{2}x^{-1/2}y^{1/2} - 2\lambda x = \frac{1}{2}x^{1/2}y^{-1/2} - 2\lambda y$$

which is satisfied if and only if

$$x^{-1/2}y^{-1/2}(y-x) = 4\lambda(x-y).$$

The latter is satisfied if and only if

$$x = y \quad \text{or} \quad \lambda = -\frac{1}{4}x^{-1/2}y^{-1/2}.$$

However, for any choice of $(x, y) \in U^*$, the value $-\frac{1}{4}x^{-1/2}y^{-1/2}$ is negative and therefore we must have $x = y$. Thanks to (KKT-2), we then find the point $(14, 14)$ for which we have $f(14, 14) = 14$.
(c) For $i \in \{1, 2\}$, define $D_i = \{(x, y) \in \mathbb{R}^2 : g_1(x, u)\}$. Then D_1 is bounded and D_1 and D_2 are both closed. Since U is also closed, it follows that $D_1 \cap D_2 \cap U$ is closed and bounded, thus compact. The function u being continuous, it achieves both a maximum and a minimum on $D_1 \cap D_2 \cap U$ and therefore $(14, 14)$ is a global maximum of f on the given domain.