OQE - PROBLEM SET 3 - SOLUTIONS

Exercise 1. Let (X, d) be a metric space and let $x \in X$ and $\epsilon > 0$.
(a) We show that the closed ball $C = \overline{\mathrm{B}_\epsilon(x)}$ is closed in X, by showing that $X \setminus C$ is open. Let y be an element in $X \setminus C$ and define $\delta = d(x, y)$. Since y does not belong to C, the number $\nu = \frac{\delta - \epsilon}{2}$ is positive. We claim that $\mathrm{B}_\nu(y)$ is contained in $X \setminus C$. Let $z \in \mathrm{B}_\nu(y)$. Then

$$d(z, x) \geq |d(x, y) - d(y, z)| \geq \delta - \frac{\delta - \epsilon}{2} = \frac{\delta + \epsilon}{2} > \frac{\epsilon + \epsilon}{2} = \epsilon.$$

The element z being arbitrary, we have proven that $\mathrm{B}_\nu(y)$ has trivial intersection with C, which gives us the claim. As the same argument applies for each element of $X \setminus C$, we have proven that, for all $y \in X \setminus C$, there exists $\nu > 0$ such that $\mathrm{B}_\nu(y)$ is contained in $X \setminus C$. As a consequence, $X \setminus C$ is open.
(b) Let $B = \mathrm{B}_\epsilon(x)$ denote the open ball with centre x and radius ϵ. We prove that the closure $\overline{B}$ of B is contained in C, as defined in (a). By Theorem 1.3.6 from the notes, $\overline{B}$ is the smallest closed subset of X that contains B. Since C contains B and, by (a), the subset C is closed, it follows that $\overline{B} \subseteq C$.
(c) To prove that in general the closure of an open ball is not equal to the corresponding closed ball, take $X = \mathbb{R}^2$ and $d = d_T$, the trivial metric. Take x to be any point of X and take $\epsilon = 1$. Then $B = \{x\} = \overline{B}$, but $C = X$.

Exercise 2. Let (X, d) be a metric space and, for each $x \in X$ and $\emptyset \neq A \subseteq X$, define $d(x, A) = \inf\{d(x, a) : a \in A\}$.
(a) Let $x, y \in X$ and let $\emptyset \neq A \subseteq X$. We show that

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

We work by contradiction assuming that $|d(x, A) - d(y, A)| > d(x, y)$. Without loss of generality, $d(x, A) > d(y, A)$ and so $|d(x, A) - d(y, A)| = d(x, A) - d(y, A)$. Since

$$\inf\{d(x, a) : a \in A\} - d(y, A) = d(x, A) - d(y, A) > d(x, y),$$

we have that, for all $a \in A$, one has $d(x, a) - d(y, A) > d(x, y)$ and there exists therefore $b \in A$ such that, for all $a \in A$, one has $d(x, a) - d(y, b) > d(x, y)$. Fix now such b. Then it follows from the inverse triangle inequality that

$$d(x, y) < d(x, b) - d(y, b) \leq d(x, y)$$

which is a contradiction.
(b) Let A be a non-empty subset of X. We show that the map $f : X \to \mathbb{R}_{>0}$, defined by $x \mapsto f(x) = d(x, A)$, is continuous. We will show that, for each $x \in X$

and for each $\epsilon > 0$, there exists $\delta > 0$ such that $f(\mathrm{B}_\delta(x)) \subseteq \mathrm{B}_\epsilon(f(x))$. Fix $x \in X$ and $\epsilon > 0$. Set $\delta = \epsilon$. Then, for each $y \in \mathrm{B}_\delta(x)$, thanks to (a), one has

$$|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y) < \delta = \epsilon$$

and therefore $f(y) \in \mathrm{B}_\epsilon(f(x))$. Both x and y being arbitrary, we have proven that f is continuous.

Exercise 3. Let (X, d) be a metric space and let A be a closed subset of X. Let $x \in X \setminus A$. We show that $d(x, A) > 0$. Assume by contradiction that $d(x, A) = 0$. As a consequence, for each $n \in \mathbb{N}$, there exists $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. Then $(a_n)_{n\in\mathbb{N}}$ is a sequence in A which conerges to x. But x does not belong to A, which is a closed subset of X. Contradiction to Theorem 1.3.8.

Exercise 4. Let (X, d) and (Y, ρ) be metric spaces and assume that $d = d_T$ is the discrete metric. We show that any function $X \to Y$ is continuous. To this end, let $f : X \to Y$ be a function. By taking balls of radius $1/2$, one shows that, for each $x \in X$, the set $\{x\}$ is open in X. Let now U be an open subset of Y. Then $f^{-1}(U) = \bigcup_{x\in f^{-1}(U)}\{x\}$ and therefore, thanks to Theorem 1.3.1, the set $f^{-1}(U)$ is open in X. The choice of U being arbitrary, it follows from Theorem 1.4.6 that f is continuous.

Exercise 5. None. See Exercise 4 from Problem Set 2.

Exercise 6. Let (X, d) be a metric space and let $x_0 \in X$. We show that the function $f : X \to \mathbb{R}$, defined by $x \mapsto f(x) = d(x, x_0)$ is uniformly continuous. We have to show that, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each $x, y \in X$ with $d(x, y) < \delta$, one has $|f(x) - f(y)| < \epsilon$. Fix $\epsilon > 0$. We set $\delta = \epsilon$. Then, for each $x, y \in X$ with $d(x, y) < \delta$, the inverse triangle inequallity yields

$$|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y) < \delta = \epsilon.$$

The choice of ϵ being arbitrary, we are done.

Exercise 7. We show that the map $f : (0, 1) \to \mathbb{R}$, defined by $x \mapsto f(x) = \frac{1}{1-x}$, is not uniformly continuous. To this end, take $\epsilon = 1$. We will show that, for each $\delta > 0$, there exist $x, y \in X$ with $|x - y| < \delta$ such that $|f(x) - f(y)| \geq 1$. Fix $\delta > 0$ and define $x = \max\{\frac{1-\delta}{3}, \frac{2}{3}\}$ and $y = \frac{1-x}{2}$. Then we have

$$|x - y| = \left|x - \frac{1-x}{2}\right| = \frac{1}{2}|3x - 1| < \delta$$

but we also have

$$|f(x) - f(y)| = \left|\frac{1}{1-x} - \frac{1}{1 - \frac{1-x}{2}}\right| = \left|\frac{1}{1-x} - \frac{2}{1+x}\right| = \left|\frac{3x-1}{1-x^2}\right|.$$

Since $x \in (0,1)$, the element x^2 is smaller than 1 and so $\frac{1}{1-x^2} > 1$. It follows that

$$|f(x) - f(y)| > |3x - 1| = |3 \max\left\{\frac{1-\delta}{3}, \frac{2}{3}\right\} - 1| = \max\{\delta, 1\} \geq 1.$$

Exercise 8. Respecting the notation from Theorem 1.6.3, we show that, for each $n \in \mathbb{N}$, one has

$$d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_1, x_0).$$

We work by induction on n and we first show that $d(x_1, x^*) \leq \frac{\beta}{1-\beta} d(x_1, x_0)$. The map T being a contraction and using the triangle inequality, we have

$$d(x_1, x^*) = d(Tx_0, Tx^*) \leq \beta d(x_0, x^*) \leq \beta d(x_0, x_1) + \beta d(x_1, x^*)$$

and so, as a consequence, we derive

$$(1-\beta) d(x_1, x^*) \leq \beta(x_0, x_1)$$

giving the base case. Assume now that $n > 1$ and that

$$d(x_{n-1}, x^*) \leq \frac{\beta^{n-1}}{1-\beta} d(x_1, x_0).$$

Then we have

$$d(x_n, x^*) = d(Tx_{n-1}, Tx^*) \leq \beta d(x_{n-1}, x^*) \leq \beta \frac{\beta^{n-1}}{1-\beta} d(x_1, x_0) = \frac{\beta^n}{1-\beta} d(x_1, x_0)$$

and the proof is complete.

Exercise 9. Let $F : [1,2] \to \mathbb{R}$ be defined by $x \mapsto F(x) = \frac{x^2+2}{2x}$. We show that F is a contraction of modulus $\frac{1}{2}$. Indeed, for each $x, y \in [1,2]$, one has

$$|F(x) - F(y)| = \left|\frac{x^2+2}{2x} - \frac{y^2+2}{2y}\right| = \frac{1}{2}\left|\frac{xy-2}{xy}(x-y)\right| = \frac{|x-y|}{2}\left|1 - \frac{2}{xy}\right|.$$

As x, y range between 1 and 2, we have that $\left|1 - \frac{2}{xy}\right| \leq 1$, from which it follows that $|F(x) - F(y)| \leq \frac{|x-y|}{2}$. We have proven that F is a contraction of modulus $\frac{1}{2}$. Thanks to Banach's contraction theorem, we now know that F has a unique fixed point x^*, which we compute by solving the equation $x = \frac{x^2+2}{2x}$. In the end, one gets $2x^2 = x^2 + 2$ which gives $x^* = \sqrt{2}$.