OQE - PROBLEM SET 4 - SOLUTIONS

Exercise 1. We determine whether the following sequences converge.
(a) Let $X = l_1$ and let $y_n := \left(\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, 0, 0, \ldots\right)$. We claim that $(y_n)_n$ coverges to $y = (\frac{1}{2^k})_{k\in\mathbb{N}}$, which is indeed an element of X since

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

Moreover, for each $n \in \mathbb{N}$, one has $\|y - y_n\|_1 = \sum_{k=n+1}^{\infty} \frac{1}{2^k}$, which converges to zero, as the geometric series is convergent.
(b) Let $X = l_1$ and $y_n = \left(\frac{n+1}{n^2}, \frac{n+2}{n^2}, \ldots, \frac{2n}{n^2}, 0, 0, \ldots\right)$. We claim that $(y_n)_n$ does not converge in X and we will prove so by contradiction. Let $x = (x_k)_{k\in\mathbb{N}}$ be the limit of $(y_n)_n$. Then, by definition of convergence, one has that $\lim_{n\to\infty} \|y_n - x\|_1 = 0$. However, for each $n \in \mathbb{N}$, one has

$$\|y_n - x\|_1 = \sum_{k=1}^{\infty} |y_{n,k} - x_k|$$

and so, for each $k \in \mathbb{N}$, one has $\lim_{n\to\infty} |y_{n,k} - x_k| = 0$. From the definitions of the y_n's, one derives that, for all k, the element x_k is equal to 0. It follows therefore that $\lim_{n\to\infty} \|y_n\|_1 = 0$, but

$$\|y_n\|_1 = \sum_{k=1}^{n} \frac{n+k}{n^2} = \frac{1}{n} + \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n} + \frac{n(n+1)}{2n^2} = \frac{n+3}{2n} = \frac{1}{2} + \frac{3}{n}$$

converges to $\frac{1}{2}$. Contradiction.

(c) Let $X = l_1$ and let $y_n = \left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ times}}, 0, 0, \ldots\right)$. We claim that $(y_n)_n$ is not convergent. As in the previous point, one could show that, if $(y_n)_n$ had a limit, then it would have to be the zero sequence. However, for each $n \in \mathbb{N}$, one has $\|y_n\|_1 = 1$ and so $\|y_n\|_1$ does not tend to zero as $n \to \infty$.
(d) Let $X = l_1$ and $y_n = \left(\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, \frac{1}{n^\sigma}, \frac{1}{(n+1)^\sigma}, \ldots\right)$, with $\sigma > 1$. We claim that $(y_n)_n$ converges to the zero sequence $(0)_{k\in\mathbb{N}}$. Indeed, we compute

$$\lim_{n\to\infty} \|y_n - (0)_{k\in\mathbb{N}}\|_1 = \lim_{n\to\infty} \|y_n\|_1 = \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{k^\sigma} = 0.$$

(e) Let $X = l_2$, $y_n := \Bigg(\underbrace{\frac{1}{n}, 0, \dots, 0, 1}_{n}, 0, 0, \dots\Bigg)$. We claim that $(y_n)_n$ is not Cauchy and therefore not convergent. Let $n \in \mathbb{N}$. Then one computes

$$\|y_n - y_{n+1}\|_2 = \|\Big(\underbrace{\frac{1}{n} - \frac{1}{n+1}, 0, \dots, 0, 1}_{n}, -1, 0, 0, \dots\Big)\|_2 =$$

$$= \Big(\Big(\frac{1}{n(n+1)}\Big)^2 + 2\Big)^{1/2} \longrightarrow \sqrt{2} > 0.$$

(f) Let $X = l_2$ and $y_n = \Bigg(\underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n^2 \text{ times}}, 0, 0, \dots\Bigg)$. We claim that $(y_n)_n$ is not convergent. Assume indeed by contradiction that $(y_n)_n$ has a limit point $x = (x_k)_{k\in\mathbb{N}}$. One then has that

$$0 = \lim_{n\to\infty} \|y_n - x\|_2 = \lim_{n\to\infty} \Big(\sum_{k=1}^{n^2} \Big|\frac{1}{n} - x_k\Big|^2 + \sum_{k=n^2}^{\infty} |x_k|^2\Big)^{1/2}$$

and so, as a consequence of Theorem 1.4.4 from the notes, one has

$$0 = \lim_{n\to\infty} \sum_{k=1}^{n^2} \Big|\frac{1}{n} - x_k\Big|^2 + \lim_{n\to\infty} \sum_{k=n^2}^{\infty} |x_k|^2 = \sum_{k=1}^{\infty} |x_k|^2 = \|x\|_2$$

and so x equals the zero sequence $(0)_{k\in\mathbb{N}}$. However,

$$\lim_{n\to\infty} \|y_n\|_2 = \Big(\sum_{k=1}^{\infty} \frac{1}{n^2}\Big)^{1/2} = \frac{\pi}{\sqrt{6}} > 0.$$

(g) Let $X = l_3$ and define $y_n = \big(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\big)$. We claim that $(y_n)_n$ converges to the sequence $x = (x_k)_{k\in\mathbb{N}}$ that is defined by $x_k = \frac{1}{k}$. Indeed, one has

$$\lim_{n\to\infty} \|y_n - x\|_3 = \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \Big|\frac{1}{k} - \frac{1}{k}\Big|^3 + \sum_{k=n+1}^{\infty} \Big|\frac{1}{k}\Big|^3\Big)^{1/3} = \lim_{n\to\infty} \Big(\sum_{k=n+1}^{\infty} \Big|\frac{1}{k}\Big|^3\Big)^{1/3}$$

and so, as a consequence of Theorem 1.4.4, we have $\lim_{n\to\infty} \|y_n - x\| = 0$. Moreover, x is indeed an element of X because

$$\|x\|_3 = \sum_{k=1}^{\infty} \frac{1}{k^3} \sim 1.202$$

Exercise 2. Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that is Cauchy. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_n$ and let $x \in X$. Assume that $(x_{n_k})_k$ converges to x. We will show that $(x_n)_n$ converges to x. To this end, we will show that , for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that, for each $n > N_\epsilon$, one has $d(x_n, x) < \epsilon$. Fix $\epsilon > 0$. The subsequence $(x_{n_k})_k$ being convergent, there exists $N_s \in \mathbb{N}$ such that, for each $n_k > N_s$, one has $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Moreover, since $(x_n)_n$ is Cauchy, there exists $N_c \in \mathbb{N}$ such that, for each $n, m > N_c$, one has $d(x_n, x_m) < \frac{\epsilon}{2}$. Define $N_\epsilon = \max\{N_s, N_c\}$. Then, thanks to the triangle inequality, for all $n, n_k > N_\epsilon$, one has

$$d(x_n, x) \leq d(x_n, x_{k_n}) + d(x_{k_n}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The choice of ϵ being arbitrary, we are done.

Exercise 3. Respecting the notation from Theorem 1.6.3, we show that, for each $n \in \mathbb{N}$, one has

$$d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_1, x_0).$$

We work by induction on n and we first show that $d(x_1, x^*) \leq \frac{\beta}{1-\beta} d(x_1, x_0)$. The map T being a contraction and using the triangle inequality, we have

$$d(x_1, x^*) = d(Tx_0, Tx^*) \leq \beta d(x_0, x^*) \leq \beta d(x_0, x_1) + \beta d(x_1, x^*)$$

and so, as a consequence, we derive

$$(1-\beta) d(x_1, x^*) \leq \beta(x_0, x_1)$$

giving the base case. Assume now that $n > 1$ and that

$$d(x_{n-1}, x^*) \leq \frac{\beta^{n-1}}{1-\beta} d(x_1, x_0).$$

Then we have

$$d(x_n, x^*) = d(Tx_{n-1}, Tx^*) \leq \beta d(x_{n-1}, x^*) \leq \beta \frac{\beta^{n-1}}{1-\beta} d(x_1, x_0) = \frac{\beta^n}{1-\beta} d(x_1, x_0)$$

and the proof is complete.

Exercise 4. Let (X, d) be a complete metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that there exists $0 < \beta < 1$ such that, for all $n \in \mathbb{N}$, one has

$$d(x_{n+2}, x_{n+1}) \leq \beta d(x_{n+1}, x_n).$$

We will show that $(x_n)_n$ is convergent. To do so, we will prove that $(x_n)_n$ is Cauchy: the space (X, d) being complete it will follow that $(x_n)_n$ is convergent.

We first claim that, for all $s \in \mathbb{N}$, one has

$$(*) \quad d(x_{s+1}, x_s) \leq \beta^{s-1} d(x_2, x_1).$$

We work by induction on s. If $s = 1$, we get

$$d(x_{s+1}, x_s) = d(x_2, x_1) = \beta^0 d(x_2, x_1) = \beta^{s-1} d(x_2, x_1)$$

and so the base case is checked. Assume now that $s > 1$ and that

$$d(x_s, x_{s-1}) \leq \beta^{s-2} d(x_2, x_1).$$

Then, by assumption $d(x_{s+1}, x_s) \leq \beta d(x_s, x_{s-1})$ and therefore it follows from the induction hypothesis that

$$d(x_{s+1}, x_s) \leq \beta d(x_s, x_{s-1}) \leq \beta \beta^{s-2} d(x_2, x_1) = \beta^{s-1} d(x_2, x_1).$$

So the claim is proven. We now show that $(x_n)_n$ is Cauchy. To this end, let $\epsilon > 0$ and define $c = \frac{d(x_1,x_2)}{1-\epsilon}$. Since $0 < \beta < 1$, the sequence $(c\beta^n)_{n\in\mathbb{N}}$ converges to 0 and therefore there exists $N_\epsilon \in \mathbb{N}$ such that, for each $n > N_\epsilon$, one has $c\beta^n < \epsilon$. Let now $n, m > N_\epsilon + 1$ and, without loss of generality, assume that $m > n$. We write $m = n + t$. As a consequence of the triangle inequality, we have

$$d(x_m, x_n) = d(x_{n+t}, x_n) \leq \sum_{i=0}^{t-1} d(x_{n+i+1}, x_{n+i})$$

and therefore, thanks to $(*)$, we get

$$d(x_m, x_n) \leq \sum_{i=0}^{t-1} d(x_{n+i+1}, x_{n+i}) \leq \sum_{i=0}^{t-1} (\beta^{n+i-1} d(x_2, x_1)) = \beta^{n-1} d(x_2, x_1) \Big(\sum_{i=0}^{t-1} \beta^i \Big).$$

Recall now that, β being smaller than 1, one has that $\left(\sum_{i=0}^{\infty} \beta^i \right) = \frac{1}{1-\beta}$ and so, each β^i being positive, we have that $\left(\sum_{i=0}^{t-1} \beta^i \right) \leq \frac{1}{1-\beta}$. As a result, we get

$$d(x_m, x_n) \leq \beta^{n-1} d(x_2, x_1) \Big(\sum_{i=0}^{t-1} \beta^i \Big) \leq \beta^{n-1} \frac{d(x_2, x_1)}{1-\beta} = \beta^{n-1} c.$$

Now, since $n > N_\epsilon + 1$, we get that $d(x_m, x_n) < \epsilon$. The choice of ϵ being arbitrary, we have proven that $(x_n)_n$ is Cauchy and thus convergent.

Exercise 5. Let (X, d) be a complete metric space and let $T \colon X \to X$ be a map. Let n be a positive integer and assume that T^n is a contraction. We prove that T has a unique fixed point. Since T^n is a contraction, Banach's fixed point theorem assures that T^n has a fixed point, x^* say. Call moreover β the modulus of T^n. As a consequence of the definition of a contraction, we get

$$d(x^*, Tx^*) = d(T^n x^*, T(T^n x^*)) = d(T^n x^*, T^n(Tx^*)) \leq \beta d(x^*, Tx^*)$$

and so, β being smaller than 1, we derive $d(x^*, Tx^*) = 0$. It follows from the defining properties of a metric that $Tx^* = x^*$ and so x^* is a fixed point of T. We now prove that x^* is also the unique fixed point of T. Let y be a fixed point of T. Then, since T^n is the composition of n copies of T, the element y is also a fixed point of T^n. From the uniqueness of the fixed point in Banach's theorem, it follows that $y = x^*$ and thus T has also a unique fixed point.

Exercise 6. We show that the map F defined by

$$f \mapsto F(f),\ [F(f)](t) = \frac{1}{2} \int_0^1 t s f(s)\, ds + \frac{5}{6} t,\ t \in [0, 1],$$

is a contraction in $C([0,1])$. We will do so by showing that, for each $f, g \in C[0,1]$, one has

$$\|F(f) - F(g)\| \leq \frac{1}{2}\|f - g\|.$$

Let $f, g \in C[0,1]$. Then one computes

$$\|F(f) - F(g)\| = \max_{t\in[0,1]} \Big|\frac{1}{2}\int_0^1 tsf(s)\,ds + \frac{5}{6}t - \Big(\frac{1}{2}\int_0^1 tsg(s)\,ds + \frac{5}{6}t\Big)\Big| =$$

$$\frac{1}{2}\max_{t\in[0,1]}\Big|\int_0^1 ts(f(s) - g(s))\,ds\Big| = \frac{1}{2}\max_{t\in[0,1]}|t|\Big|\int_0^1 s(f(s) - g(s))\,ds\Big|.$$

Now, the absolute value of t being at most 1, we have

$$\|F(f) - F(g)\| \leq \frac{1}{2}\Big|\int_0^1 s(f(s) - g(s))\,ds\Big| \leq \frac{1}{2}\int_0^1 |s(f(s) - g(s))|\,ds.$$

The integration interval being $[0,1]$, we have that

$$\|F(f) - F(g)\| \leq \frac{1}{2}\int_0^1 |(f(s) - g(s))|\,ds \leq \frac{1}{2}\max_{t\in[0,1]}|f(t) - g(t)| = \frac{1}{2}\|f - g\|$$

and therefore F is a contraction in $C[0,1]$ of modulus $\frac{1}{2}$. Thanks to Banach's contraction theorem, we know that F has exactly one fixed point, f^* say, which we now compute using Banach's algorithm. Let $f_0 = 0$ be the constant function $[0,1] \to \{0\}$. We define $f_1(t) = [F(f_0)](t)$ and, in general, $f_{n+1}(t) = F(f_n)(t)$. We compute

$$f_1(t) = \frac{1}{2}\int_0^1 tsf_0(s)\,ds + \frac{5}{6}t = \frac{5}{6}t.$$

Setting $c_1 = \frac{5}{6}t$, one gets that $f_1(t) = c_1 t$. We claim that, for all $n \in \mathbb{N}$, defining

$$c_n = 5\Big(\sum_{i=1}^{n}\Big(\frac{1}{6}\Big)^i\Big)$$

gives $f_n(t) = c_n t$. Since we know the claim to be true for $n = 1$, we assume that $n > 1$ and that $f_{n-1}(t) = c_{n-1}t$. As a consequence, we have

$$f_n(t) = \frac{1}{2}\int_0^1 tsf_{n-1}(s)\,ds + \frac{5}{6}t = \frac{1}{2}\int_0^1 tsc_{n-1}s\,ds + \frac{5}{6}t = \frac{c_{n-1}}{2}t\int_{0^1} s^2\,ds + \frac{5}{6}t =$$

$$\frac{c_{n-1}}{2}t\Big(\frac{s^3}{3}\Big)_0^1 + \frac{5}{6}t = \frac{c_{n-1}}{6}t + \frac{5}{6}t = \frac{c_{n-1}+5}{6}\,t$$

and therefore we can compute

$$c_n = \frac{c_{n-1}+5}{6} = \frac{1}{6}c_{n-1} + \frac{5}{6} = \frac{5}{6}\Big(\sum_{i=1}^{n-1}\Big(\frac{1}{6}\Big)^i\Big) + \frac{5}{6} =$$

$$5\Big(\sum_{i=2}^{n}\Big(\frac{1}{6}\Big)^i\Big) + \frac{5}{6} = 5\Big(\sum_{i=1}^{n}\Big(\frac{1}{6}\Big)^i\Big)$$

giving the claim. Thanks to Banach's theorem, we know that $f^* = \lim_{n\to\infty} f_n$ and therefore, for all $t \in [0,1]$, we get

$$f^*(t) = \Big(\lim_{n\to\infty} f_n\Big)(t) = \Big(\lim_{n\to\infty} c_n\Big)t = 5\Big(\sum_{i=1}^{\infty}\Big(\frac{1}{6}\Big)^i\Big)t = 5\frac{\frac{1}{6}}{1-\frac{1}{6}}t = t.$$

Exercise 7. Let $F : [1,2] \to \mathbb{R}$ be defined by $x \mapsto F(x) = \frac{x^2+2}{2x}$. We show that F is a contraction of modulus $\frac{1}{2}$. Indeed, for each $x, y \in [1,2]$, one has

$$|F(x) - F(y)| = \Big|\frac{x^2+2}{2x} - \frac{y^2+2}{2y}\Big| = \frac{1}{2}\Big|\frac{xy-2}{xy}(x-y)\Big| = \frac{|x-y|}{2}\Big|1 - \frac{2}{xy}\Big|.$$

As x, y range between 1 and 2, we have that $\Big|1 - \frac{2}{xy}\Big| \leq 1$, from which it follows that $|F(x) - F(y)| \leq \frac{|x-y|}{2}$. We have proven that F is a contraction of modulus $\frac{1}{2}$. Thanks to Banach's contraction theorem, we now know that F has a unique fixed point x^* and so, in particular, the expression $F(x) - x = 0$ has excatly one root in $[1,2]$, namely x^*. To conclude, note that

$$F(x) - x = \frac{x^2+2}{2x} - x = \frac{x}{2} + \frac{1}{x} - x = \frac{1}{x} - \frac{x}{2}.$$

Exercise 8. We show that $C([-1,1])$ is not complete with respect to the metric

$$d(f,g) := \left(\int_{-1}^{1} |f(t) - g(t)|^2\, dt\right)^{1/2}.$$

We define the sequence $(f_n)_{n\in\mathbb{N}}$ by means of

$$f_n(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -\frac{1}{n}, \\ nt & \text{if } -\frac{1}{n} \leq t \leq \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1. \end{cases}$$

and we claim that $(f_n)_n$ is Cauchy but not convergent in $C[-1,1]$. We first show it is Caucy. To this end, let $n, m \in \mathbb{N}$, with $n > m$. Then

$$d(f_n, f_m) = \left(\int_{-1}^{1} |f_n(t) - f_m(t)|^2\, dt\right)^{1/2} =$$

$$\left(\int_{-1/m}^{-1/n} |f_n(t) - f_m(t)|^2\, dt + \int_{-1/n}^{1/n} |f_n(t) - f_m(t)|^2\, dt + \int_{1/n}^{1/m} |f_n(t) - f_m(t)|^2\, dt\right)^{1/2} =$$

$$\left(\int_{-1/m}^{-1/n} |-1 - mt|^2\, dt + \int_{-1/n}^{1/n} |nt - mt|^2\, dt + \int_{1/n}^{1/m} |1 - mt|^2\, dt\right)^{1/2} =$$

$$\left(\int_{-1/m}^{-1/n} (1 + 2mt + m^2t^2)\, dt + (n-m)^2 \int_{-1/n}^{1/n} t^2\, dt + \int_{1/n}^{1/m} (1 - 2mt + m^2t^2)\, dt\right)^{1/2} =$$

$$\left(\int_{-1/m}^{-1/n}(1+2mt+m^2t^2)\,dt+\int_{1/n}^{1/m}(1-2mt+m^2t^2)\,dt\right)^{1/2}=-\frac{2}{n}-\frac{2m^2}{n^3}+\frac{2}{3m}.$$

Now, to show that $(f_n)_{n\geq 1}$ is a Cauchy sequence, let $\epsilon > 0$ be given and let N be an integer greater than $\frac{14}{3\epsilon}$ (this can be found by 'working backwards' by first finding a bound on $d(f_n, f_m)$ in terms of N and then defining N in terms of ϵ).

For all $m, n > N$, we have $d(f_n, f_m) = -\frac{2}{n} - \frac{2m^2}{n^3} + \frac{2}{3m} < \frac{2}{n} + \frac{2m^2}{n^3} + \frac{2}{3m}$ (because $m, n > 0$). As $m, n > N$, we know that $\frac{1}{n}, \frac{1}{m} < \frac{1}{N}$, and as we have chosen $n \geq m$, we know that $\frac{m^2}{n^3} \leq \frac{n^2}{n^3} = \frac{1}{n} < \frac{1}{N}$, so we have $d(f_n, f_m) < \frac{2}{N} + \frac{2}{N} + \frac{2}{3N} = \frac{14}{3N} < \epsilon$.

To show that the sequence is not convergent in $C[-1,1]$, one observes that, if $(f_n)_{n\geq 1}$ had a limit f in $C[-1,1]$, then $\lim_{n\to\infty}\|f - f_n\|$ would have to be zero and so

$$\int_{-1}^{1/n} |f_n(t) - f(t)|^2\,dt + \int_{1/n}^{1} |f_n(t) - f(t)|^2\,dt \to 0.$$

Using the definition of $(f_n)_{n\geq 1}$, one would then have

$$\int_{-1}^{1/n} |-1 - f(t)|^2\,dt + \int_{1/n}^{1} |1 - f(t)|^2\,dt \to 0$$

which which would only be possible if f would satisfy

$$f(t) = \begin{cases} -1 & \text{if } -1 \leq t < 0, \\ 1 & \text{if } 0 < t \leq 1. \end{cases}$$

but then f wouldn't be continuous at $x = 0$.

Exercise 9. Let the sequence $(f_n)_{n\in\mathbb{N}}$, consisting of functions $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$, be defined by

$$f_n : t \mapsto \cos\left(\frac{t}{n}\right) e^{-t}.$$

(a) We claim that the sequence $(f_n)_n$ converges pointwise to $f : [0,\infty) \to \mathbb{R}$, defined by $t \mapsto f(t) = e^{-1}$. Indeed, we have that, for each $t \in [0,\infty)$, the limit $\lim_{n\to\infty}\frac{t}{n}$ is equal to 0 and so, cos being a continuous function, we have, for each $t \in [0,\infty)$, that $\lim_{n\to\infty}\cos(t/n) = 1$. It follows that, for each $t \in [0,\infty)$, we get $\lim_{n\to\infty}\cos(t/n)e^{-t} = e^{-t}$.

(b) Let now f be as in (a). We show that $(f_n)_n$ converges uniformly to f. Let $n \in \mathbb{N}$. One can prove that, for each $t \in [0,\infty)$, one has $0 \leq \frac{t^2}{e^t} \leq 1$ and so, using that $\cos(x) \geq 1 - \frac{x^2}{2}$, we compute

$$\|f - f_n\| = \sup_{t\in[0,\infty)} |f(t) - f_n(t)| = \sup_{t\in[0,\infty)} |e^{-t}(1 - \cos(t/n))| \leq \sup_{t\in[0,\infty)} |e^{-t}\frac{t^2}{2n^2}| =$$

$$\sup_{t\in[0,\infty)}\left(e^{-t}\frac{t^2}{2n^2}\right) = \frac{1}{2n^2}\sup_{t\in[0,\infty)}\frac{t^2}{e^t} \leq \frac{1}{2n^2}.$$

Since, for each $n \in \mathbb{N}$, one has

$$0 \leq \|f - f_n\| \leq \frac{1}{2n^2}$$

and $\lim_{n\to\infty} \frac{1}{2n^2} = 0$, we have that $\lim_{n\to\infty} \|f - f_n\| = 0$.