OQE - PROBLEM SET 5 - SOLUTIONS

Exercise 1. Let (X, d) be a metric space, where d is the discrete metric. We claim that (X, d) is

(a) complete;

(b) separable if and only if X is countable.

(a) Similarly to Exercise 4 from Problem Set 2, one shows that Cauchy sequences in (X, d) are stationary and therefore convergent.
(b) We saw multiple times in class that each subset of X is open, since the metric is discrete, and therefore each subset of X is closed. It follows from Theorem 1.3.6(iv) that X is separable if and only if it is countable.

Exercise 2. Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to Y$ be a continuous function. Let A be a dense subset of X. We show that $f(A)$ is dense in $f(X)$. Define $Z = f(X)$. We will show that, for each $z \in Z$ and for each $\epsilon > 0$, the intersection $\mathrm{B}_\epsilon(z) \cap f(A)$ is non-empty. To this end, let $z \in Z$ and $\epsilon > 0$. Let moreover $x \in X$ be such that $f(x) = z$. By the continuity of f, there exists $\delta > 0$ such that $f(\mathrm{B}_\delta(x)) \subseteq \mathrm{B}_\epsilon(z)$. Moreover, since A is dense in X, there exists $a \in A \cap \mathrm{B}_\delta(x)$. Fix such element a. Then

$$f(a) \in f(A) \cap f(\mathrm{B}_\delta(x)) \subseteq f(A) \cap \mathrm{B}_\epsilon(z)$$

and therefore $f(A) \cap \mathrm{B}_\epsilon(z) \neq \emptyset$.

Exercise 3. Let $1 \leq p < \infty$ and let $r \in \mathbb{R}$. We claim that the sequence $(r^n)_{n\in\mathbb{N}}$ belongs to l_p if and only if $|r| < 1$, in which case $\|(r^n)_n\|_p = \left(\frac{|r|^p}{(1-|r|^p)}\right)^{1/p}$. We recall that $(r^n)_{n\in\mathbb{N}}$ belongs to l_p if and only if $\sum_{n\geq 1} |r^n|^p$ is convergent. One can rewrite

$$\sum_{n\geq 1} |r^n|^p = \sum_{n\geq 1} |r^p|^n$$

which is a geometric series with 'base' $|r^p|$ and therefore convergent if and only if $|r^p| < 1$. However, $|r|^p = |r^p|$ is smaller than 1 if and only if $|r|$ itself is smaller than 1 and therefore our first claim is proven. For $|r| < 1$, we now compute

$$\|(r^n)_n\|_p = (\sum_{n\geq 1} |r^p|^n)^{1/p} = \left(\frac{|r|^p}{1 - |r|^p}\right)^{1/p}$$

and therefore the second claim is proven too.

Exercise 4. We claim that

(i) $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ does not belong to l_1;

(ii) $\left(\frac{\sin(\pi n)}{n^2}\right)_{n\in\mathbb{N}}$ belongs to l_1.

(i) One has

$$\sum_{n\geq 1}\Big|\frac{1}{n}\Big| = \sum_{n\geq 1}\frac{1}{n} = \infty$$

and so $\left(\frac{1}{n}\right)_n$ does not belong to l_1.

(ii) One computes

$$\sum_{n\geq 1}\Big|\frac{\sin(\pi n)}{n^2}\Big| \leq \sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6}$$

and therefore $\left(\frac{\sin(\pi n)}{n^2}\right)_n$ belong to l_1.

Exercise 5. We show that the map

$$f \mapsto F(f) = \frac{1}{2}\int_0^t f(s)ds + 1$$

is a contraction in $C[0,1]$ of modulus $\frac{1}{2}$ and we find the fixed point of F using Banach's algorithm. For each $f, g \in C[0,1]$, we have

$$\|F(f) - F(g)\| = \max_{t\in[0,1]}\Big|\frac{1}{2}\int_0^t (f(s) - g(s))ds\Big| = \frac{1}{2}\max_{t\in[0,1]}\Big|\int_0^t (f(s) - g(s))ds\Big|$$

$$\leq \frac{1}{2}\max_{t\in[0,1]}\Big(t\max_{s\in[0,t]}|f(s) - g(s)|\Big) \leq \frac{1}{2}\max_{t\in[0,1]}\Big(t\max_{s\in[0,1]}|f(s) - g(s)|\Big) \leq \frac{1}{2}\|f - g\|$$

and so F is a contraction of modulus $\frac{1}{2}$. We now compute f^*, the unique fixed point of F, using Banach's algorithm. We set $f_0(t) = 0$ and $f_{n+1}(t) = F(f_n)(t)$. We claim that, for each $n \in \mathbb{N}$, one has

$$f_{n+1}(t) = \sum_{k=0}^{n}\frac{(t/2)^k}{k!}.$$

We prove the claim by induction on n. One easily computes that, for all $t \in [0,1]$, one has $f_1(t) = 1$ and $f_2(t) = \frac{1}{2}t + 1$. We now assume that $n \geq 2$ and that $f_n(t) = \sum_{k=0}^{n-1}\frac{(t/2)^k}{k!}$. For each $t \in [0,1]$, we then have

$$f_{n+1}(t) = F(f_n)(t) = \frac{1}{2}\int_0^t \Big(\sum_{k=0}^{n-1}\frac{(s/2)^k}{k!}\Big)ds + 1 = \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{k!2^k}\Big(\frac{t^{k+1}}{k+1}\Big) + 1 =$$

$$\sum_{k=0}^{n-1}\frac{t^{k+1}}{(k+1)!2^{k+1}} + 1 = \sum_{k=1}^{n}\frac{t^k}{(k)!2^k} + 1 = \sum_{k=0}^{n}\frac{t^k}{(k)!2^k} = \sum_{k=0}^{n}\frac{(t/2)^k}{(k)!}$$

and so the claim is proven. To conclude, we compute the point-wise limit of the sequence $(f_n)_n$. For each $t \in C[0,1]$, we get

$$\Big(\lim_{n\to\infty} f_n\Big)(t) = \sum_{k=0}^{\infty}\frac{(t/2)^k}{(k)!} = e^{t/2}.$$

Exercise 6. (i) We claim that the sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n \in C[0,1]$ is defined by $t \mapsto f_n(t) = t^{2n}$, is not convergent in $C[0,1]$. Indeed, for each $t \in [0,1)$, the limit $\lim_{n\to\infty} t^{2n}$ is equal to 0, while $1^{2n} = 1$ for any choice of n.
(ii) We claim that the sequence $(f_n)_{n\in\mathbb{N}}$, defined by $t \mapsto f_n(t) = te^{-nt}$ is convergent in $C[0,1]$ to the function $t \mapsto 0$. Indeed, one has

$$\|f_n\| = \max_{t\in[0,1]} |te^{-nt}| = \max_{t\in[0,1]} |t(e^t)^{-n}|$$

where $t \in [0,1]$ and therefore $e^t \geq 1$. It follows that, for each $t \in [0,1]$, one has

$$0 \leq te^{-nt} \leq e^{-nt}$$

and so, since $\lim_{n\to\infty}(e^t)^{-n} = 0$, we get $\lim_{n\to\infty} te^{-nt} = 0$.
(iii) We claim that the sequence $(f_n)_{n\in\mathbb{N}}$, defined by $t \mapsto f_n(t) = te^{-t/n}$ is convergent in $C[0,1]$ to the function $t \mapsto t$. Indeed, for each $n \in \mathbb{N}$, we can compute

$$\|f_n - \mathrm{id}_{[0,1]}\| = \max_{t\in[0,1]} |te^{-t/n} - t| = \max_{t\in[0,1]} |t(e^{-t/n} - 1)| \leq \max_{t\in[0,1]} |e^{-t/n} - 1|$$

and, since, for each $t \in [0,1]$, one has $\lim_{n\to\infty} e^{-t/n} = 1$, the limit $\lim_{n\to\infty} \|f_n - \mathrm{id}\|$ is equal to 0. As a consequence, $f_n \to \mathrm{id}$ in $C[0,1]$.
(iv) We claim that the sequence $(f_n)_{n\in\mathbb{N}}$, defined by $t \mapsto f_n(t) = n^{-1}\sin(\pi nt)$ is convergent in $C[0,1]$ to the function $t \mapsto 0$. For each natural number n, one has indeed

$$\|f_n\| = \max_{t\in[0,1]} \left|\frac{\sin(\pi nt)}{n}\right| \leq \frac{1}{n}$$

which converges to 0 as $n \to \infty$.
(v) We claim that the sequence $(f_n)_{n\in\mathbb{N}}$, where $t \mapsto f_n(t) = \sin(\pi t)^n$, is not convergent in $C[0,1]$. Indeed, if $t \neq 1/2$, then $|\sin(\pi t)| < 1$ and therefore, for each $t \neq 1/2$, one has $\lim_{n\to\infty} f_n(t) = 0$. However, for each $n \in \mathbb{N}$, we have $f_n(1/2) = 1$ and therefore $(f_n)_n$ cannot converge to a continuous function.
(vi) The sequence $(f_n)_{n\in\mathbb{N}}$, defined by $t \mapsto f_n(t) = \frac{1}{1+nt^2}$, is not convergent in $C[0,1]$. As in the previous example, consider the case $t = 0$ and $t \in (0,1]$ separately.

Exercise 7. We claim that $C_0(\mathbb{R})$, together with the sup-norm, is a Banach space. We first claim that $C_0(\mathbb{R})$ is a subset of $C_b(\mathbb{R})$, i.e. the collection of bounded continuous functions $\mathbb{R} \to \mathbb{R}$. To this end, let $\epsilon > 0$ and let $f \in C_0(\mathbb{R})$. Since $\lim_{|t|\to\infty} f(t) = 0$, there exists $r \in \mathbb{R}_{>0}$ such that, for each $x \in (-\infty, -r) \cup (r, +\infty)$, one has $|f(x)| < \epsilon$. Moreover, we know by real analysis that, being continuous, f is bounded on the closed interval $[-r, r]$ and therefore $f \in C_b(\mathbb{R})$. The choices of f and ϵ being arbitrary, we have proved the claim. Our next claim is that $C_0(\mathbb{R})$ is closed in $C_b(\mathbb{R})$. To show that, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_0(\mathbb{R})$ converging to an element $f \in C_b(\mathbb{R})$. Then, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, one has $\|f_n - f\| < \epsilon$. In particular, for each $x \in \mathbb{R}$, one has $|f_n(x) - f(x)| < \epsilon$ and therefore, for each $x \in \mathbb{R}$

$$-\epsilon + f_n(x) < f(x) < f_n(x) + \epsilon.$$

If we now let $|x| \to \infty$, we get that $-\epsilon < \lim_{|x|\to\infty} f(x) < \epsilon$ and so, the choice of ϵ being arbitrary, it follows that $\lim_{|x|\to\infty} f(x) = 0$. We have proven that $f \in C_0(\mathbb{R})$ and so, as a consequence of Theorem 1.3.8 from the notes, the subset $C_0(\mathbb{R})$ is closed in $C_b(\mathbb{R})$. Corollary 1.9.4 together with Exercise 1.5.5 from the notes yields the completeness of $C_0(\mathbb{R})$.

Exercise 8. We check continuity and compute the norms of the following operators:

(i) $l : l_1 \to \mathbb{R}$, defined by $x = (x_i)_{i\in\mathbb{N}} \mapsto l(x) = \sum_{i=1}^{\infty} \Big(1 - \frac{1}{i}\Big) x_i$;
(ii) $l : l_2 \to \mathbb{R}$, defined by $x = (x_i)_{i\in\mathbb{N}} \mapsto l(x) = \sum_{i=1}^{\infty} \frac{1}{i} x_i$;
(iii) $l : l_2 \to \mathbb{R}$, defined by $x = (x_i)_{i\in\mathbb{N}} \mapsto l(x) = \sum_{i=1}^{\infty} \Big(\frac{(1-(-1)^i)(i-1)}{i}\Big) x_i$;
(iv) $l : C[0,1] \to \mathbb{R}$, defined by $f \mapsto l(f) = \int_0^1 f(t) \operatorname{sign}\Big(t - \frac{1}{2}\Big) dt$.

Throughout the whole exercise we will make constant use of Theorem 1.10.1 from the notes.

(i) We prove continuity at 0. To this end, let $\epsilon > 0$ and let $x = (x_i)_{i\in\mathbb{N}} \in l_1$ be such that $\|x\|_1 < \epsilon$. Then one has

$$|l(x)| = \Big|\sum_{i=1}^{\infty}\Big(1 - \frac{1}{i}\Big)x_i\Big| \le \sum_{i=1}^{\infty}\Big(1 - \frac{1}{i}\Big)|x_i| \le \sum_{i=1}^{\infty}|x_i| = \|x\|_1 < \epsilon$$

and so, the choice of ϵ being arbitrary, l is continuous. We now claim that

$$\|l\| = \sup_{\|x\|_1 \le 1} |l(x)| = 1.$$

We prove that, for each $x \in l_1$, one has $|l(x)| \le \|x\|_1$ and therefore $\|l\| \le 1$. Define now, for each $k \in \mathbb{N}$, the element $x_k = (x_{k,i})_{i\in\mathbb{N}}$ of l_1 by

$$x_{k,i} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise .} \end{cases}$$

Then, for each $k \in \mathbb{N}$, the norm of e_k is equal to 1 and the value of $|l(e_k)|$ is $1 - \frac{1}{k}$. We conclude by observing that

$$1 = \lim_{k\to\infty} |l(e_k)| \le \sup_{\|x\|_1 \le 1} |l(x)| = \|l\| \le 1$$

and therefore $\|l\| = 1$.

(ii) We prove continuity at 0. To this end, let $\epsilon > 0$ and define $\delta = \frac{\sqrt{6}\epsilon}{\pi}$. Then, using Hölder's inequality, one has, for all $x \in l_2$ with $\|x\|_2 < \delta$, that

$$\Big|\sum_{i=1}^{\infty}\frac{1}{i}x_i\Big| \le \sum_{i=1}^{\infty}\Big|\frac{1}{i}x_i\Big| \le \Big(\sum_{i=1}^{\infty}\frac{1}{i^2}\Big)^{1/2}\Big(\sum_{i=1}^{\infty}|x_i|^2\Big)^{1/2} = \frac{\pi}{\sqrt{6}}\|x\|_2 < \frac{\pi}{\sqrt{6}}\delta = \epsilon.$$

It follows that l is continuous and that, for each $x \in l_2$, one has $|l(x)| \leq \frac{\pi}{\sqrt{6}}\|x\|_2$. In particular, $\|l\| \leq \frac{\pi}{\sqrt{6}}$. Define now the element $x = (x_i)_{i\in\mathbb{N}}$ of l_2 by $x_i = 1/i$. Then one has

$$\frac{|l(x)|}{\|x\|_2} = \Big|\sum_{i=1}^{\infty} \frac{1}{i^2}\Big| \Big/ \Big(\sum_{i=1}^{\infty} \frac{1}{i^2}\Big)^{1/2} = \Big(\sum_{i=1}^{\infty} \frac{1}{i^2}\Big)^{1/2} = \frac{\pi}{\sqrt{6}}.$$

It follows that $\|l\| = \pi/\sqrt{6}$.

(iii) We claim that the operator l is not continuous. We will exhibit $x \in l_2$ such that $l(x)$ is not finite. Let $x = (x_i)_{i\geq 1}$ be defined by $x_i = 1/i$. Then x is an element of l_2 of norm $\pi/\sqrt{6}$. We now claim that $l(x)$ is not a convergent series. Indeed, playing with the parities of the indices, one computes

$$l(x) = \sum_{i=1}^{\infty} \frac{(1-(-1)^i)(i-1)}{i^2} = \sum_{i=1}^{\infty} \frac{2(2i+1-1)}{(2i+1)^2} = \sum_{i=1}^{\infty} \frac{4i}{(2i+1)^2}.$$

We now will 'fill in the slots of the even indices' to show that $\sum_{i=1}^{\infty} \frac{4i}{(2i+1)^2}$ is divergent. Indeed, for each $i \geq 1$, one has that

$$\frac{3i}{(2i+1)^2} \geq \frac{i}{(2i)^2}$$

and therefore

$$\sum_{i=1}^{\infty} \frac{4i}{(2i+1)^2} \geq \sum_{i=1}^{\infty} \Big(\frac{i}{(2i)^2} + \frac{i}{(2i+1)^2}\Big) = \frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{i} + \sum_{i=1}^{\infty} \frac{i}{(2i+1)^2} = \infty.$$

(iv) We prove continuity at 0. To this end, let $\epsilon > 0$ and take $f \in C[0,1]$ with $\|f\| < \epsilon$. Since the image of sign belongs to $[-1,1]$, we have

$$\Big|\int_0^1 f(t)\,\mathrm{sign}\Big(t-\frac{1}{2}\Big)dt\Big| \leq \int_0^1 \Big|f(t)\,\mathrm{sign}\Big(t-\frac{1}{2}\Big)\Big|dt \leq \int_0^1 |f(t)|dt \leq \|f\| < \epsilon.$$

The choice of ϵ being arbitrary, we have proven that f is continuous. Moreover, we have proven that, for each $f \in C[0,1]$, one has $|l(f)| \leq \|f\|$ and therefore $\|l\| \leq 1$. Define now $f : [0,1] \to \mathbb{R}$ to be $t \mapsto f(t) = 4t+2$, which is a continuous map. Then one can compute

$$|l(f)| = \Big|\int_0^1 (4t-2)\,\mathrm{sign}\Big(t-\frac{1}{2}\Big)dt\Big| = \Big|\int_0^{1/2} (-4t+2)dt + \int_{1/2}^1 (4t-2)dt\Big| = 1$$

and therefore $\|l\| = 1$.