OQE - PROBLEM SET 6 - SOLUTIONS

Exercise 1. Let A and B be compact subsets of a metric space (X, d). We show that $A \cap B$ and $A \cup B$ are also compact. We start from $A \cup B$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $A \cup B$. Then $\mathcal{U}$ is also an open cover of A and B. Because of their compactness, there exist finite subsets I_A and I_B of I such that $\{U_i\}_{i \in I_A}$ is an open cover of A and $\{U_i\}_{i \in I_B}$ is an open cover of B. It follows then that $\{U_i\}_{i \in I_A \cup I_B}$ is an open cover of $A \cup B$. Since $I_A \cup I_B$ is finite and $\mathcal{U}$ is arbitrary, the set $A \cup B$ is compact. We now move to $A \cap B$. By Theorem 1.11.4, both A and B are closed in X and so $A \cap B$ is closed. It follows from Theorem 1.11.5 that $A \cap B$ is compact.

Exercise 2. Let (X, d) be a discrete metric space. We claim that X is compact if and only if it is finite. The implication from right to left is clear, so we prove the other one. Assume that X is compact. Then each open cover $\mathcal{U}$ of X admits a finite subcover. Define

$$\mathcal{U} = \{\mathrm{B}_1(x) : x \in X\}.$$

Since, for each $x \in X$, one has $\mathrm{B}_1(x) = \{x\}$, the only subcover of $\mathcal{U}$ is $\mathcal{U}$ itself. The set X being compact, it follows that $\mathcal{U}$ is finite and therefore so is X.

Exercise 3. Let $f : X \to Y$ be a continuous map of metric spaces. Assume that X is compact. We show that $f(X)$ is compact. To this end, let $\mathcal{V}$ be an open cover of $f(X)$. The map f being continuous, it follows from Theorem 1.4.6 that

$$\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$$

is an open cover of X. Thanks to the compactness of X, there exists a finite open subcover $\mathcal{U}_f$ of X. We get a finite subcover of $\mathcal{V}$ by taking the collection $\{f(U) : U \in \mathcal{U}_f\}$.

Exercise 4. In this exercise, we will determine whether the following subsets of $\mathbb{R}^2$ are compact

(a) $A = (\mathbb{Q} \cap [0, 1]) \times [0, 1]$;
(b) $B = \{(x, y) \in \mathbb{R}^2 : x = 0\}$;
(c) $C = (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}) \times [0, 1]$;
(d) $D = \{(\frac{1}{n}, \frac{n-1}{n}) : n \in \mathbb{N}\}$.

We will be using Theorem 1.11.6.

(a) We claim that A is not compact. To show so, we will prove, by contradiction, that A is not closed. Assume it is. Since the y-axis $A_y = \mathbb{R} \times \{0\}$ is closed in $\mathbb{R}^2$, the intersection $A \cap A_y$ is also closed. However, we have that

$$A \cap A_y = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{Q} \cap [0, 1]\}$$

and it is not difficult to produce a sequence in $A \cap A_y$ with limit outside $A \cap A_y$. Contradiction to Theorem 1.3.8.

(b) We claim that B is not compact. For each $n \in \mathbb{Z}$, define U_n to be the open ball $\mathrm{B}_1((0, n))$. Then $\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}}$ is an open covering of B, which however does not admit a subcover. Indeed, for each $n \in \mathbb{Z}$, the only element of $\mathcal{U}$ to which $(0, n)$ belongs is U_n and therefore each element of U_n is necessary to cover B.

(c) We claim that C is compact and we will prove it by showing that C is closed and bounded in $\mathbb{R}^2$. The set C is contained in the closed square $S = [0, 1] \times [0, 1]$ and it is therefore bounded. To show that C is closed, we prove that its complement is open. To this end, let $x \in \mathbb{R}^2 \setminus C$. If x is not in S, then, as a consequence of exercise 3 from Problem Set 3, the distance $d(x, S)$ is positive. Because of its definition, $\mathrm{B}_{d(x,S)}(x)$ is then contained in $\mathbb{R}^2 \setminus C$. Assume now that $x \in S$. Then x has coordinates (x_1, x_2), where $0 \leq x_1, x_2 \leq 1$ and, since x does not belong to C, there exists $n \in \mathbb{N}$ such that $\frac{1}{n+1} < x_1 < \frac{1}{n}$. We define $\epsilon = \min\{x_1 - \frac{1}{n+1}, \frac{1}{n} - x_1\}$ and we note that, by construction, the ball $\mathrm{B}_\epsilon(x)$ is entirely contained in S. We have proven that, for each $x \in \mathbb{R}^2 \setminus C$, there exists $\epsilon > 0$ such that $\mathrm{B}_\epsilon(x) \subseteq \mathbb{R}^2 \setminus C$ and therefore $\mathbb{R}^2 \setminus C$ is open.

(d) We show that D is not compact by showing that it is not closed. Indeed, define the sequence $(x_n)_{n\in\mathbb{N}}$ in D by $x_n = (\frac{1}{n}, \frac{n-1}{n})$. Then $(x_n)_n$ is a sequence in D, whose limit point $(0, 1)$ does not lie in D.

Exercise 5. We show that the closed ball $\overline{\mathrm{B}_1(0)}$ is not a sequentially compact subspace of l_∞. We will do so by constructing a sequence in $\overline{\mathrm{B}_1(0)}$ which does not admit a converging subsequence. We define, for each $n \in \mathbb{N}$, the element $x_n = (x_{n,k})_{k\in\mathbb{N}}$ of l_∞ by means of

$$x_{n,k} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise} . \end{cases}$$

Then, for each $n \in \mathbb{N}$, one has $\|x_n\|_\infty = 1$ and so the sequence $(x_n)_{n\in\mathbb{N}}$ is in $\overline{\mathrm{B}_1(0)}$. Assume now by contradiction that $(x_{n_r})_{r\in\mathbb{N}}$ is a subsequence of $(x_n)_n$ that is convergent. In particular $(x_{n_r})_r$ is Cauchy and so, for each $1 > \epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $r, s > N$, one has $\|x_{n_r} - x_{n_s}\| < \epsilon$. However, if $r \neq s$, the element $x_{n_r} - x_{n_s} = (y_i)_{i\geq 1}$ of l_∞ is given by

$$y_i = \begin{cases} 1 & \text{if } \ i = r, \\ -1 & \text{if } \ i = s, \\ 0 & \text{otherwise} \end{cases}$$

and therefore $\|x_{n_r} - x_{n_s}\| = 1$. Contradiction.

Exercise 6. Let K be a non-empty compact subset of $\mathbb{R}^2$. We will show that there exists $x^* \in K$ such that

$$d(x^*, 0) = \sup_{x \in K} d(x, 0).$$

We know from Exercise 5 from Problem Set 3 that the function $f : K \to \mathbb{R}$ that is defined by $x \mapsto f(x) = d(x, 0)$ is uniformly continuous and, in particular, continuous. Applying Theorem 1.12.3, we get that there exists an element $x^* = x_{\max} \in K$ such that $f(x^*) = \sup_{x \in K} f(x)$.

Exercise 7. We will show that $A = \{f \in C[0, 1] : f(0) = 1\}$ is a closed subset of $C[0, 1]$. To do so, we will show that each sequence of elements of A that converges in $C[0, 1]$ actually converges in A. To this end, let $(f_n)_{n \in \mathbb{N}}$ be a sequence in A with limit $f \in C[0, 1]$. Then, as n goes to infinity, we have that $\|f_n - f\| \to 0$. As a consequence, for each $t \in [0, 1]$, one has $\lim_{n \to \infty} |f_n(t) - f(t)| = 0$ and so in particular,

$$\lim_{n \to \infty} |1 - f(0)| = \lim_{n \to \infty} |f_n(0) - f(0)| = 0.$$

It follows that $\lim_{n \to \infty} f(0) = 1$ and therefore $f \in A$. The choice of $(f_n)_{n \in \mathbb{N}}$ being arbitrary, the subset A is closed.