OQE - PROBLEM SET 8 - SOLUTIONS

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$(x, y) \mapsto f(x, y) = \begin{cases} \left(\frac{3xy^2}{4x^2+4y^4}\right)^2 & \text{if } \ x \neq 0 \\ 0 \ \text{ otherwise} \end{cases} .$$

We claim that:

(a) if ℓ is a line through $(0, 0)$, then f is continuous on ℓ; and
(b) the function f is not continuous at $(0, 0)$.

(a) If ℓ is a line through $(0, 0)$, then there exists $m \in \mathbb{R}$ such that

$$\ell = \{(x, y) \in \mathbb{R}^2 : y = mx\}.$$

Fix $m \in \mathbb{R}$ and call ℓ the subset of $\mathbb{R}^2$ consisting of those elements (x, y) such that $y = mx$. Then, restricting f to ℓ gives

$$f(x, y) = \begin{cases} \left(\frac{3m^2x^3}{4x^2(1+m^4x^2)}\right)^2 & \text{if } \ (x, y) \in \ell \setminus \{(0, 0)\} \\ 0 \ \text{ if } \ (x, y) = (0, 0) \end{cases} .$$

We now have that

$$\lim_{\substack{(x,y)\in\ell \\ (x,y)\to(0,0)}} f(x, y) = \lim_{x\to 0} \left(\frac{3m^2x^3}{4x^2(1 + m^4x^2)}\right)^2 = \lim_{x\to 0} \left(\frac{3m^2x^3}{4x^2}\right)^2 = 0$$

and therefore the function f is continuous on ℓ.

(b) To show that f is not continuous at $(0, 0)$, we construct a convergent sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $\lim_{n\to\infty} z_n = (0, 0)$, but $\lim_{n\to\infty} f(z_n) \neq 0$. To this end, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that, for each $n \in \mathbb{N}$, the element $y_n \neq 0$ but such that $\lim_{n\to\infty} y_n = 0$. We define $(z_n)_{n\in\mathbb{N}}$ by setting, for each $n \in \mathbb{N}$, the element z_n equal to (y_n^2, y_n). As a consequence of the definition of $(y_n)_n$, the first component of each z_n is different from 0, but $\lim_{n\to\infty} z_n = (0, 0)$. However, when we compute the images of z_n under f, we get

$$\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(y_n^2, y_n) = \lim_{n\to\infty} \left(\frac{3y_n^4}{8y_n^4}\right)^2 = \frac{9}{64}.$$

We have proven that $\lim_{n\to\infty} f(z_n) \neq 0$ and so, in view of Theorem 1.4.4, the function f is not continuous at $(0, 0)$.

Exercise 2. Let $g, h : \mathbb{R}^2 \to \mathbb{R}$ be respectively defined by $(x, y) \mapsto x + y^2$ and $(x, y) \mapsto x^2y$. Define moreover $f : \mathbb{R}^2 \to \mathbb{R}$ by means of $(a, b) \mapsto ab^2$. We want to compute, using the chain rule, the gradient of the function $F : \mathbb{R}^2 \to \mathbb{R}$ that is defined by

$$(x, y) \mapsto F(x, y) = f(g(x, y), h(x, y)).$$

To do so, we define an additional function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by means of $(x, y) \mapsto (g(x, y), h(x, y))$ and we note that $F = f \circ \phi$. By the chain rule, we then have that

$$DF(x, y) = Df(\phi(x, y))D\phi(x, y) = Df(g(x, y), h(x, y))D\phi(x, y)$$

so we compute the gradient of f:

$$Df(a, b) = \nabla f(a, b) = (b^2, 2ab)$$

and also the Hessian of ϕ:

$$D\phi(x, y) = \begin{bmatrix} 1 & 2y \\ 2xy & x^2 \end{bmatrix}.$$

As a consequence we have

$$\begin{aligned} DF(x, y) &= Df(g(x, y), h(x, y))D\phi(x, y) \\ &= \begin{bmatrix} h(x, y)^2 & 2g(x, y)h(x, y) \end{bmatrix} D\phi(x, y) \\ &= \begin{bmatrix} x^4y^2 & 2x^3y + 2x^2y^3 \end{bmatrix} \begin{bmatrix} 1 & 2y \\ 2xy & x^2 \end{bmatrix} \\ &= (5x^4y^2 + 4x^3y^4, 4x^4y^3 + 2x^5y). \end{aligned}$$

Exercise 3. Let x, a, p, w, t be real variables, each of which identifies:

$x =$ demand of some good;
$p =$ price of the good;
$a =$ amount the producer spends on advertising the good;
$w =$ weather (measured in some way);
$t =$ tax rate.

The interaction between the variables is governed by the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ in the following way:

(a) $x = f(p, a)$ and, for all $a, p \in \mathbb{R}$, one has $\partial_p f(p, a) < 0$ and $\partial_a f(p, a) > 0$;
(b) $p = g(w, t)$ and, for all $w, t \in \mathbb{R}$, one has $\partial_w g(w, t) > 0$ and $\partial_t g(w, t) < 0$;
(c) $a = h(t)$ and, for all $t \in \mathbb{R}$, one has $h'(t) > 0$.

We claim that, if the tax rate (t) increases, then the demand for the good (x) necessarily increases. To prove so, we will write x as a function F of w and t and show that, for any choice of w and t, one has $\partial_t F(w, t) > 0$. We define $F : \mathbb{R}^2 \to \mathbb{R}$

by means of $(w,t) \mapsto F(w,t) = f(g(w,t),h(t))$ and we note that F expresses x in terms of w and t. To compute $\nabla F(w,t)$, we use the chain rule and get

$$\begin{aligned}\nabla F(w,t) &= \nabla f(g(w,t),h(t)) D(g,h)(w,t)\\ &= \begin{bmatrix}\partial_p f(g(w,t),h(t)) & \partial_a f(g(w,t),h(t))\end{bmatrix}\begin{bmatrix}\partial_w g(w,t) & \partial_t g(w,t)\\ 0 & h'(t)\end{bmatrix}.\end{aligned}$$

It follows then that

$$\partial_t F(w,t) = \partial_p f(g(w,t))\partial_t g(w,t) + \partial_a f(g(w,t),h(t))h'(t)$$

which is, thanks to $(a-c)$, positive for any choice of w,t.

Exercise 4. Let U be the open subset of $\mathbb{R}^2$ that is defined by

$$U = \{(x,y) \in \mathbb{R}^2 : x,y > 0\}.$$

Let moreover $a \in (0,1)$ and define $f : U \to \mathbb{R}$ by $(x,y) \mapsto f(x,y) = x^a y^{1-a}$. We compute the second order Taylor polynomial of f at $(1,1)$. The gradient and Hessian of f are $\nabla f(x,y) = (ax^{a-1}y^{1-a}, (1-a)x^a y^{-a})$ and

$$D^2 f(x,y) = \begin{bmatrix}a(a-1)x^{a-2}y^{1-a} & a(1-a)x^{a-1}y^{-a}\\ a(1-a)x^{a-1}y^{-a} & a(a-1)x^a y^{-1-a}\end{bmatrix}.$$

Let now $h = (h_1,h_2)$ be an element of $\mathbb{R}^2$. We compute

$$\begin{aligned}f((1,1)+h) =& f(1,1) + \partial_x f(1,1)h_1 + \partial_y f(1,1)h_2 + \frac{1}{2}\partial_x^2 f(1,1)h_1^2 +\\ &\partial_x\partial_y f(1,1)h_1h_2 + \frac{1}{2}\partial_y^2 f(1,1)h_2^2 + o(\|h\|^2) =\\ &1 + ah_1 + (1-a)h_2 + \frac{a(a-1)}{2}(h_1^2 - 2h_1h_2 + h_2^2) + o(\|h\|^2).\end{aligned}$$

Exercise 5. We calculate the second order Taylor polynomial of the function $f : \mathbb{R}^3 \to \mathbb{R}$ that is defined by

$$(x,y,z) \mapsto f(x,y,z) = xe^{-y} + y + z + 1$$

at the point $(1,0,0)$. To do so, we determine both gradient and Hessian of f, getting $\nabla f(x,y,z) = (e^{-y}, -xe^{-y}+1, 1)$ and

$$D^2 f(x,y,z) = \begin{bmatrix}0 & -e^{-y} & 0\\ -e^{-y} & xe^{-y} & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Pick now $h = (h_1, h_2, h_3) \in \mathbb{R}^3$. Then we get

$$
\begin{aligned}
f((1,0,0)+h) = & f(1,0,0) + \partial_x f(1,0,0)h_1 + \partial_y f(1,0,0)h_2 + \partial_z f(1,0,0)h_3 + \\
& \partial_x\partial_y f(1,0,0)h_1h_2 + \frac{1}{2}\partial_y^2 f(1,0,0)h_2^2 + o(\|h\|^2) = \\
& 2 + h_1 + h_3 - h_1h_2 + \frac{1}{2}h_2^2 + o(\|h\|^2).
\end{aligned}
$$