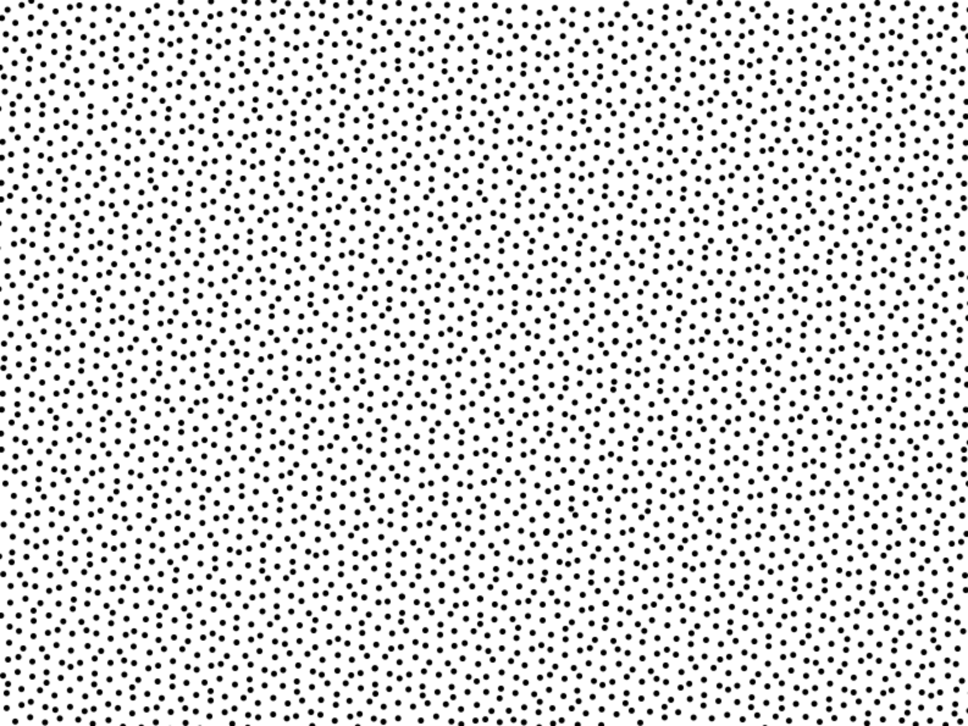
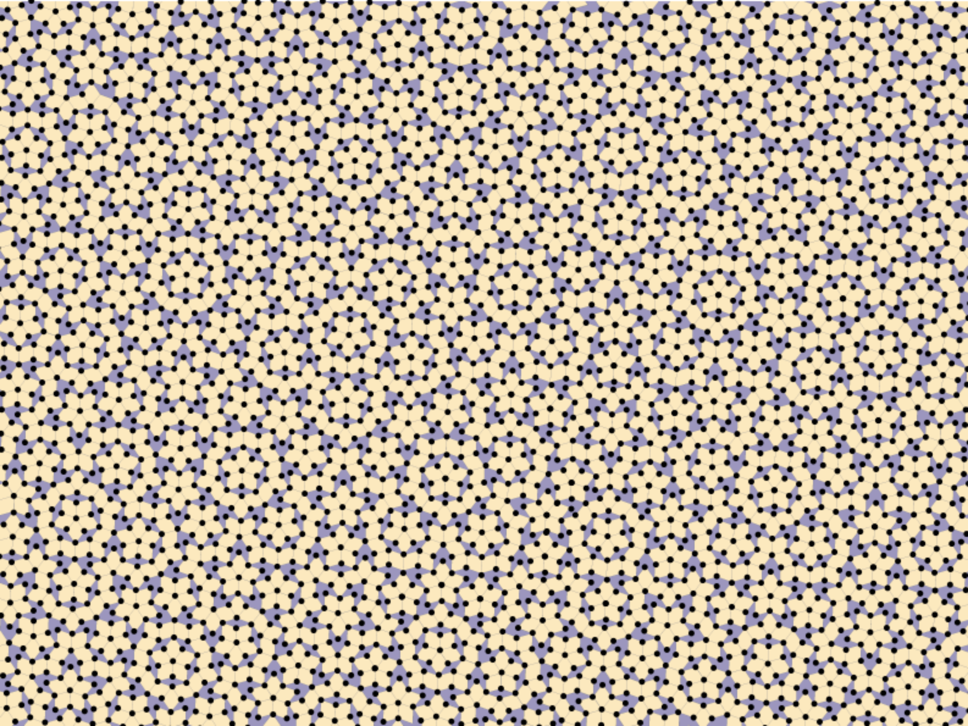
The strange topology of aperiodic tilings and their cohomology: Grout

Dan Rust & Scott Balchin

University of Leicester

- Dan Shechtman won the 2011 Nobel Prize in Chemistry
- Not a periodic lattice – Has no global translational symmetries
- Does have ‘long range order’ and ‘local symmetries’ on arbitrarily scales
- Lattices are classified by their symmetry group (and their quotient orbifolds)
- We want to go the other way
- Form the moduli space which parametrises ‘isomorphic’ aperiodic crystals and then assign a group which measures the symmetry of the space





Outline

1. Tiling spaces
   - Substitutions
   - Tiling Spaces
2. Čech Cohomology
   - Properties of Čech Cohomology
   - Proof
   - Examples

- $\mathcal{A}$ - alphabet
- $\mathcal{A}^* = \bigcup_{n\geq 1} \mathcal{A}^n$ - set of words
- $\phi\colon \mathcal{A} \to \mathcal{A}^*$ - substitution
- M_ϕ - incidence matrix

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$$\phi^n(1) = 1011010110110101101\ldots$$

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The *discrete hull* Σ_ϕ of ϕ is the set of all bi-infinite sequences $T \in \mathcal{A}^{\mathbb{Z}}$ such that if w is a subword of T, then $w \in \mathcal{L}$.

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$$\Xi_\phi = \{T \in \mathcal{A}^{\mathbb{Z}} \mid \text{ every subword of } T \text{ is in the language of } \phi\}$$

Substitution Tiling Spaces

Shift map - $\sigma\colon \Sigma \to \Sigma$ is a homeomorphism.

Definition

The *tiling space* Ω of ϕ is the mapping cylinder of σ

$$\Omega = (\Sigma \times [0,1])/(T,1) \sim (\sigma(T),0)$$

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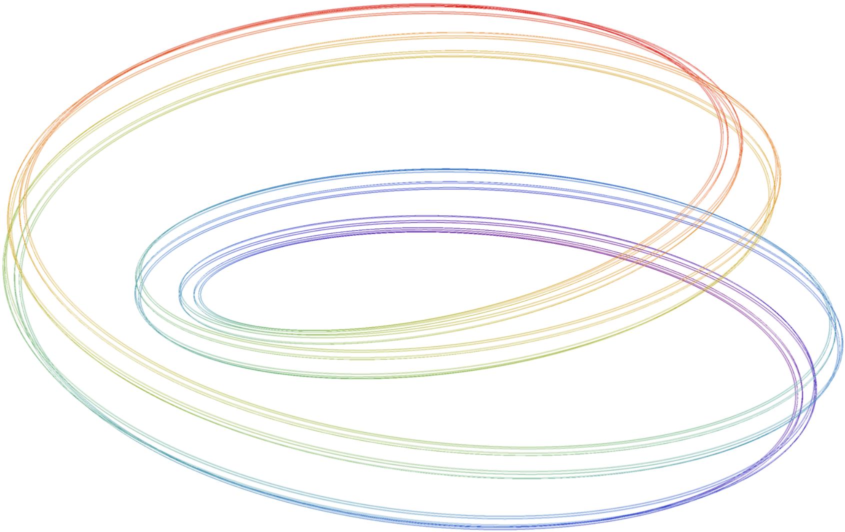
Proposition (Tiling spaces are weird)

For "most" σ

- *Σ is a σ-minimal Cantor set (σ-orbit of every point is dense)*
- *Ω is a connected, non-locally connected, non-path connected, compact metric space, and all homotopy groups and singular (co)homology groups independent of ϕ*
- *Ω is a non-homogeneous circle bundle with Cantor fiber*



Čech cohomology

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Let X be a topological space. The functor $\check{H}^\bullet : \mathbf{Top} \to \mathbf{Ab}$ satisfies

- $\check{H}^\bullet$ *is a cohomology theory*
- $\check{H}^\bullet(X) = H^\bullet_{sing}(X)$ *for X homotopy equivalent to a CW complex*
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Let X be the dyadic solenoid.

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We want to calculate the Čech cohomology of tiling spaces.

Remark

- Sturmian sequences (incl. Fibonacci) – $\check{H}^1 = \mathbb{Z}^2$
- Thue-Morse $(0 \mapsto 01,\ 1 \mapsto 10)$ – $\check{H}^1 = \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]$
- Period Doubling $(0 \mapsto 00,\ 1 \mapsto 10)$ – $\check{H}^1 = \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]$
- Maloney – R. $(a \mapsto ab,\ b \mapsto abbbb)$ – $\check{H}^1 \subsetneq \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}[\frac{1}{3}]$
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Theorem (R. '15)

Fix a system (F, s). Under reasonable conditions, the Čech cohomology $\check{H}^1$ of $\Omega_{F,s}$ fits into an exact sequence

$$0 \to \mathbb{Z}^l \to \varinjlim M_{s_i}^T \to \check{H}^1 \to \mathbb{Z}^k \to 0$$

Idea of Proof

- Modification of tools appearing in Barge & Diamond ('08)
- Given (F, s)
- Build sequence of systems $(F, \sigma^n(s))_{n\geq 0}$. [Forget the last n substitutions]
- Each has a language $\mathcal{L}_{F,\sigma^n(s)} = \mathcal{L}_n$
- Build oriented graph G_n from 1 and 2 letter words in $\mathcal{L}_n$
- Induced substitution maps

$$G_0 \xleftarrow{\phi_{s_0}} G_1 \xleftarrow{\phi_{s_1}} G_2 \xleftarrow{\phi_{s_2}} \cdots$$

- Identify $\Omega_{F,s}$ " $\cong$ " $\varprojlim(G_n, \phi_{s_n})$ – (*Shape equivalence, not homeomorphism, but good enough for cohomology*)
- Use LES of relative cohomology

Toy Example for BD Complex

Example (Fibonacci)

$$\phi_0(a) = b, \quad \phi_0(b) = ba$$

$$F = \{\phi_0\}, \quad s = (\phi_0, \phi_0, \ldots)$$

$$\ldots babbababbabba \ldots$$

$$\mathcal{L}^1(n) = \{a, b\}, \quad \mathcal{L}^2(n) = \{ab, ba, bb\}$$

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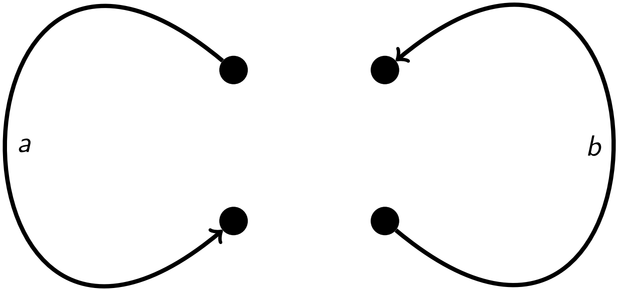

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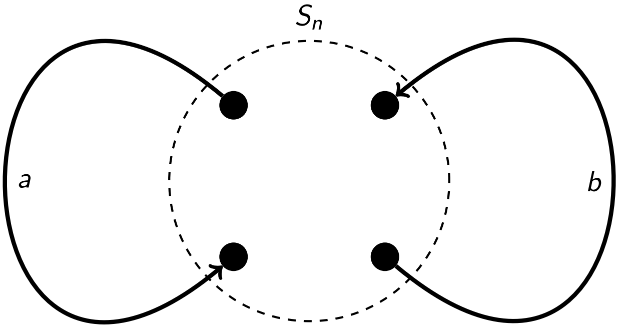

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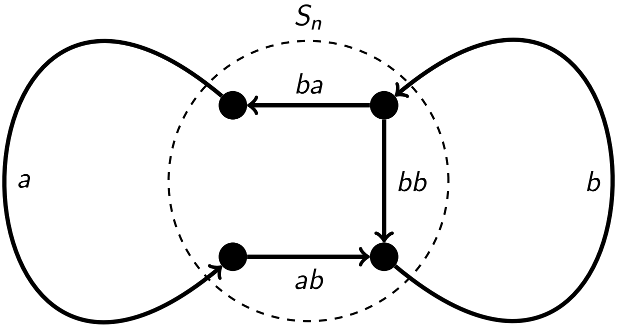

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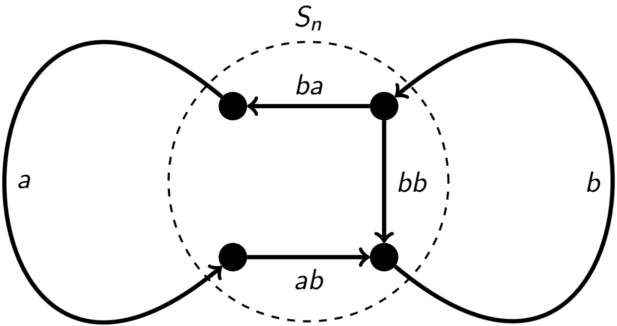

$$\phi_0 : \begin{cases} [a] \mapsto [b], & [b] \mapsto [b][ba][a] \\ [ab] \mapsto [bb], & [ba] \mapsto [ab], \quad [bb] \mapsto [ab] \end{cases}$$

$\mathcal{A} = \{a, b, c\}$