Quasicrystals, hierarchies and entropy

Dan Rust

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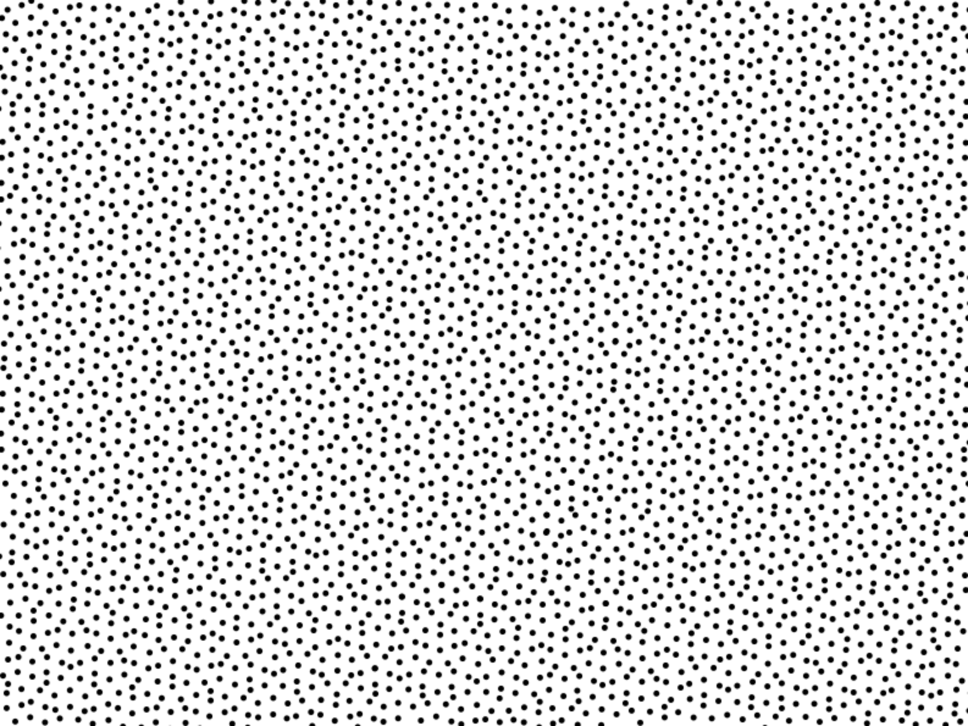

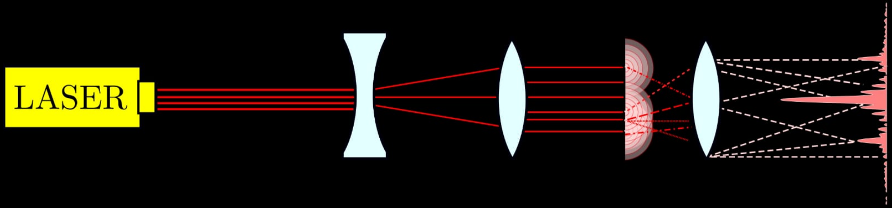
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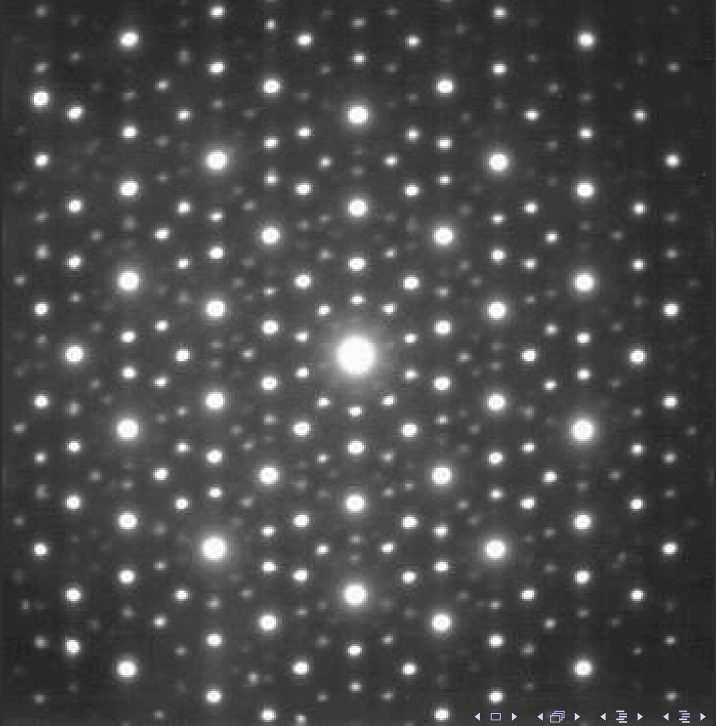

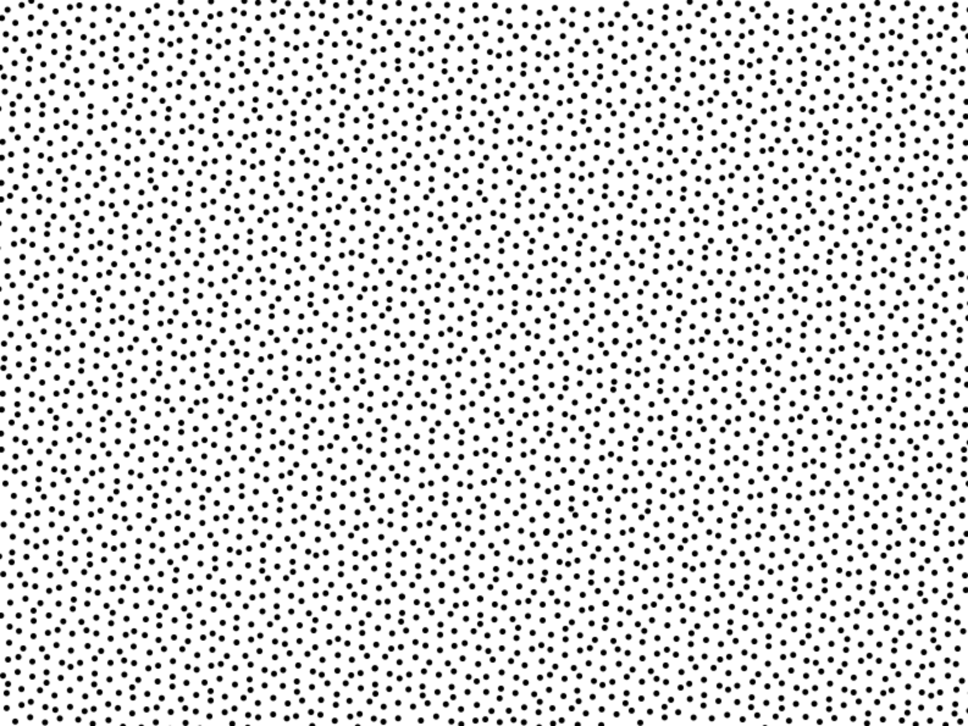

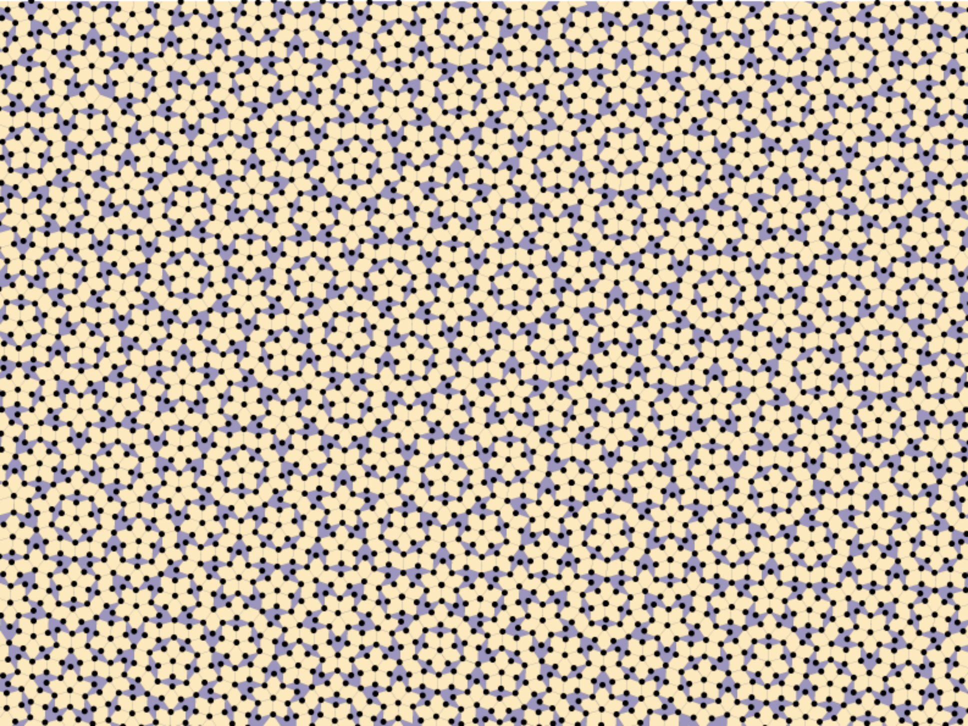

Main research interests

Mathematical Quasicrystals

Point sets/tilings in $\mathbb{R}^n$ without translational symmetry.

Aspects I study:

- Methods for constructing quasicrystals

  substitution method, cut-and-project method, random tilings, local matching rules, ...

- Notions of order and disorder

  dynamics, entropy, diffraction spectrum, limit periodicity, symmetry groups, topological features, ...

- Computational methods for calculating characteristic properties of quasicrystals.

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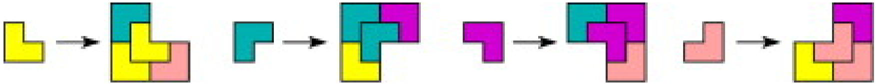

(Chair substitution)

Substitution tilings

By repeating the substitution, (in the limit) we get a **substitution tiling**.

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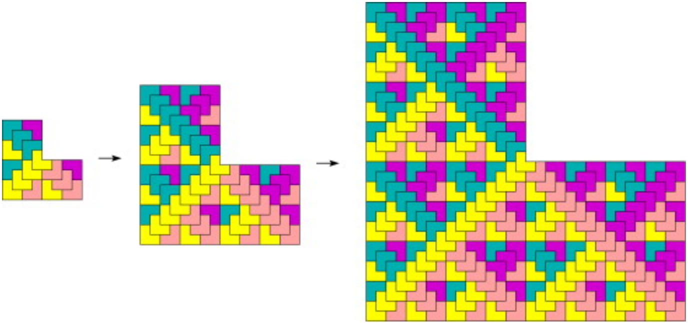

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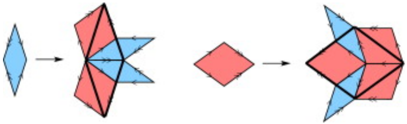

(Penrose substitution)

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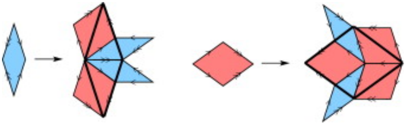

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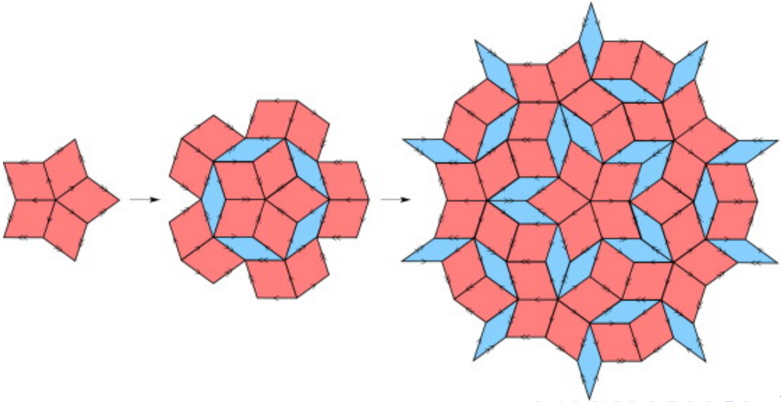

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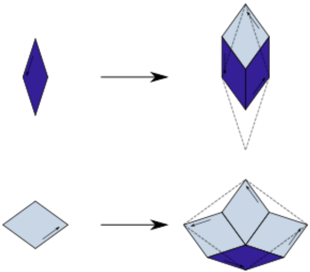
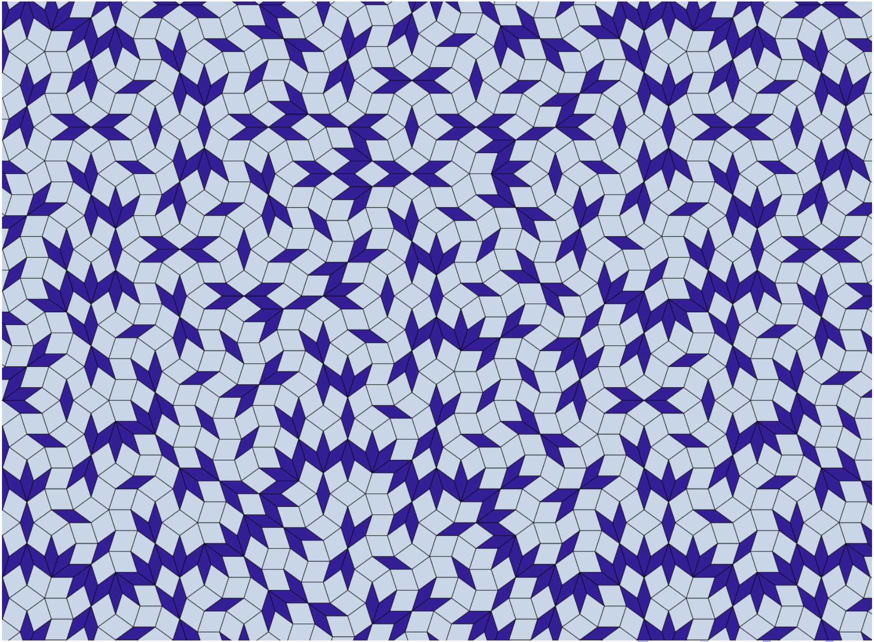
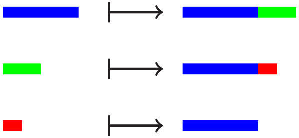


(Godrèche–Lançon–Billard substitution)

Substitution tilings



(Tribonacci substitution)



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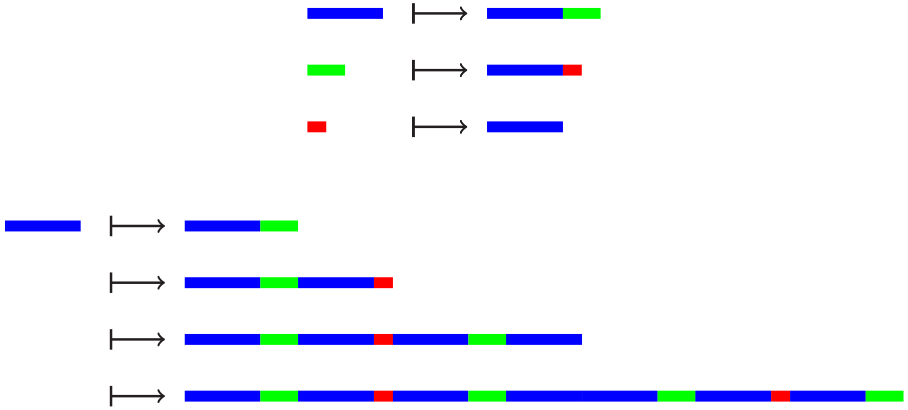

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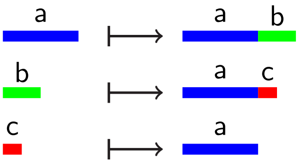

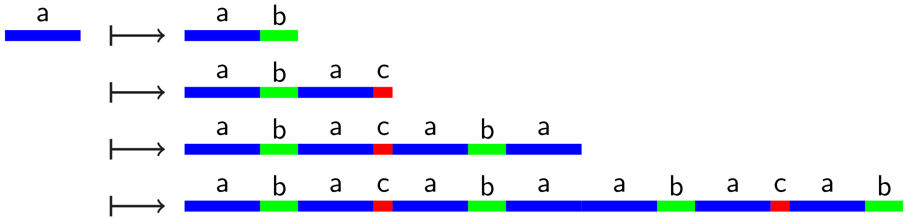

Symbolic dynamics

- $\mathcal{A} = \{a_1, a_2, \ldots, a_d\}$ — **Finite alphabet** on d letters
- $\mathcal{A}^{\mathbb{Z}} = \{\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots \mid x_i \in \mathcal{A}\}$ — **Full shift** on $\mathcal{A}$, two sequences x, y are 'close' if they agree on a large central subword

$$x_{[-n,n]} = y_{[-n,n]}$$

- $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \colon x_i \mapsto x_{i+1}$ — **Left shift** map, homeomorphism
- $X \subseteq \mathcal{A}^{\mathbb{Z}}$, closed, σ-invariant — **Subshift**
- We want to study the discrete dynamical system (X, σ)

$$\tau\colon \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$

$$w = \cdots abacabaabacababacabaabac.abacabaabacababacabaabacabaca\cdots$$

$$X_\tau := \overline{\{\sigma^n(w) \mid n \in \mathbb{Z}\}}$$

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Properties of X_τ: • Cantor set • aperiodic • uniquely ergodic • minimal • zero topological entropy • discrete dynamical spectrum • discrete diffraction spectrum • C-balanced

Last three properties are specific to tribonacci and depend on statistical properties of τ.

In practice, we go the other way.

Symbolic substitution $\longrightarrow$ Geometric substitutions

Question: How do we find the correct tile lengths?

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Symbolic substitution $\longrightarrow$ Geometric substitutions

Question: How do we find the correct tile lengths?
Answer: Abelianise the substitution.

$$\tau\colon \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$

Define M by $m_{ij} = [\text{number of } j\text{s in } \tau(i)]$

$$M_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Use square brackets $[u]$ to represent abelianisation of the word u.

Ex. $[abaacab] = (4, 2, 1)^T$.

Then

$$[\tau^n(u)] = M^n[u].$$

Let $l_i > 0$ to be the geometric length of the ith tile and $\mathbf{L} = (l_1, l_2, l_3)$. The geometric length of u is therefore $\mathbf{L}[u] = \langle \mathbf{L}, [u] \rangle$.

The lengths are consistent with a geometric substitution if and only if there is an expansion factor $\lambda > 1$ such that $\mathbf{L}[\tau(u)] = \lambda \mathbf{L}[u]$ for any word u.

Hence, $\mathbf{L}M[u] = \mathbf{L}\lambda[u]$ for every u, and so λ is an eigenvalue with eigenvector $\mathbf{L}$.

Thankfully, if M (or M^k, $k \geq 1$) has positive entries, then we can use the Perron–Frobenius theorem. We call τ *primitive* if $M^k > 0$ for some $k \geq 1$.

Proposition

If τ is a primitive substitution, then

- *M has a real eigenvalue $\lambda_{PF} > 1$ such that $|\lambda| < \lambda_{PF}$ for all other eigenvalues λ,*
- *There is a unique left eigenvector $\mathbf{L} > 0$ for λ_{PF} and $\mathbf{L}$ encodes the geometric lengths of the tiles.*
- *There is a unique right eigenvector $\mathbf{R} > 0$ for λ_{PF} and $\mathbf{R}$ encodes the relative frequencies of letters.*

Rauzy fractals

Let w be an infinite word with well-defined letter frequencies $\mathbf{R}_i$. Then w is C-**balanced** if there is a $C > 0$ such that for all subwords u,

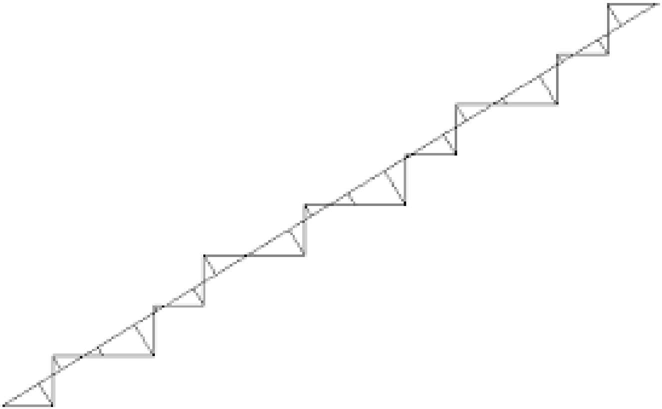
$$\big||u|_{a_i} - \mathbf{R}_i|u|\big| \leq C.$$

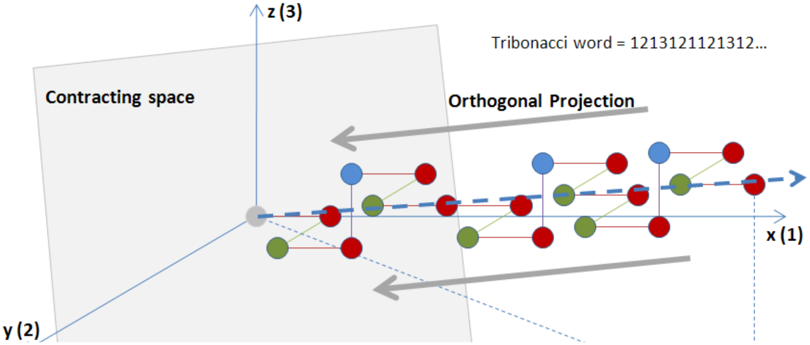
That is, the number of a_is in u never deviates more than C from the expected number.

If w is C-balanced, then the lattice vectors

$$[w_1], [w_1 w_2], [w_1 w_2 w_3], \ldots$$

all remain within a small neighbourhood of the ray spanned by the frequency vector $\mathbf{R}$.

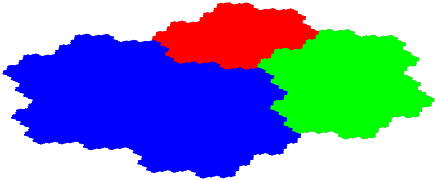




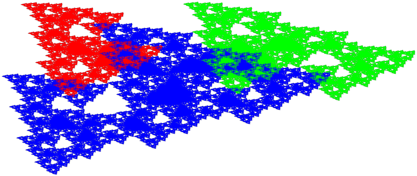
[*Shameless stolen from wikipedia*]

Define $\mathcal{R}(\tau) = \mathcal{R}(w) := \overline{\{\mathrm{proj}_{\mathbf{R}}(w_{[0,n]}) \mid n \geq 0\}}$, the **Rauzy fractal** of w.

$$\tau : \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases} \qquad \tilde{\tau} : \begin{cases} a & \mapsto & {\color{red}ba} \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$



Tribonacci



Twisted Tribonaci

Rauzy fractals for substitutions

- $\mathcal{R}(\tau)$ is a compact subset of $\mathbb{R}^{d-1}$ equal to closure of its interior. Not always connected.
- For $w, w' \in X_\tau$, $\mathcal{R}(w) = \mathcal{R}(w') + \mathbf{x}$ so $\mathcal{R}(\tau)$ makes sense.
- $\mathcal{R}(\tau)$ is the unique attractor of an associated GIFS.
- Properties of $\mathcal{R}(\tau)$ correspond to properties of τ and X_τ.

$\mathcal{R}(\tau)$	X_τ
Tiles the plane, or equiv.	Discrete spectrum, equiv.
Markov partition of dual $\mathbb{T}$ auto.	Measure iso. with $\mathbb{T}$ translation

- Useful tools for studying *Pisot substitutions* (λ_{PF} is a Pisot number).
- **Pisot conjecture**: τ (irreducible) Pisot $\implies$ $\mathcal{R}(\tau)$ tiles the plane.
- If true, then $\mathcal{R}(\tau)$ is a *window* for a cut-and-project scheme whose model sets are exactly the geometric realisations of elements in X_τ .

Random substitutions

Now, letters have choices for how they are substituted.

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Locally mix tribonacci with twisted tribonacci (Random tribonacci)

$$\tau\colon \begin{cases} a & \mapsto & \{ab, ba\} & \quad \text{with probabilities } (p, 1-p) \\ b & \mapsto & \{ac\} \\ c & \mapsto & \{a\} \end{cases}$$

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Choices are independent for each letter:

$$a \mapsto ab \mapsto \overbrace{ba}^{\tau(a)}\ ac \mapsto ac\ \overbrace{ab}^{\tau(a)}\ \overbrace{ba}^{\tau(a)}\ a \mapsto baabaacacabba \mapsto \cdots$$

$$w = \cdots baabaacacabba.acababacbaababaaabacacab \cdots$$

$$X_\tau := \overline{\{\sigma^n(w) \mid n \in \mathbb{Z}, w \text{ generic}\}}.$$

Results

Theorem (R., Spindeler '18)

Properties of X_τ for primitive τ:

- *Cantor set*
- *Either no periodic points or periodic points are dense*
- *Uncountably many minimal components*
- *Uncountably many invariant prob. measure*
- *Canonical measure μ_p induced by probabilities*
- *Almost all orbits are dense (in particular topologically transitive)*
- *Positive topological entropy (with bounds given by* [Gohlke, '19]*)*

Even though X_τ has positive topological entropy, we can still get long-range correlations. Allows us to form **entropic quasicrystals**.

Theorem (Baake, Spindeler, Strungaru, '18. Godrèche, Luck, '89)

For random Fibonacci $\vartheta_{Fib}\colon a \mapsto \{ab, ba\}, b \mapsto \{a\}$,

- *The diffraction spectrum has a non-trivial pp component (long-range correlations) and non-trivial ac component (pos. entropy).*
- *Eigenfunctions are continuous on a set of full measure (very close to having discrete dynamical spectrum).*

[For the experts] BSS used the fact that the geometric realisation of $w \in X_{\vartheta_{Fib}}$ is a relatively dense subset of a regular cut-and-project set. The window is the union of the windows for deterministic Fibonacci and its reverse with the same C+P scheme. Seems to hold for general Pisot random substitutions. Potentially very powerful method.

Open questions

Just a few of the questions still to be fully tackled

- What kinds of subshifts can appear as RS-subshifts? - already know that we can get all mixing SFTs [Gohlke, R., Spindeler, '19])
- Can we classify when $\mathrm{Per}(X_\tau) = \varnothing$? - Some criteria [R., 19']
- When $\mathrm{Per}(X_\tau) \neq \varnothing$, how many periodic points of period p are there?
- Are 'suitably nice' RS-subshifts intrinsically ergodic? (unique mme)
- Essentially nothing has been done in $\mathbb{R}^2$ or higher.

Let τ be a primitive random substitution (+ mild technical conditions). Write $\lambda_{PF} > |\lambda_2| \geq \cdots \geq |\lambda_d|$.

Theorem (Miro, R., Sadun, Tadeo '20)

(1) If $|\lambda_2| < 1$, then X_τ is not top. mixing.

(2) X_τ top. mixing $\implies$ $\gcd\{|\tau^k(a)| : a \in \mathcal{A}\} = 1, \forall k \geq 1$.

(3) If $|\lambda_2| > 1$ and $\#\mathcal{A} = 2$, then '$\Longleftarrow$' holds also.

(4) If $|\lambda_2| = 1$, then both can happen. [Dekking–Keane '78]

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- In the deterministic case, (1) is much simpler because

$$|\lambda_2| < 1 \quad \implies \quad \text{not top. weak mixing} \quad \overset{*}{\implies} \quad \text{not top. mixing}$$

but $*$ requires X_τ to be minimal.

- The proof of (3) required developing a new classification of periodic 2-letter substitutions.

Open: Classify mixing when (i) $|\lambda_2| > 1$, $\#\mathcal{A} > 2$ (ii) $|\lambda_2| = 1$.

Why λ_2?

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The length of the word $\tau^n(a_j)$ is given by

$$|\tau^n(a_j)| = \sum_{i=1}^{d} \langle e_i, M^n e_j \rangle$$

and the number of times a_i appears in $\tau^n(a_j)$ is $|\tau^n(a_j)|_{a_i} = \langle e_i, M^n e_j \rangle$. To make things simple, suppose M is diagonalisable. Then,

$$e_j = C_j \mathbf{R} + \sum_{k=2}^{d} c_{kj} v_k \implies M^n e_j = \lambda_{PF} C_j \mathbf{R} + O(\lambda_2^n)$$

$$\implies |\tau^n(a_j)| = \lambda_{PF}^n C_j \mathbf{R}_i + O(\lambda_2^n)$$

$$\implies \left| |\tau^n(a_j)|_{a_i} - |\tau^n(a_j)| \mathbf{R}_i \right| = |\lambda_{PF}^n C_j \mathbf{R}_i - \lambda_{PF}^n C_j \mathbf{R}_i| + O(\lambda_2^n).$$

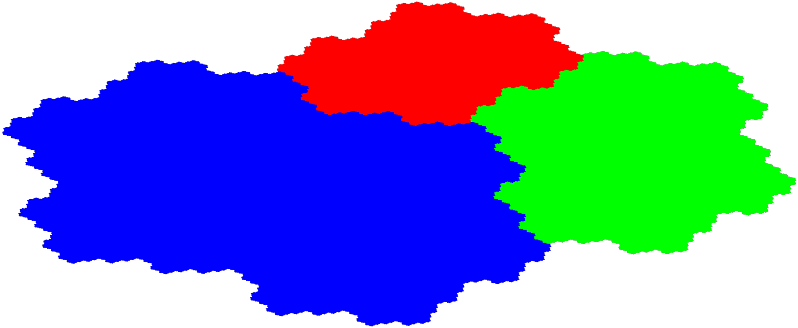
So, λ_2 regulates how far letter-counts can deviate from expected counts.

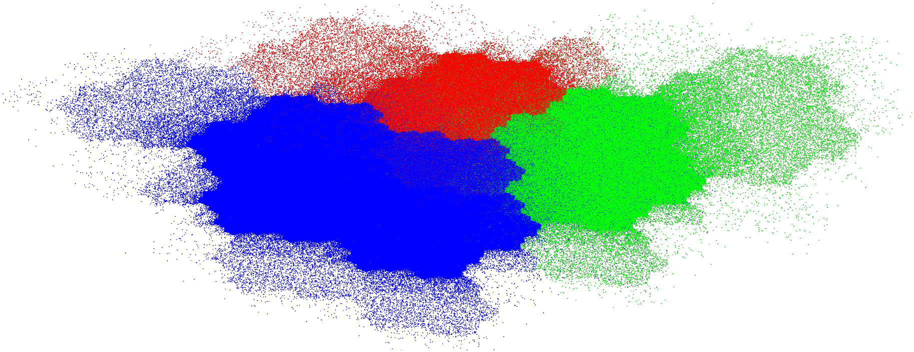
Theorem (Miro, R., Sadun, Tadeo '20)

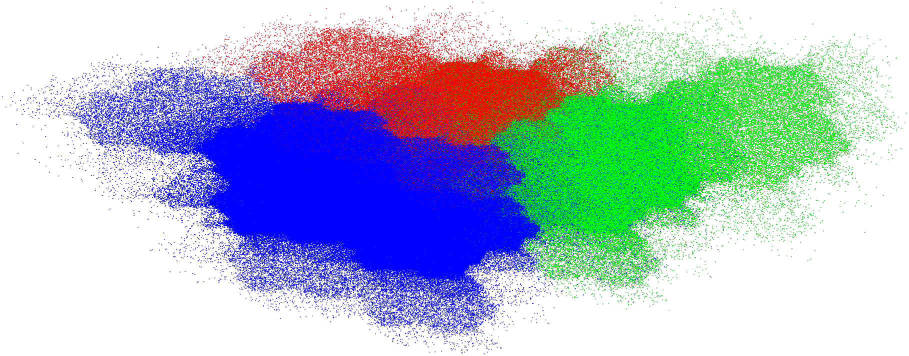
- *If $|\lambda_2| < 1$, then X_τ is C-balanced.*
- *If $|\lambda_2| > 1$, then X_τ is not C-balanced.*
- *If $|\lambda_2| = 1$, then both can happen (i.e., "M is not enough").*

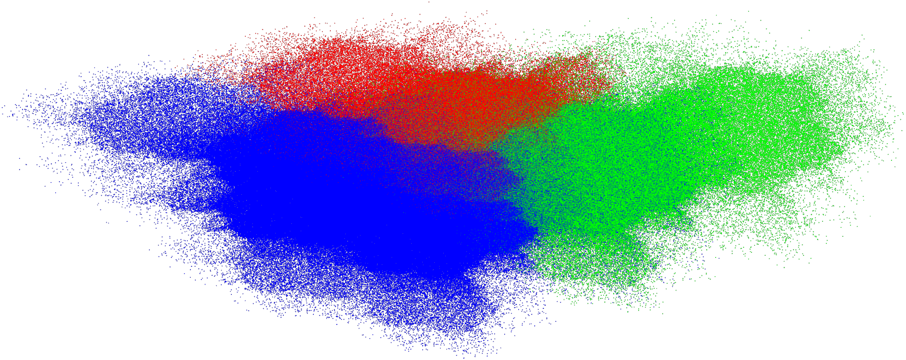
Open: Classify C-balancedness when $|\lambda_2| = 1$.

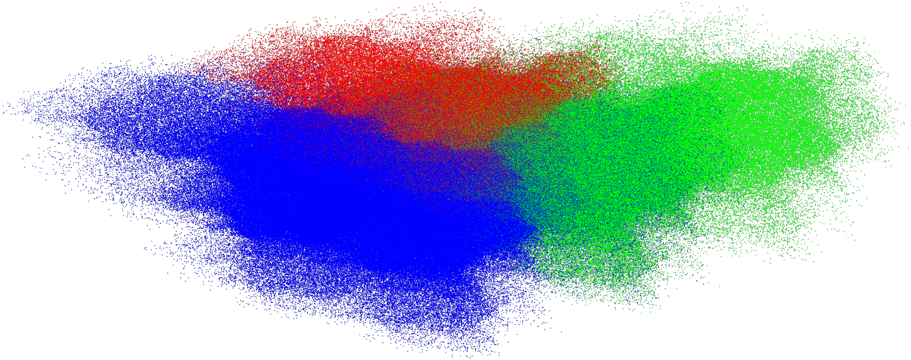
(With T. Samuel in Birmingham)
We can therefore generate approximate Rauzy fractals $\mathcal{R}(\tau, p)$.

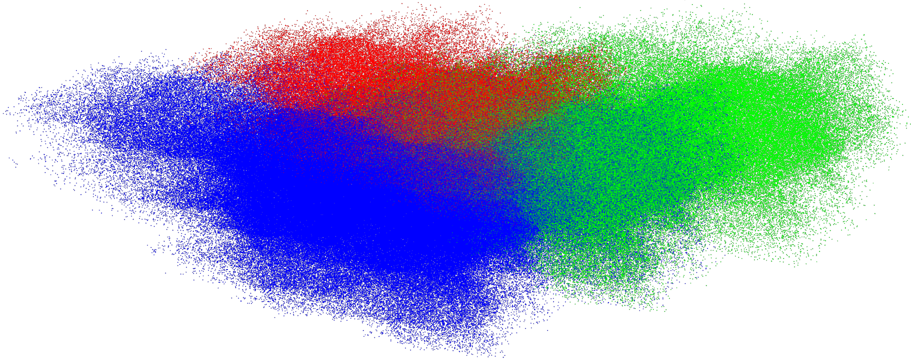


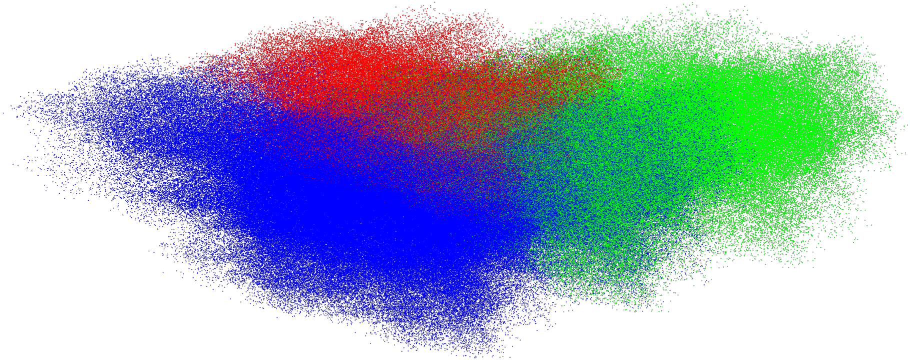


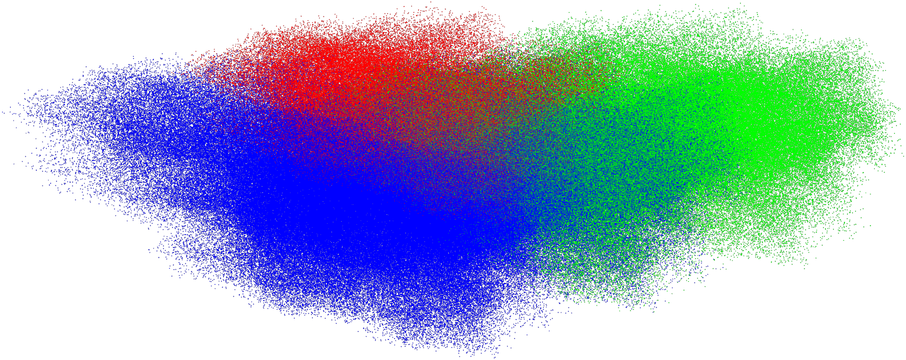


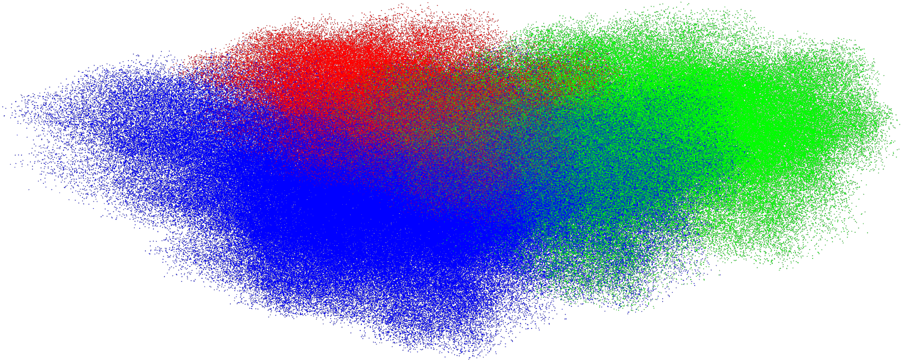


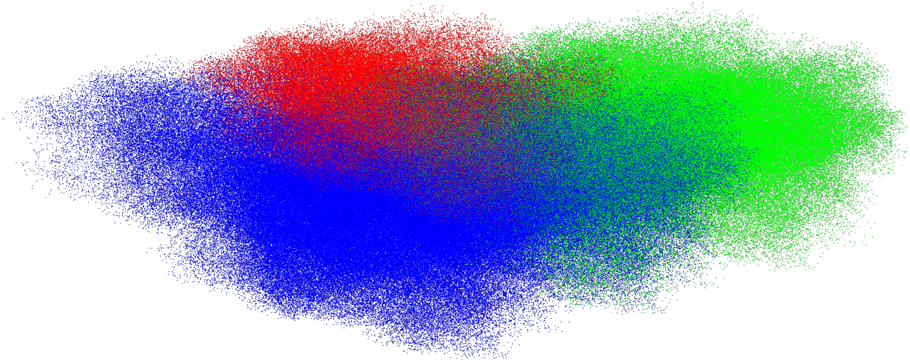


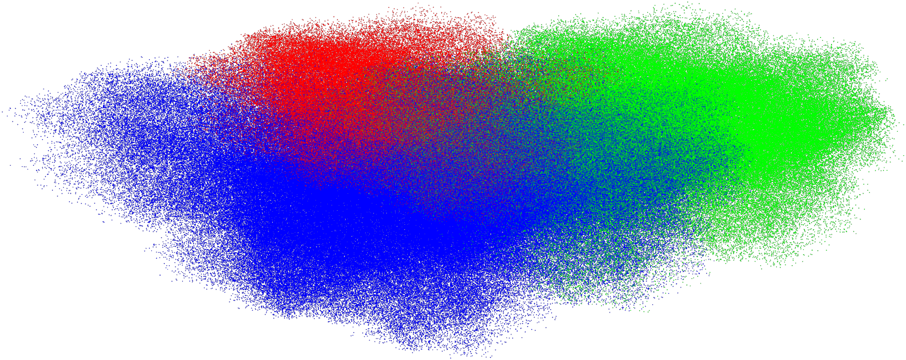


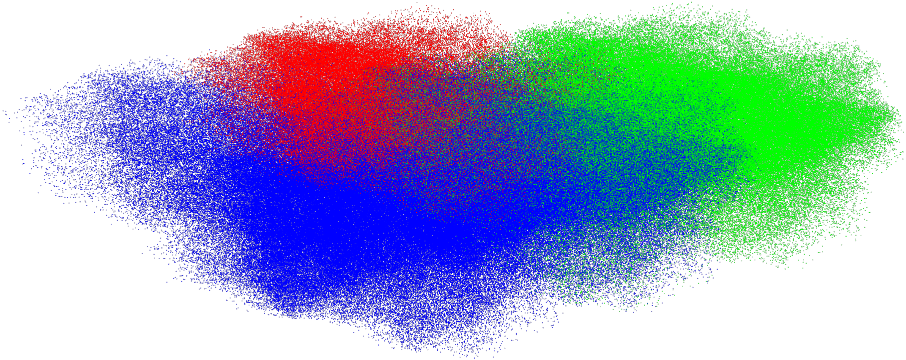


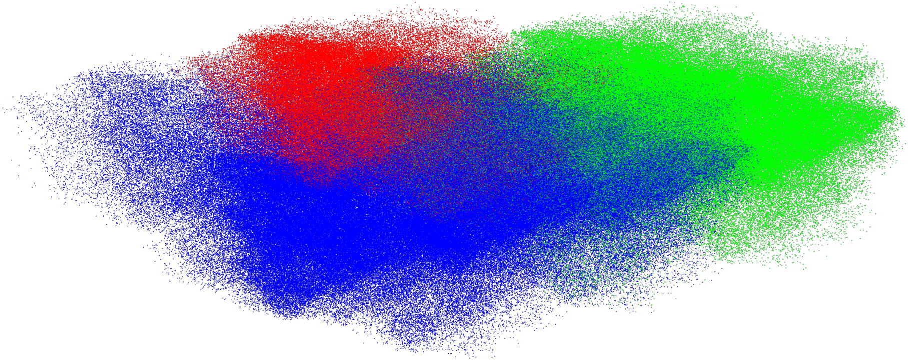


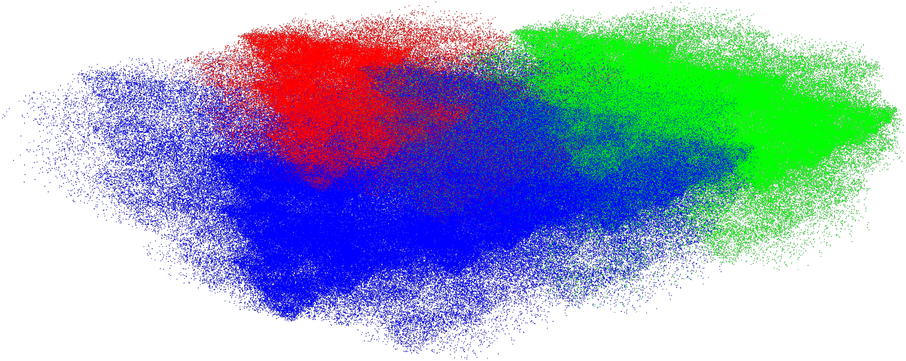


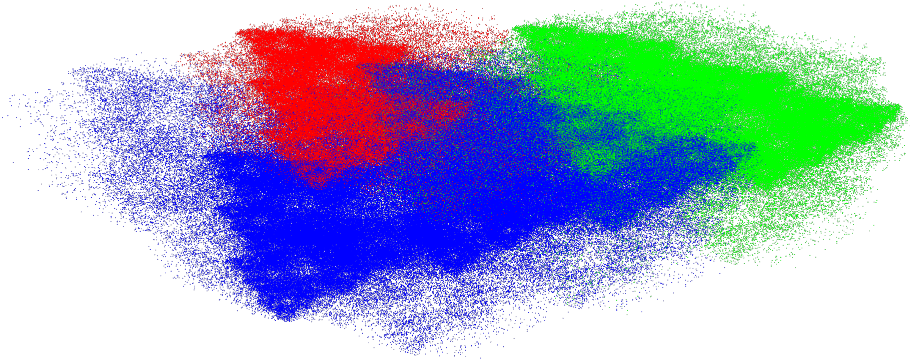


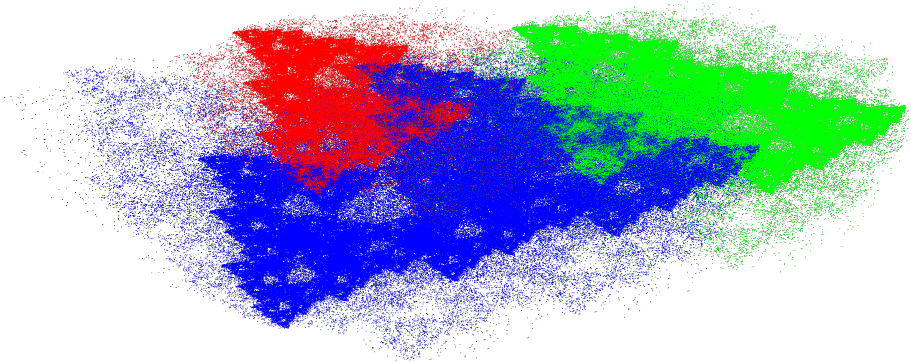


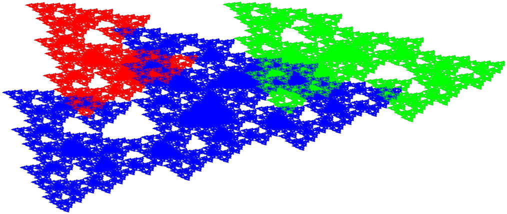


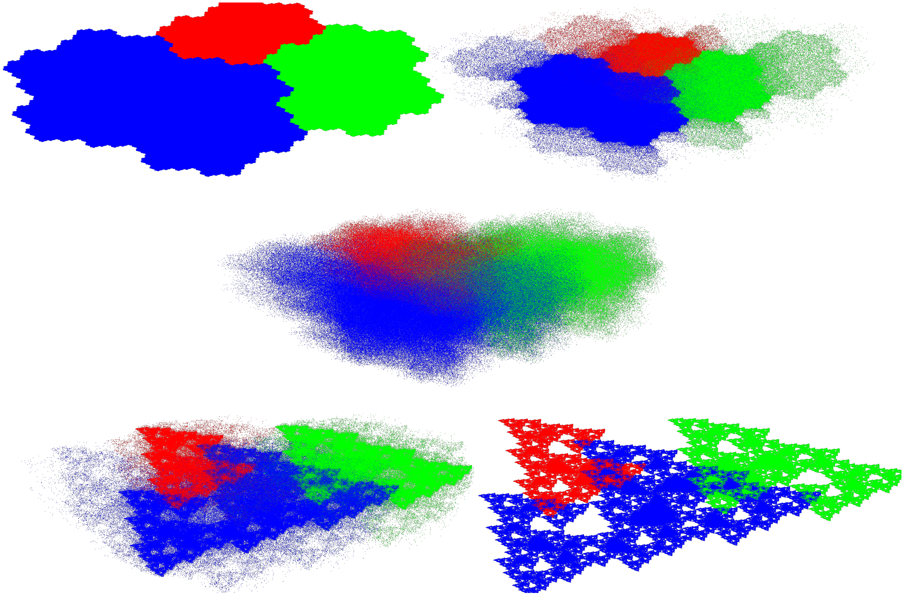












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• Difference in images is therefore indicating the differences of the induced measures on $\mathcal{R}(\tau, p)$ as p changes.

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- Topologically, $\mathcal{R}(\mathtt{\tau}, p)$ does not depend on p.
- Difference in images is therefore indicating the differences of the induced measures on $\mathcal{R}(\mathtt{\tau}, p)$ as p changes.
- No longer the attractor of a GIFS, but rather a ‘Galton–Watson’ GIFS.
- Potentially offers a new approach to the Pisot conjecture:
  - Construct $\tilde{\tau}$ which is of ‘Barge type’ and $M_\tau = M_{\tau'}$ (always exists).
  - Construct random substitution $\mathtt{\tau}$ which is a local mixture of τ and $\tilde{\tau}$.
  - We know that $\mathcal{R}(\tilde{\tau})$ tiles the plane [Barge, '16].
  - Show that tilability of $\mathcal{R}(\mathtt{\tau})$ (or an analogous property) is invariant as p ranges smoothly from 1 to 0.
  - Conclude that $\mathcal{R}(\tau)$ tiles the plane.

We'll see!


Thank you!