Dynamics of stochastic substitution subshifts

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Subshifts

$\mathcal{A} = \{0, \ldots, n-1\}$ – Alphabet. $\mathcal{A}^{\mathbb{Z}}$ – full (two-sided) shift.
$\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ – shift map.
A (two-sided) *subshift* X is a closed, shift invariant subspace of the full shift $\mathcal{A}^{\mathbb{Z}}$. Generally define X in one of a few ways:

- **Forbidden words** – $X_{\mathcal{F}} = \{w \in \mathcal{A}^{\mathbb{Z}} \mid u \subset w \implies u \notin \mathcal{F}\}$.
  Example: shift of finite type – $|\mathcal{F}| < \infty$
- **Language** – $X_{\mathcal{L}} = \{w \in \mathcal{A}^{\mathbb{Z}} \mid u \subset w \implies u \in \mathcal{L}\}$.
  Example: substitution ϕ – $\mathcal{L} = \{u \mid \exists n \geq 1, i \in \mathcal{A} \text{ s.t. } u \subset \phi^n(i)\}$.
- **Orbit closure** – $X_A = \overline{\{\sigma^n(w) \mid w \in A\ subset \mathcal{A}^{\mathbb{Z}}, n \in \mathbb{Z}\}}$.
  Example: $A = \{w = \cdots 000111 \cdots\}$ –
  $X_A = \mathcal{O}w \cup \{\cdots 000000 \cdots, \cdots 111111 \cdots\}$.
- $\cdots$

Substitutions

Substitutions (deterministic)

$\phi\colon \mathcal{A} \to \mathcal{A}^+$ – assign a finite non-empty word to every letter in $\mathcal{A}$.
Example: [*Period doubling substitutions*]

$$\phi\colon 0 \mapsto 01, 1 \mapsto 00, \qquad M_\phi = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

Iterate

$$0 \mapsto 01 \mapsto 0100 \mapsto 01000101 \mapsto 0100010101000100 \mapsto \cdots$$

$$w = \cdots 01000101010001000100010101000101 \cdots$$

$X_\phi = X_{\{w\}}$ or
$X_\phi = X_{\mathcal{L}}$ where $\mathcal{L} = \{u \mid \exists n \text{ s.t. } u \subset \phi^n(0)\}$.

Primitivity

Definition

A substitution ϕ is *primitive* if there exists a $k \geq 1$ such that for all $i, j \in \mathcal{A}$ the letter j appears in the word $\phi^k(j)$.

Equivalently, if there exists a $k \geq 1$ such that the matrix M_ϕ^k has positive entries.

Proposition

Let ϕ be a primitive substitution. Then:

- X_ϕ *is non-empty;*
- *Either every element of* X_ϕ *is periodic or every element is non-periodic;*
- X_ϕ *is either finite or a Cantor set;*
- $\phi(X_\phi) \subset X_\phi$ *(closed under substitution);*
- $x \in X_\phi \implies \exists n \geq 1, y \in X_\phi$ *such that* $\sigma^n(\phi(y)) = x$ *(preimages);*
- (X_ϕ, σ) *is minimal;*
- (X_ϕ, σ) *is linearly recurrent (there exists an* $L \geq 1$ *such that* $u \in \mathcal{L}^n, v \in \mathcal{L}^{Ln}$ *then* u *is a subword of* v*);*
- $h_{top}(X_\phi, \sigma) = 0$. *(zero topological entropy).*

Stochastic substitutions

$\vartheta\colon \mathcal{A} \to \mathcal{P}(\mathcal{A}^+)$ – assign a non-empty set of finite non-empty words to every letter in $\mathcal{A}$.
ϑ is *deterministic* if $|\vartheta(i)| = 1$ for all $i \in \mathcal{A}$.
For every $i \in \mathcal{A}$, $\boldsymbol{P}_i\colon \vartheta(i) \to [0,1]$ such that $\sum_{u\in\vartheta(i)} \boldsymbol{P}_i(u) = 1$.
Example: [*Random period doubling substitution*]

$$\vartheta\colon 0 \mapsto \{01, 10\}, 1 \mapsto \{00\}$$

$$\boldsymbol{P}_0(01) = p, \quad \boldsymbol{P}_0(10) = 1 - p, \quad \boldsymbol{P}_1(00) = 1$$

$$M_\vartheta = \begin{bmatrix} p + (1-p) & 2 \\ p + (1-p) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\vartheta\colon 0 \mapsto \{01, 10\}, 1 \mapsto \{00\}$$

$$0 \mapsto \{01, 10\} \mapsto \{\overbrace{0100, 1000}^{01}, \overbrace{0001, 0010}^{10}\}$$

$$\mapsto \left\{ \begin{array}{l} \overbrace{01000101, 01000110, 01001001, 10000101,}^{0100} \\ 01001010, 10000110, 10001001, 10001010, \\ \overbrace{00010101, 00010110, 00011001, 00100101,}^{1000} \\ 00011010, 00100110, 00101001, 00101010, \\ \overbrace{\cdots}^{0001}, \overbrace{\cdots}^{0010} \end{array} \right\}$$

$$\mapsto \cdots$$

Definition

$u \stackrel{\bullet}{=} \vartheta^k(v)$ if $u \in \vartheta^k(v)$. u is a *realisation* of $\vartheta^k(v)$.
$u \blacktriangleleft \vartheta^k(v)$ if u is a subword of some realisation of $\vartheta^k(v)$.

Example: [Random period doubling] $000100110 \stackrel{\bullet}{=} \vartheta^3(0)$, $010011 \blacktriangleleft \vartheta^3(0)$.

Definition

A stochastic substitution ϑ is *primitive* if $\exists k \geq 1$ such that for all $i, j \in \mathcal{A}$, $i \blacktriangleleft \vartheta^k(j)$.

Proposition

Let ϑ be a non-degenerate stochastic substitution ($0 \notin \operatorname{Im} \boldsymbol{P}_i$).
ϑ is primitive if and only if M_ϑ is primitive.

Language of ϑ is given by

$$\mathcal{L}_\vartheta = \{u \in \mathcal{A}^* \mid u \blacktriangleleft \vartheta^k(i) \text{ for some } k \geq 1, i \in \mathcal{A}\}.$$

Subshift given by $X_\vartheta = X_{\mathcal{L}_\vartheta}$.

Example

$$\vartheta \colon 0 \mapsto \{00, 01, 10, 11\}, 1 \mapsto \{00, 01, 10, 11\}$$

Question:

What is X_ϑ?

Answer:

$$X_\vartheta = \mathcal{A}^{\mathbb{Z}}$$

Example

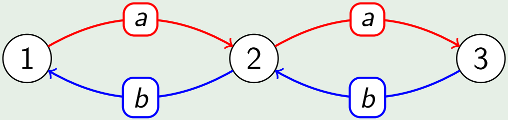
$$\vartheta\colon 0 \mapsto \{010, 0\}, 1 \mapsto \{01, 1\}$$

$$X_\vartheta = X_{\mathcal{F}=\{11\}} \quad \text{(Golden mean shift)}$$

Example

$$\vartheta\colon a \mapsto \{ab, ba\}, b \mapsto \{ab, ba\}$$

$$X_\vartheta = X_{\mathcal{G}} \quad \text{(Sofic shift)}$$



Question

- *Which shifts of finite type/sofic shifts can be realised as primitive stochastic substitution subshifts?*
- *Given a primitive stochastic substitution ϑ, can we determine if X_ϑ is topologically conjugate or semi-conjugate to a shift of finite type?*

Results [R., Spindeler (2017)]

Proposition (Deterministic + Stochastic)

- $\vartheta(X_\vartheta) \subset X_\vartheta$ *(for all realisations of* ϑ*).*
- *For all* $w \in X_\vartheta$*, there exists a* $w' \in X_\vartheta$ *such that* $w \doteq \sigma^n(\vartheta(w'))$ *for some* $n \geq 0$*.*

Proposition (Deterministic)

If ϕ is primitive, then X_ϕ is non-empty

Proposition (Stochastic)

If ϑ is primitive, then X_ϑ is non-empty if and only if $\lambda_{PF} > 1$.

Example

$$\vartheta\colon a \mapsto \{a, b\}, b \mapsto \{a\}$$

Primitive for all non-degenerate values of $\boldsymbol{P}_a$. $\lambda_{PF} = 1$. $X_\vartheta = \emptyset$.

Proposition (Deterministic)

If ϕ is primitive, then (X_ϕ, σ) is minimal.
Equivalently, every element has a dense orbit.
Equivalently, for all $w \in X_\phi$, $X_\phi = \overline{\{\sigma^n(w) \mid n \in \mathbb{Z}\}}$.

Proposition (Stochastic)

If ϑ is primitive, then (X_ϑ, σ) is topologically transitive.
Equivalently, there exists an element with dense orbit.
Equivalently, there exists an element $w \in X_\vartheta$, $X_\vartheta = \overline{\{\sigma^n(w) \mid n \in \mathbb{Z}\}}$.

Proposition (Stochastic)

If ϑ is primitive, then for all $w \in X_\vartheta$,

$$X_\vartheta = \overline{\{\sigma^n(\vartheta^k(w)) \mid n \in \mathbb{Z}, k \geq 0\}}.$$

Let $\mathrm{Per}(X)$ be the set of periodic elements in X under the shift σ. $\mathrm{Per}(X) = \{x \in X \mid \exists n \geq 1, \sigma^n(x) = x\}$.

Proposition (Deterministic)

If ϕ is primitive, then either $\mathrm{Per}(X_\phi) = \emptyset$ or $\mathrm{Per}(X_\phi) = X_\phi$.

Proposition (Stochastic)

If ϑ is primitive, then either $\mathrm{Per}(X_\vartheta) = \emptyset$ or $\mathrm{Per}(X_\vartheta) \overset{dense}{\subseteq} X_\phi$.

Example

The random period doubling subshift has periodic points

$$001 \mapsto \overbrace{10}^{0}\overbrace{01}^{0}\overbrace{00}^{1} \mapsto 001001001001$$
$$\mapsto 100100100100100100100100 \mapsto \cdots$$

Question

Find a property P such that, for a primitive substitution ϑ, $\mathrm{Per}(X_\vartheta) = \emptyset$ if and only if ϑ is P.

Proposition (Deterministic)

If ϕ is primitive, then X_ϕ is either finite or a Cantor set.

Proposition (Stochastic)

If ϑ is primitive, then X_ϑ is either finite or a Cantor set.

(If ϕ is primitive, then X_ϕ is a Cantor set if and only if ϕ is *recognisable*)

Question

Find a property P such that, for a primitive substitution ϑ, X_ϑ is a Cantor set if and only if ϑ is P.

We say w is *linearly repetitive* if there exists an $L \geq 1$ such that $u \in \mathcal{L}^n(w), v \in \mathcal{L}^{Ln}(w)$ then u is a subword of v. Let $\mathrm{Lin}(X)$ be the set of linearly repetitive elements of X.

Proposition (Deterministic)

If ϕ is primitive, then $\mathrm{Lin}(X_\phi) = X_\phi$.

Proposition (Stochastic)

If ϕ is primitive, then $\mathrm{Lin}(X_\vartheta) \overset{dense}{\subseteq} X_\vartheta$.

Let

$$h_{\text{top}}(X) = \lim_{n\to\infty} \frac{\log|\mathcal{L}^n(X)|}{n}$$

denote the topological entropy of X.

Proposition (Deterministic)

$$h_{top}(X_\phi) = 0$$

Proposition (Stochastic)

Let ϑ be primitive. If there exists a letter $i \in \mathcal{A}$ and two words $u, v \in \vartheta(i)$ such that u is either not a prefix of v or u is not a suffix of v, then

$$h_{top}(X_\vartheta) > 0.$$

Question (Conjecture)

If ϑ is primitive, then $h_{top}(X_\vartheta) > 0$ if and only if there exists a power k so that the above hypothesis holds for ϑ^k.

More questions

We have found h_{top} explicitly for some examples e.g. Random Fibonacci $(0 \mapsto \{01, 10\}, 1 \mapsto \{0\})$, Random periodic doubling, any SFT/sofic shift.

Question

Is there an effective method for calculating h_{top}?

We know how to calculate the Artin-Mazur zeta-function ζ for SFTs/sofic shifts. We don't know the zeta function of any other type of non-trivial stochastic substitution.

Question

Is there an effective method for calculating ζ_ϑ?

Question (Long-term goal)

Given two primitive stochastic substitutions ϑ, φ, is there an effective method for deciding if X_ϑ and X_φ are topologically conjugate?

Thank you!