Tilings with transcendental inflation factor

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Aperiodic Order

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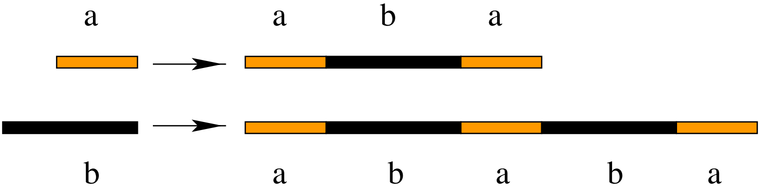
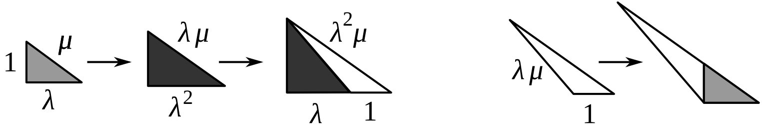
Substitution tiling in dimension $d = 1$:



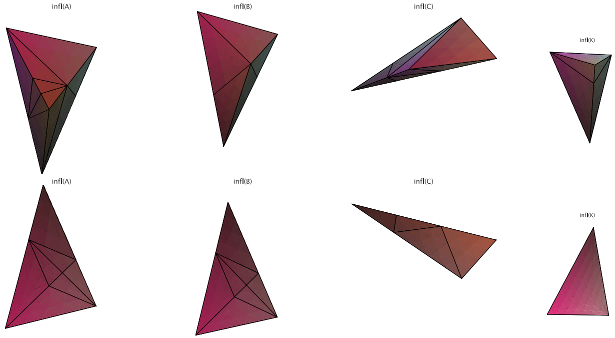
- substitution matrix $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$,
- inflation factor $\lambda = 2 + \sqrt{3}$,
- minimal polynomial $x^2 - 4x + 1$.

In dimension $d = 2$:



- substitution matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$,
- inflation factor $\lambda = 1.3247\ldots$ (the *plastic number*),
- minimal polynomial $x^3 - x - 1$.

In dimension $d = 3$:



- substitution matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}$,
- inflation factor $\lambda = \frac{1}{2}(\sqrt{5} + 1)$ (the *golden mean*),
- minimal polynomial $x^2 - x - 1$.

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What if there are infinitely many prototiles?

In most examples with infinitely many prototiles studied so far (Ferenczi, Sadun, Frank-Sadun, Smilansky-Solomon...):

- tiles of length 1, infinitely many labels, or
- no proper inflation factor

Mañibo-Rust-Walton (preprint 2022): conditions for unique ergodicity of the dynamical systems arising from substitutions in dimension $d = 1$ for infinitely many prototiles with distinct lengths.

Their example: Prototiles $0, 1, 2, 3, \ldots$ and ∞.

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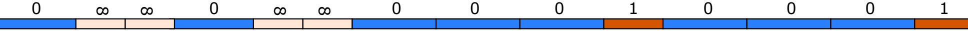

The tiles have indeed well-defined (distinct) lengths ℓ_i:

$$\ell_i = 1 + \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\right)^i,$$

and a proper inflation factor: $\lambda = 3 + \frac{1}{\sqrt{2}}$

Their substitution "matrix":

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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When we saw this example we tried to find more.

But: unlike in the finite case one cannot just turn any "matrix" into a proper substitution
(negative lengths, lengths $\to \infty$, all tile frequencies 0, ...)

There is also no simple analogue of Perron-Frobenius.

And in order to establish unique ergodicity they (Neil-Dan-Jamie) need to work a lot:

- The alphabet $\{0, 1, 2, \ldots, \} \cup \{\infty\}$ needs to be compact,
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- the symbolic substitution needs to be continuous,
- and primitive,
- but what means primitive here?

However, they solve all this.

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Theorem (F-Garber-Mañibo 2022+)

For any $\lambda > 2$ there is a primitive substitution with infinitely many prototiles having λ as inflation factor.

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Theorem (F-Garber-Mañibo 2022+)

For any $\lambda > 2$ there is a primitive substitution with infinitely many prototiles having λ as inflation factor.

Corollary

There are a lot of substitution tilings with transcendental inflation factor.

[1] that is, not algebraic

Proof: (idea, simplified) Generalize the example above:

Let $\mathbf{a} = (a_i)_i = a_0, a_1, a_2, \ldots$ with $a_i \in \{1, 2, \ldots, N\}$ for some $N \in \mathbb{Z}^+$.

$$\text{Let } A = \begin{pmatrix} a_0 & 1 + a_1 & a_2 & a_3 & a_4 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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For instance, $a_0 = 3$ and $a_i = 1$ for $i \geq 1$ is the example above.

$$\varrho_{\mathbf{a}} = \begin{cases} 0 \mapsto & 0^{a_0}\ 1 \\ i \mapsto & 0^{a_i}\ i-1\ i+1 \\ \cdots & \cdots \end{cases}$$

In order to show that this defines nice substitution tilings ("good" tile lengths and frequencies etc) we apply Mañibo-Rust-Walton:

We need to turn the set $\{0, 1, 2, \ldots\}$ (corr. to the prototiles) into a compact alphabet $\mathcal{A}$. (Amazingly sophisticated)

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...and show that

- The substitution $\varrho_{\mathbf{a}}$ is a continuous map $\varrho_{\mathbf{a}} : \mathcal{A} \to \mathcal{A}^+$,
- $\varrho_{\mathbf{a}}$ is primitive,
- $\varrho_{\mathbf{a}}$ is recognizable,
- the substitution operator (roughly, the "matrix") is quasicompact

It remains to realize all inflation factors $\lambda > 2$.

Ansatz:
Let $(a_i)_i$ be fixed, and let $\mu \in (0, \frac{1}{2}]$ be the unique number with

$$\frac{1}{\mu} = \sum_{i=0}^{\infty} a_i \mu^i$$

Claim:
$\lambda = \mu + \frac{1}{\mu}$ is an eigenvalue with eigenvector $\mathbf{v} = (1, \mu, \mu^2, \dots)^T$.

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Row by row:

- 1^{st} row: $\mu + \sum_{i=0}^{\infty} a_i \mu^i = \mu + \frac{1}{\mu} = \lambda \cdot 1.$ ✓
- i^{th} row: $\mu^{i-2} + \mu^i = (\mu^{-1} + \mu)\mu^{i-1} = \lambda \mu^{-1}.$ ✓

It follows that λ is the inflation factor (by some infinite equivalent of Perron-Frobenius: eigenvector in the positive cone), and **v** (normalized) is the vector of tile frequencies.

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- First, we don't. We need to allow $a_i = 0$.
- But to keep it simple, let us assume $a_i \neq 0$.
- Then we get all values $\lambda > \frac{5}{2}$.

Now we fix $\mu \in (0, \frac{1}{2}]$. We have to find $(a_i)_i$ such that

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Plugging it into $\varrho_{\mathbf{a}}$ yields a transcendental inflation factor $\lambda = \mu + \frac{1}{\mu}$ which we can compute (approximately).

Why?

Consider the classical Thue–Morse sequence $t_n := (-1)^{s(n)}$, where $s(n)$ is the number of ones in the binary expansion of n.

$$1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, \dots$$

Theorem (Mahler 1929)

- *Consider the generating function* $T(z) := \sum_{n \geqslant 0} t_n z^n$.
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The generating function of the 1-2-Thue-Morse sequence is

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{3}{2} \cdot \frac{1}{1-z} + \frac{1}{2} T(z)$$

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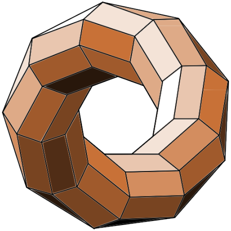

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