

PISOT SUBSTITUTION SEQUENCES, ONE DIMENSIONAL CUT-AND-PROJECT SETS AND BOUNDED REMAINDER SETS WITH FRACTAL BOUNDARY

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ABSTRACT. This paper uses a connection between bounded remainder sets in \mathbb{R}^d and cut-and-project sets in \mathbb{R} together with the fact that each one-dimensional Pisot substitution sequence is bounded distance equivalent to some lattice in order to construct several bounded remainder sets with fractal boundary. Moreover it is shown that there are cut-and-project sets being not bounded distance equivalent to each other even if they are locally indistinguishable, more precisely: even if they are contained in the same hull.

1. INTRODUCTION

Bounded remainder sets (BRS) have been studied intensely by several authors, including early works by Hecke [14], Ostrowski [20], Szűsz [25] and Kesten [18]; see for instance [10, 12] and references therein for an overview. BRS are sets $P \subset [0, 1]^d$ with uniformly bounded discrepancy with respect to some (discrete) toral rotation; see below for details. BRS have been studied in the context of Diophantine approximation, dynamical systems or quasi Monte Carlo methods.

Cut-and-project sets (CPS) and substitution sequences are (infinite) discrete point sets in \mathbb{R} . They are central objects of study in the theory of aperiodic order, see [4] and references therein. There is a strong connection between BRS and (windows of) CPS. This connection was implicit in several papers (see for instance [7, 16]) but up to the knowledge of the authors the connection was explicitly spelled out only in [11, 12].

Here we use this connection together with several other results in order to obtain many explicit examples of BRS with fractal boundary. Along the way we re-prove some of these results, namely the connection between CPS and BRS ([12] respectively Theorem 4.5), and the fact that the point sets generated by Pisot substitution are bounded distance equivalent to a lattice ([16] respectively Theorem 3.5). We also use partial answers to the Pisot conjecture (Conjecture 5.3).

Independently we prove that two CPS are bounded distance equivalent if their windows are equidecomposable by appropriate vectors (Theorem 6.1), but that there are CPS with the same data but shifted windows that are not bounded distance equivalent to each other (Theorem 6.4). This implies that bounded distance equivalence of CPS is not preserved under local indistinguishability, or more precisely: two CPS in the same hull are not necessarily bounded distance equivalent to each other.

Because our results are dealing mainly with discrete point sets in one dimension, most terminology is stated for this one-dimensional case only. Note that most objects mentioned in this paper (substitution sequences, tilings, CPS) have higher dimensional analogues. Section 6 contains some results for higher dimensions, namely, Lemma 6.2, Lemma 6.3 and Theorem 6.1.

But mainly the objects of study in this paper are discrete point sets on the line. One important example is the set \mathbb{Z} of integers, or more general: the set $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$ for some $a \in \mathbb{R} \setminus \{0\}$. Such a set $a\mathbb{Z}$ is *periodic*, that is, there is a non-zero $t \in \mathbb{R}$ such that

$t + a\mathbb{Z} = a\mathbb{Z}$ (namely, all $t \in a\mathbb{Z}$ will do). We are more interested in studying *nonperiodic* discrete point sets $\Lambda \subset \mathbb{R}$, that is, sets Λ such that $\Lambda + t = \Lambda$ implies $t = 0$. Two interesting classes of nonperiodic discrete point sets are substitution sequences and CPS. They are introduced in Section 2.

The main question motivating this paper is “when are two given discrete point sets on the line bounded distance equivalent to each other?” Roughly speaking, this means that there is a perfect matching between the point sets such that the distance between any two matched points is uniformly bounded. More precisely:

Definition 1.1. Two discrete point sets $\Lambda, \Lambda' \in \mathbb{R}^d$ are called *bounded distance equivalent* ($\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$) if there is some $c > 0$ and some bijection $\varphi: \Lambda \rightarrow \Lambda'$ such that $|x - \varphi(x)| < c$ for all $x \in \Lambda$.

Remark 1. It is straight-forward to see that bounded distance equivalence is indeed an equivalence relation.

A cut-and-project set (CPS) is a discrete point set on the line that is defined by projecting the points of some point lattice in \mathbb{R}^{d+1} that are contained in some strip $W \times \mathbb{R}$ to the line \mathbb{R} , where W is some (usually compact) subset of \mathbb{R}^d . Compare Figure 1 for an example with $d = 1$, see Section 2 for a precise explanation. It turns out that the question whether a given (nonperiodic) CPS is bounded distance equivalent to some (periodic) point lattice is strongly related to the question whether W is a bounded remainder set.

Remark 2. Throughout the paper μ denotes Lebesgue measure in the appropriate dimension.

Definition 1.2. Let $\alpha \in \mathbb{R}^d$, $W \subset [0, 1]^d$. W is a *bounded remainder set* (BRS) with respect to α , if for the *discrepancy*

$$D_n(W, x) = \sum_{k=0}^{n-1} 1_W((x + k\alpha) \bmod 1) - n\mu(W)$$

holds: there is $c > 0$ such that for all n and for almost all $x \in [0, 1[$ holds $|D_n(W, x)| < c$.

The parameter α in the definition above is some irrational number (if $d = 1$) respectively irrational vector (if $d > 1$) which in our setup depends on the point lattice defining the cut-an-project set under consideration. The x in the definition above is negligible for our purposes.

2. SUBSTITUTIONS AND CUT-AND-PROJECT SETS

A *point lattice* (or shortly *lattice*) in \mathbb{R}^d is the integer span of d linearly independent vectors. Hence a lattice in \mathbb{R} is a set $a\mathbb{Z} = \{na \mid n \in \mathbb{Z}\}$, where $a \in \mathbb{R} \setminus \{0\}$.

The *density* of a discrete point set Λ in \mathbb{R} is the average number of points of Λ per unit. It is defined as the limit $\lim_{n \rightarrow \infty} \frac{1}{2n} |\Lambda \cap [-n, n]|$, if it exists. (In fact, the definition of the density of a discrete point set is much more subtle, compare for instance [4, Section 1]. For our purposes this definition, sometimes called *central density*, is sufficient.)

The density of a lattice in \mathbb{R}^d is $\frac{1}{|\det(M)|}$, where M is the matrix whose columns are the spanning vectors of the lattice. Hence the density of a lattice $L = \{na \mid n \in \mathbb{Z}\}$ is $\frac{1}{|a|}$.

Let $\mathcal{A}_n = \{a_1, \dots, a_n\}$ denote an alphabet of n letters. Let \mathcal{A}_n^* denote all finite words over \mathcal{A}_n . A symbolic substitution is a map $\sigma: \mathcal{A}_n^* \rightarrow \mathcal{A}_n^*$ such that for all $u, v \in \mathcal{A}_n^*$ holds $\sigma(u)\sigma(v) = \sigma(uv)$. Here uv denotes the concatenation of u and v . Hence a symbolic substitution is uniquely defined by $\sigma(a_1), \dots, \sigma(a_n)$. The *substitution matrix* of

a substitution σ is the matrix $M_\sigma = (|\sigma(a_j)|_i)_{1 \leq i, j \leq n}$, where $|\sigma(a_j)|_i$ denotes the number of letters a_i in $\sigma(a_j)$. A substitution σ is called *primitive*, if there is k such that M_σ^k contains positive entries only. By the Perron-Frobenius theorem [21], the matrix M_σ of a primitive substitution σ has a unique real eigenvalue $\lambda > 0$, such that all other eigenvalues of M_σ are strictly smaller in absolute value. Furthermore, M_σ has an eigenvector v for λ with positive entries only. In the following, we refer to λ as Perron-Frobenius eigenvalue of M_σ and to v as Perron-Frobenius eigenvector of M_σ . The *symbolic hull* of σ is

$$X_\sigma := \{u \in \mathcal{A}_n^{\mathbb{Z}} \mid \text{each finite subword of } u \text{ is contained in some } \sigma^k(a_i)\}.$$

Remark 3. In general, the hull X_u of any symbolic sequence $u = (\dots, u_{-1}, u_0, u_1, \dots) \in \mathcal{A}^{\mathbb{Z}}$ is the closure of $\{S^k u \mid k \in \mathbb{Z}\}$ with respect to the metric

$$d(u, u) := 0, \quad d(u, v) := 2^{-\min\{|i| \mid u_i \neq v_i\}} \quad \text{where } v = (\dots, v_{-1}, v_0, v_1, \dots) \neq u$$

Here S denotes the shift operator $S(\dots, u_{-1}, u_0, u_1, \dots) = (\dots, v_{-1}, v_0, v_1, \dots)$ with $v_i = u_{i+1}$. If σ is a primitive substitution then $X_\sigma = X_u$ for all $u \in X_\sigma$ [9].

A tile substitution in \mathbb{R} is a collection of intervals $[0, \ell_1], \dots, [0, \ell_n]$, an *inflation factor* λ , and a rule how to partition each $\lambda[0, \ell_i]$ into translates of the original intervals $[0, \ell_1], \dots, [0, \ell_n]$. Translates of the $[0, \ell_1], \dots, [0, \ell_n]$ are called *tiles*. A tile substitution can be repeatedly applied to a single tile, covering larger and larger portions of the line. The iterate $\sigma^k([0, \ell_i])$ is called (*level- k*) *supertile* (for convenience, we will sometimes use a_i for the tile $[0, \ell_i]$ below).

Trivially, any tile substitution can be transformed into a symbolic substitution by identifying translates of intervals $[0, \ell_i]$ with letters a_i . Because of the following folklore result (see e.g. [4]) each primitive symbolic substitution can be turned into a tile substitution, too.

Theorem 2.1. *Let σ be a primitive symbolic substitution. Let λ be the Perron-Frobenius eigenvalue of M_σ , and let $w = (\ell_1, \dots, \ell_n)$ be a left eigenvector of M for λ . Identifying a_i with $[0, \ell_i]$ yields a tile substitution σ with inflation factor λ .*

The ℓ_i in the result above are called *natural tile lengths*. The choice is unique up to an overall scaling of the ℓ_i . Any symbolic infinite sequence in the hull of some symbolic substitution can be turned into a tiling of the line by identifying the symbol a_i with a translate of the tile $[0, \ell_i]$. For the sake of convenience we want to identify the symbol $u_0 = a_i$ in $u = \dots u_{-1} u_0 u_1 \dots$ with the tile $[0, \ell_i]$ with left endpoint 0. Since we will consider discrete point sets in the sequel, we consider the point sets $\{\dots, x_{-1}, x_0 = 0, x_1, \dots\}$ consisting of the left endpoints of the tiles in the tiling rather than the tilings itself. The set of all point sets arising from some primitive substitution (symbolic or tile) in this way is the *geometric hull* and is denoted by \mathbb{X}_σ . Theorem 2.1 has the following consequences for these point sets $\Lambda \in \mathbb{X}_\sigma$ [4].

Theorem 2.2. *Let σ be a primitive substitution, and let $u \in \mathbb{X}_\sigma$. Let λ be the Perron-Frobenius eigenvalue of M_σ , and let $v = (v_1, \dots, v_n)^T$ the right normalized eigenvector of M for λ . The relative frequency of symbol a_i in u*

$$\text{freq}(a_i) = \lim_{n \rightarrow \infty} \frac{|\{u_k \mid u_k = a_i, -n < k \leq n\}|}{2n}$$

exists and equals v_i . The density of each point set $\Lambda = \{\dots, x_{-1}, x_0 = 0, x_1, \dots\} \in \mathbb{X}_\sigma$ using natural tile lengths $w = (\ell_1, \dots, \ell_n)$ exists and equals

$$\text{dens}(\Lambda) = \left(\sum_{i=1}^n v_i \ell_i \right)^{-1} = \frac{1}{w \cdot v}.$$

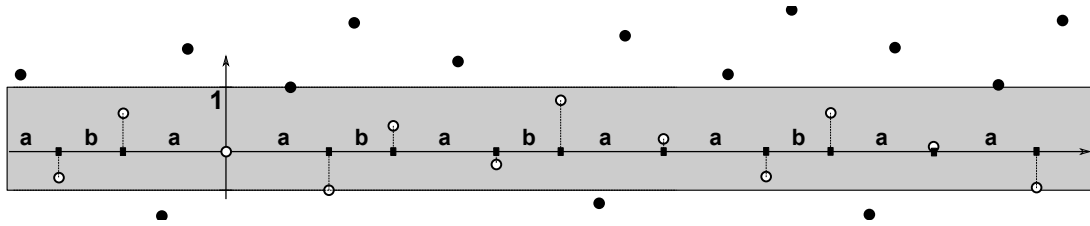


FIGURE 1. An illustration of the cut-and-project scheme for the Fibonacci sequence. The symbolic Fibonacci sequence is indicated by letters a and b . The vertex set of the resulting tiling are the black marks on the x -axis. They are projections of all the lattice points contained in the gray strip (white round points). The gray strip is $\mathbb{R} \times [-\frac{1}{\tau}, 1[$.

Remark 4. By the normalized vector v we always mean that the sum of its coordinates is 1. Particularly the theorem implies that the sum of relative frequencies of all symbols is 1, as it ought to be.

Example 2.3. The Fibonacci substitution σ is given by $a \mapsto ab$, $b \mapsto a$. It produces a bi-infinite sequence by repeatedly applying it to the pair of letters $a|a$: $\sigma(a|a) = ab|ab$, $\sigma_f^2(a|a) = aba|aba$, $\sigma_f^3(a|a) = abaab|abaab$, $\sigma_f^4(a|a) = abaababa|abaababa$, \dots . Its substitution matrix is $M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with Perron-Frobenius eigenvalue $\lambda = \frac{1}{2}(1 + \sqrt{5}) =: \tau$ and second eigenvalue $\frac{1}{2}(1 - \sqrt{5})$. The normalized eigenvector of M_σ for λ is $(\frac{\tau}{1+\tau}, \frac{1}{1+\tau})^T$, hence the relative frequency of a is $\frac{\tau}{1+\tau}$ and the relative frequency of b is $\frac{1}{1+\tau}$.

The sequence can be transformed into a discrete point set in \mathbb{R} by assigning an interval of length τ to a and an interval of length 1 to b . We then consider the endpoints of the intervals as our point set Λ_F . Then the density of Λ_F is $\left((\tau, 1) \cdot \left(\frac{\tau}{1+\tau}, \frac{1}{1+\tau} \right)^T \right)^{-1} = \frac{1+\tau}{2+\tau}$.

Definition 2.4. A *cut-and-project set* (CPS, aka *model set*) Λ is given by a collection of maps and spaces:

$$\begin{array}{ccccc} G = \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times \mathbb{R}^d & \xrightarrow{\pi_2} & H = \mathbb{R}^d \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

where G and H are locally compact abelian groups, Γ is a lattice (i.e., a discrete cocompact subgroup) in $G \times H$, W is a relatively compact set in H , and π_1 and π_2 are projections to G , respectively to H , such that $\pi_1|_\Gamma$ is one-to-one, and $\pi_2(\Gamma)$ is dense in W . Then

$$\Lambda = \{ \pi_1(x) \mid x \in \Gamma, \pi_2(x) \in W \}$$

is a CPS.

Remark 5. For our purposes it does not really matter whether if we replace “lattice” by “translate of a lattice” in the definition above: translating the lattice is equivalent to translating the window — and the CPS Λ , if needed — by some small amount.

Throughout the paper G usually equals \mathbb{R} , and H always equals \mathbb{R}^d for some $d \geq 1$. The following result is standard, see for instance [4].

Lemma 2.5. *The density of a CPS Λ is $\frac{\mu(W)}{|\det(M_\Gamma)|}$, where M_Γ is the matrix whose columns are the spanning vectors of the lattice Γ .*

Similar to a point set arising from a substitution, a CPS $\Lambda \subset \mathbb{R}$ can be written as a sequence $\{\dots, x_{-1}, x_0 = 0, x_1, x_2, \dots\}$ with $x_i < x_j$ for $i < j$.

Example 2.6. (Example 2.3 continued) A Fibonacci point set Λ_F , as constructed by the substitution above, can as well be generated as a CPS. This CPS has $d = 1$, $W = [-\frac{1}{\tau}, 1]$, lattice $\Gamma = \langle (\frac{1}{1}), (\frac{\tau}{\tau}) \rangle_{\mathbb{Z}}$, and π_1 and π_2 are orthogonal projections; compare Figure 1. Consequently the density of Λ_F is $\frac{1+\frac{1}{\tau}}{\tau+\frac{1}{\tau}} = \frac{\tau+1}{\tau+2}$.

3. SUBSTITUTION SEQUENCES IN BOUNDED DISTANCE TO A LATTICE

Definition 3.1. Let $\mathcal{A} = \{a_1, \dots, a_m\}$ and let σ be a primitive symbolic substitution over \mathcal{A} . The substitution is called a *Pisot substitution* if the matrix M_σ of the substitution has one (simple) positive real eigenvalue $\lambda > 1$, and all other eigenvalues (real or complex) have absolute values less than 1.

Note that sometimes a Pisot substitution is defined to obey a stronger condition, namely, that in addition to the definition above all eigenvalues different from λ are non-zero, see for instance [9, 1.2.5]. Since we are dealing with integer matrices this condition implies the characteristic polynomial of M_σ being irreducible. Theorem 3.5 below holds for the more general case of eigenvalues zero, hence we stick to the more general definition here.

The goal of this section is to give an elementary proof that the set of vertices of a one-dimensional primitive Pisot substitution is bounded distance equivalent to some lattice. This was done before, for instance by Holton and Zamboni [16, Theorem A.2]. (Probably this result was already known to Rauzy, but we are not aware of any reference). A similar result holds for Pisot substitution tilings in higher dimension, shown by Solomon [24]. The proof here uses much simpler techniques than the ones by Holton and Zamboni, and by Solomon. Although we need to require additionally that the substitution matrix is diagonalizable, we gain that our result also includes the case of eigenvalue(s) zero.

We proceed to show that any $\Lambda' \in \mathbb{X}_\sigma$ is bounded distance equivalent to $\text{dens}^{-1}(\Lambda')\mathbb{Z}$ by showing this for some fixed point of σ . That means we choose Λ such that for the biinfinite symbolic sequence u corresponding to Λ holds $\sigma(u) = u$. Such a u can always be found by the pigeonhole principle, possibly by using some power σ^k of σ instead of σ itself. Using the following result it suffices to show the claim for Λ . Since it holds in higher dimensions as well we state it for the general case.

Theorem 3.2. *Let Λ be a discrete point set in \mathbb{R}^d generated by some primitive substitution, or some CPS, such that $\Lambda \stackrel{\text{bd}}{\sim} a\mathbb{Z}^d$ for some a . Then for any Λ' in the hull of Λ holds: $\Lambda' \stackrel{\text{bd}}{\sim} a\mathbb{Z}^d$.*

Proof. We give only a sketch of the proof based on the criterion of Laczkovich [19]. The criterion says that given a set $M \subset \mathbb{R}^d$ then $M \stackrel{\text{bd}}{\sim} a\mathbb{Z}^d$ if and only if the number of points of M in any union of parallel unit cubes differs from the volume of that union by some constant (which depends on M only) times the area of that union.

Since $\Lambda \stackrel{\text{bd}}{\sim} a\mathbb{Z}^d$ this condition holds for Λ . Since Λ and Λ' are in the same hull, any pattern in Λ' can be found in Λ and therefore the condition of Laczkovich holds for Λ' as well. \square

The argument in the sequel goes along the following lines: First we bound the distance between the endpoint of a k -level supertile in Λ and the corresponding point in $\text{dens}^{-1}(\Lambda)\mathbb{Z}$ by some term $c \cdot c_\sigma^k$ with $c_\sigma < 1$. Then we show that the error between any point in Λ and its corresponding point in $\text{dens}^{-1}(\Lambda)\mathbb{Z}$ is bounded by $C(c_\sigma^0 + c_\sigma^1 + c_\sigma^2 + \dots) + C'$ for some constants C, C' . Hence the distance is uniformly bounded.

In the sequel let x_j^i denote the j th point in the point set $\{x_0^i = 0, x_1^i, \dots, x_{n_k^i}^i\}$ representing the level- k supertile $\sigma^k(a_i)$ of type i . This supertile consists of $n_k^i = (1, \dots, 1)M_\sigma^k \mathbf{e}_i$ intervals of length ℓ_1, \dots, ℓ_m where \mathbf{e}_i is the i th vector of the standard basis in \mathbb{R}^m . By Theorem 2.2 for any $i \in \{1, \dots, m\}$ the average length of intervals in the sequence $\{x_j^i\}_j$ equals the scalar product $\text{dens}^{-1}(\Lambda)$ of the left and right λ -eigenvectors of the matrix M_σ (the right eigenvector normalized). Hence we have by the definition of the density

$$\lim_{j \rightarrow \infty} \left| \frac{x_j^i}{j} - \text{dens}^{-1}(\Lambda) \right| = 0. \quad (1)$$

By Theorem 2.2 a left eigenvector is just (ℓ_1, \dots, ℓ_m) , and the normalized right eigenvector $(v_1, \dots, v_m)^T$ contains the frequencies of the tiles.

Lemma 3.3. *Let the substitution matrix $M_\sigma \in \mathbb{R}^{m \times m}$ be diagonalizable over \mathbb{C} . There is a number $c_\sigma \in [0, 1[$ such that for every $i \in \{1, \dots, m\}$ there is some positive constant c_i such that*

$$\left| x_{n_k^i}^i - n_k^i \cdot \text{dens}^{-1}(\Lambda) \right| < c_i c_\sigma^k.$$

Proof. Let's fix i . We will write down the left hand side using the substitution matrix. The number $x_{n_k^i}^i$ is the length of level- k supertile of type i . After every substitution step the length of each basic tile multiplies by λ , thus $x_{n_k^i}^i = \lambda^k \ell_i$.

We know that $n_k^i = (1, \dots, 1)M_\sigma^k \mathbf{e}_i$. Since the matrix M_σ is diagonalizable, there is a matrix C (possibly with complex entries) such that $M_\sigma = C \text{diag}(\lambda, \lambda_2, \dots, \lambda_m)C^{-1}$ where λ_i are all algebraic conjugates of λ . Then

$$M_\sigma^k = C \text{diag}(\lambda^k, \lambda_2^k, \dots, \lambda_m^k)C^{-1} = \lambda^k C \text{diag}(1, 0, \dots, 0)C^{-1} + B_k$$

for some matrix B_k with entries represented by linear forms of $\lambda_2^k, \dots, \lambda_m^k$.

Plugging this back in the formula for n_k^i we get

$$x_{n_k^i}^i - n_k^i \cdot \text{dens}^{-1}(\Lambda) = b \cdot \lambda^k + (1, \dots, 1)B_k \mathbf{e}_i \cdot \text{dens}^{-1}(\Lambda)$$

for some number b . Note, that λ is the only eigenvalue of M_σ with absolute value greater than 1, hence $\lim_{k \rightarrow \infty} B_k = 0$ and, using Equation (1):

$$0 = \lim_{k \rightarrow \infty} \left| \frac{x_{n_k^i}^i}{n_k^i} - \text{dens}^{-1}(\Lambda) \right| = b \cdot \lim_{k \rightarrow \infty} \frac{\lambda^k}{n_k^i} = b \cdot \text{dens}^{-1}(\Lambda).$$

Hence $b = 0$, and the difference $\left| x_{n_k^i}^i - n_k^i \cdot \text{dens}^{-1}(\Lambda) \right|$ can be written as a fixed linear form of k th powers of algebraic conjugates of λ which have absolute values less than 1.

Now we can choose $c_\sigma = \max(|\lambda_2|, \dots, |\lambda_m|) < 1$ and the statement of the lemma is obvious. \square

Let ε_k^i be the maximum discrepancy (or error) within the level- k supertile of type i , that is

$$\varepsilon_k^i = \max_{0 < j \leq n_k^i} |x_j^i - j \cdot \text{dens}^{-1}(\Lambda)|.$$

Our next step is to establish recurrent inequalities for the errors ε_k^i .

Lemma 3.4. *For every $k > 1$ we have*

$$\varepsilon_k^i \leq \max_{1 \leq j \leq m} \varepsilon_{k-1}^j + C \cdot c_\sigma^{k-1}$$

where C is a constant that does not depend on k or i .

Proof. Let N be the maximum number of tiles in the level-1 supertile, and let C_{max} be the maximum of constants c_j from Lemma 3.3. Then we can take $C = C_{max}N$.

Indeed, assume that the maximum for ε_k^i is attained at the point x_t^i . Then this point falls into one of n_1^i supertiles of level $k-1$ that constitute the level- k supertile. Assume it falls in the s th supertile from left. Then according to Lemma 3.3 each of the first $s-1$ level $k-1$ supertiles contributes an error of at most $C_{max}c_\sigma^{k-1}$, and the error in the last supertile is at most $\max_{1 \leq j \leq m} \varepsilon_{k-1}^j$. Since $s \leq n_1^i \leq N$ we obtain

$$\varepsilon_k^i \leq (s-1)C_{max}c_\sigma^{k-1} + \max_{1 \leq j \leq m} \varepsilon_{k-1}^j \leq \max_{1 \leq j \leq m} \varepsilon_{k-1}^j + C \cdot c_\sigma^{k-1}. \quad \square$$

Theorem 3.5. *Let σ be a primitive Pisot substitution in \mathbb{R} such that the substitution matrix M_σ is diagonalizable over C . For each $\Lambda \in \mathbb{X}_\sigma$ holds $\Lambda \stackrel{\text{bd}}{\sim} \text{dens}^{-1}(\Lambda)\mathbb{Z}$.*

Proof. By Theorem 3.2 it is sufficient to show the claim for some Λ arising from a fixed point of σ . Since Λ is a nested sequence of supertiles it is sufficient to show that all sequences ε_k^i are uniformly bounded. Lemma 3.4 implies $\varepsilon_k^i \leq \max_{1 \leq j \leq m} \varepsilon_{k-1}^j + C \cdot c_\sigma^{k-1}$ and therefore

$$\max_{1 \leq j \leq m} \varepsilon_k^j \leq \max_{1 \leq j \leq m} \varepsilon_{k-1}^j + C \cdot c_\sigma^{k-1}.$$

Applying this inequality to the errors $\varepsilon_{k-1}^j, \varepsilon_{k-2}^j$ and so on, we obtain that

$$\max_{1 \leq j \leq m} \varepsilon_k^j \leq C \cdot c_\sigma^{k-1} + C \cdot c_\sigma^{k-2} + \dots + C \cdot c_\sigma^0 + \max_{1 \leq j \leq m} \varepsilon_0^j.$$

So, each error ε_k^i does not exceed the sum of the infinite geometric series $\sum_{i=0}^{\infty} Cc_\sigma^i$ with positive quotient $c_\sigma < 1$ plus some constant. Thus all errors are bounded. \square

Remark 6. As mentioned above, some sources do not allow 0 to be an eigenvalue of the matrix of a Pisot substitution. But our approach works in that case, too. For instance, consider the substitution $a \mapsto aabb, b \mapsto ab$. The substitution matrix is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ with eigenvalues 3 and 0. The natural tile lengths are $\ell_1 = 2, \ell_2 = 1$, the frequencies are $\frac{1}{2}$ for both a and b . The discrete point set Λ arising from this substitution is nonperiodic and has density $\frac{2}{3}$. Theorem 3.5 implies that $\Lambda \stackrel{\text{bd}}{\sim} \frac{3}{2}\mathbb{Z}$.

Additionally, the same arguments will work not only if M_σ is diagonalizable but also if some power of M_σ is diagonalizable. The latter condition means that for any non-zero eigenvalue of M_σ all Jordan cells have size 1 but Jordan cells corresponding to the eigenvalue 0 can be arbitrary.

4. BRS AND BOUNDED DISTANCE EQUIVALENCE OF CPS

In the sequel we will exploit a correspondence between CPS and BRS in \mathbb{R}^d . A relevant result in this context is the following theorem by Kesten that gives a necessary and sufficient condition for an interval $[a, b] \subset [0, 1[$ to be a BRS with respect to some given slope α .

Theorem 4.1 ([18]). *Let $\alpha \in [0, 1[$, $0 \leq a < b \leq 1$. Then $[a, b]$ is a BRS with respect to α if and only if $b - a \in \alpha\mathbb{Z} + \mathbb{Z}$.*

The “if” part of Theorem 4.1 was known for a long time. It was shown by Hecke [14] (p 73) for $a = 0$, using analytic number theory (Dirichlet series of meromorphic functions). Ostrowski found a simple argument for generalizing this to arbitrary a [20]. Kesten settled the “only if” part by “heavy use of continued fraction expansions”. Several papers studied the higher dimensional analogues of the problem, see [10, 13] for an overview.

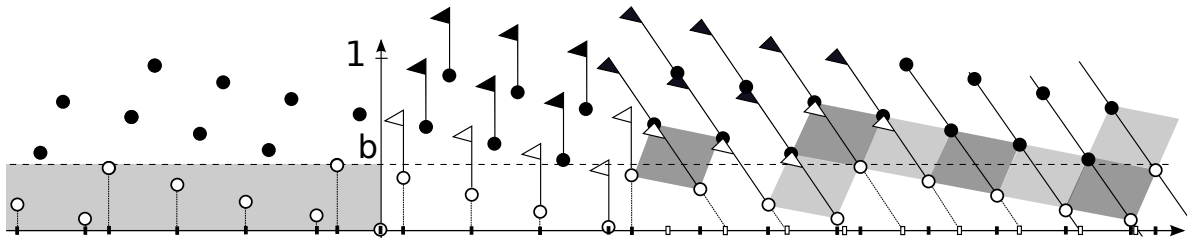


FIGURE 2. A graphical sketch of the connection between CPS and BRS. The black and white points $(k, k\alpha \bmod 1)$ are lifted versions of the points $k\alpha \bmod 1$. The white points correspond to the points $k\alpha \bmod 1 \in [0, b]$. Comparing the left part of the image with Figure 1 reveals already some similarity. Furthermore, it is shown that projecting all points in the strip to the x -axis (left part) is the same as assigning to each point a flag-stick of height b , and projecting each point whose flag-stick is hit by the horizontal line through $(0, b)$ (middle). By slanting the flag-sticks and projecting not orthogonally, but in direction of the flag-sticks, the resulting set of projected points becomes a periodic set (white rectangles, right).

The “if”-part of Theorem 4.1 can be proven by a simple geometric argument [6, 13]. Here we want to give a brief sketch of the idea on a very concrete level, since it illustrates nicely the connection between BRS and CPS that we will make more precise below.

Let $a = 0$, and consider the set

$$\Lambda = \{k \in \mathbb{Z} \mid k\alpha \bmod 1 \in [0, b]\}.$$

We lift Λ to \mathbb{R}^2 as follows: Let $\Gamma := \langle (1, \alpha)^T, (1, \alpha - 1)^T \rangle_{\mathbb{Z}}$. In particular, Γ contains all points $(k, k\alpha \bmod 1)^T$ where $k \in \mathbb{Z}$. If π_1 denotes the orthogonal projection to the first coordinate, then Λ consists of all elements $\pi_1(k, k\alpha \bmod 1)$ of Γ with $0 \leq k\alpha \bmod 1 < b$; compare Figure 2, left part: the points in the gray strip are the points that are projected to points in Λ . (Comparing this left part of the image with Figure 1 already gives an idea about the connection between CPS and BRS.)

The condition $b \in \mathbb{Z} + \alpha\mathbb{Z}$ means that there is some lattice point $(k, k\alpha \bmod 1) \in \Gamma$ such that $b = k\alpha \bmod 1$. Rather than projecting all points orthogonally to the line (by π_1) we project now in direction of $(k, k\alpha \bmod 1)$. With respect to the points in the strip this is clearly a bounded distance transformation.

Fixing $k \in \mathbb{Z} \setminus \{0\}$ and attaching to each lattice point in Γ a line segment $(k, k\alpha \bmod 1)$ (the “flag-sticks” in the image) yields a set of parallel equidistant lines. Because of the particular property of b being equal to $k\alpha \bmod 1$ for some appropriate $k \in \mathbb{Z}$, the line segments are in fact a partition of the lines (or more general, an m -fold covering of the lines). Projecting all points in some strip in direction $(k, k\alpha \bmod 1)$ then is the same as considering the intersection of the parallel lines with the x -axis.

The intersection of the parallel equidistant lines with the x -axis is a periodic set $c\mathbb{Z}$. In fact, it is almost a CPS: it fulfils all requirements but the one that the for projection π on \mathbb{R} holds that $\pi : \Gamma \rightarrow \mathbb{R}$ is one-to-one. In fact this can be fixed easily by changing the direction of projection slightly. Hence, by Lemma 2.5 the density of the resulting (almost) CPS $c\mathbb{Z}$ is $\frac{b}{|\det(M_\Gamma)|} = b$ (since $\Gamma = \langle \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \begin{pmatrix} 1 \\ \alpha - 1 \end{pmatrix} \rangle_{\mathbb{Z}}$). Thus we obtain $c\mathbb{Z} = \frac{1}{b}\mathbb{Z}$. Altogether we have $\frac{1}{b}\mathbb{Z} \stackrel{\text{bd}}{\sim} \Lambda$. The if-part of Kesten’s theorem follows now like in the proof of Theorem 4.5 below.

The argument above makes use of a relation between BRS and CPS: the set Γ above is the lattice in $\mathbb{R} \times \mathbb{R}$, the set Λ corresponds to the CPS. What is lacking is the requirement that the projection $\pi_1|_{\Lambda}$ is one-to-one. This can be achieved easily by changing the

direction of projection, or equivalently, by applying a linear transformation $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ to Γ . In the sequel we want to translate an arbitrary CPS into the canonical form used in the reasoning above, i.e. into a set that fulfils all the conditions of a CPS but one, having lattice $\Gamma' = \langle v_1, \dots, v_{d+1} \rangle_{\mathbb{Z}}$, such that the first entry of each v_i is 1. (Hence the condition not fulfilled is that $\pi_1(\Gamma')$ is one-to-one.) Furthermore, the window W should fit into $[0, 1]^d$. Let us call a point set with all these properties a *canonical quasi-CPS*, cqCPS for short. Then we can study whether W is a BRS by considering whether the corresponding cqCPS is bounded distance equivalent to some lattice.

In order to fit the window W into $[0, 1]^d$ the following two trivial observations are helpful.

Lemma 4.2. *A CPS Λ with lattice Γ , $G = \mathbb{R}$, $H = \mathbb{R}^d$, and $W \subset H$ is the union of k^{d+1} CPS with lattice translates $k\Gamma + t$ ($t \in \Gamma/k\Gamma$) and the same G , H , W .*

Lemma 4.3. *If $\Lambda_1, \dots, \Lambda_\ell$ are disjoint CPS in \mathbb{R} such that there is $a \in \mathbb{R}$ with $\Lambda_i \stackrel{\text{bd}}{\sim} a\mathbb{Z}$ for $1 \leq i \leq \ell$, then $\Lambda_1 \cup \dots \cup \Lambda_\ell \stackrel{\text{bd}}{\sim} \frac{a}{\ell}\mathbb{Z}$.*

A more interesting result on the converse of Lemma 4.3 is Lemma 6.3 in Section 6.

Lemma 4.4. *Let $\Lambda \subset \mathbb{R}$ be a CPS with window $W \subset \mathbb{R}^d$ and lattice Γ . Then Λ is bounded distance equivalent to some lattice $a\mathbb{Z}$ if and only if the corresponding cqCPS Λ' constructed above is bounded distance equivalent to some lattice $b\mathbb{Z}$.*

Of course in this case $a = \text{dens}^{-1}(\Lambda)$ and $b = \text{dens}^{-1}(\Lambda')$.

Proof. Let us make the relation sketched above more precise. Let Λ be some CPS with lattice $\Gamma \subset \mathbb{R} \times \mathbb{R}^d$ and window W . Choose a d -dimensional sublattice Γ_d of Γ that is maximal in the sense that $\Gamma_d = \Gamma \cap H'$, where H' is some d -dimensional hyperplane. This step corresponds to *choosing* the direction of projection: we want to project along some lattice vector (if $H = \mathbb{R}$), respectively along a d -dimensional sublattice (if $H = \mathbb{R}^d$, $d \geq 1$).

By Lemmas 4.2 and 4.3 we can assume without loss of generality that there is a fundamental domain F of Γ_d whose projection $\pi_2(F)$ contains W in its interior. (Otherwise we split the CPS Λ into k^{d+1} coarser ones using Lemma 4.2. Showing that each of them is bounded distance equivalent to some $k^{d+1}a\mathbb{Z}$ implies that $\Lambda \stackrel{\text{bd}}{\sim} a\mathbb{Z}$.)

The next step is to *change* the direction of projection, or alternatively: applying some linear map to the lattice Γ (and keeping π_1 being the orthogonal projection to $G = \mathbb{R}$). H' can be written as $(l(y), y)^T$ ($y \in \mathbb{R}^d$) with $l : \mathbb{R}^d \rightarrow \mathbb{R}$ linear. Hence we apply the map $\varphi : (x_1, x_2, \dots, x_{d+1})^T \mapsto (x_1 - l(x_2, \dots, x_{d+1}), x_2, \dots, x_{d+1})^T$ to Γ . This transformation maps H' to H and hence Γ_d to H . While φ is not a bounded distance transformation with respect to the entire lattice Γ , it is one with respect to $\Gamma \cap (\mathbb{R} \times W)$, since l is bounded on W . Hence the CPS Λ will be mapped to Λ' , where $\Lambda' \stackrel{\text{bd}}{\sim} \Lambda$. Since $\Gamma' := \varphi(\Gamma)$ consists of d -dimensional layers of copies of $\pi_2(\Gamma_d)$ parallel to $H = \mathbb{R}^d$, Λ' is already a subset of some periodic set $c\mathbb{Z}$. Without loss of generality let $c = 1$.

By shifting Γ' and W along H simultaneously we can achieve that the parallelepiped $\pi_2(F)$ has one vertex at 0. This shift leaves Λ' invariant. Transforming $\pi_2(F)$ into a unit cube $[0, 1]^d$ (hence $\pi_2(\Gamma')$ into \mathbb{Z}^d) is achieved by a linear map ψ that does only affect the internal space \mathbb{R}^d , hence ψ keeps Λ' unchanged. Thus Λ' is a cqCPS. \square

Theorem 4.5. *Let $\Lambda \subset \mathbb{Z}$ be a cqCPS with window $W \subset \mathbb{R}^d$, lattice Γ and slope α . Then Λ is bounded distance equivalent to the lattice $\text{dens}^{-1}(\Lambda)\mathbb{Z}$ if and only if W is a BRS with respect to α .*

Proof. Without loss of generality, let $\Lambda = \{\dots, \lambda_{-1}, \lambda_0 = 0, \lambda_1, \lambda_2, \dots\}$. (If $0 \notin \Lambda$ we may consider $\Lambda - k$ for some $k \in \Lambda$.)

By definition, W is a BRS with respect to α if and only if there is $C > 0$ such that for all n

$$|D_n(W, x)| = \left| \sum_{k=0}^{n-1} 1_W(x + k\alpha \bmod 1) - n\mu(W) \right| < C. \quad (2)$$

Let $\lambda_m \in \Lambda$ and let $n_m - 1$ be the summation index yielding λ_m ; that is, $n_m - 1 = \lambda_m$, and

$$m = \sum_{k=0}^{n_m-1} 1_W(x + k\alpha \bmod 1).$$

Using O notation, (2) becomes $m = n_m\mu(W) + O(1)$. This is equivalent to

$$\frac{1}{\mu(W)}m = n_m + O(1) = n_m - 1 + O(1),$$

hence to $\frac{1}{\mu(W)}m = \lambda_m + O(1)$. This means that the distance between the m th point λ_m of Λ and the m th point of $\frac{1}{\mu(W)}\mathbb{Z} = \text{dens}^{-1}(\Lambda)\mathbb{Z}$ is uniformly bounded in m , hence $\Lambda \stackrel{\text{bd}}{\sim} \text{dens}^{-1}(\Lambda)\mathbb{Z}$. \square

Note that the transformations in the proof of Lemma 4.4, transforming a CPS into a cqCPS, changes the direction of the slope α , as well as the shape of the window W . Whenever we use Theorem 4.5 together with Lemma 4.4 in order to show that the window W of some CPS is a BRS one has to read this as “an affine image of W is a BRS with respect to an appropriate slope α ”.

Computing the particular slope for some concrete example might become tedious. It would be very interesting if one could obtain an example of some BRS that is not covered by Theorem 5.2 below. But in all examples we tried the required affine maps always leave $\mathbb{Z}^d + \alpha\mathbb{Z}$ invariant. Hence these examples yielded nothing beyond the scope of Theorem 5.2.

5. BRS WITH FRACTAL BOUNDARY

In the sequel we need a notion of equidecomposability.

Definition 5.1. Let X and Y be two subsets of \mathbb{R}^d , and let V be a set of d -dimensional vectors. We say that X and Y are V -*equidecomposable* if there is a natural number k , a decomposition of X into k disjoint subsets

$$X = X_1 \cup \dots \cup X_k,$$

and k vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ such that

$$Y = (X_1 + \mathbf{v}_1) \cup \dots \cup (X_k + \mathbf{v}_k).$$

There are several profound results yielding conditions that imply that some given compact set in \mathbb{R}^d is a BRS. For $d = 1$ see for instance Theorem 4.1. For $d > 1$ there are several results for special cases, see [10, 13] and references therein. A pretty general result was obtained by Grepstad and Lev.

Theorem 5.2 ([10]). *Let $\alpha \in \mathbb{R}^d$ be completely irrational. That is, $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $1, \alpha_1, \dots, \alpha_d$ are linearly independent over \mathbb{Q} . Then any zonotope in \mathbb{R}^d with vertices belonging to $\mathbb{Z}^d + \alpha\mathbb{Z}$ is a BRS.*

A Riemann measurable set $S \in \mathbb{R}^d$ is a BRS if and only if S is $(\mathbb{Z}^d + \alpha\mathbb{Z})$ -equidecomposable to some parallelepiped spanned by vectors in $\mathbb{Z}^d + \alpha\mathbb{Z}$.

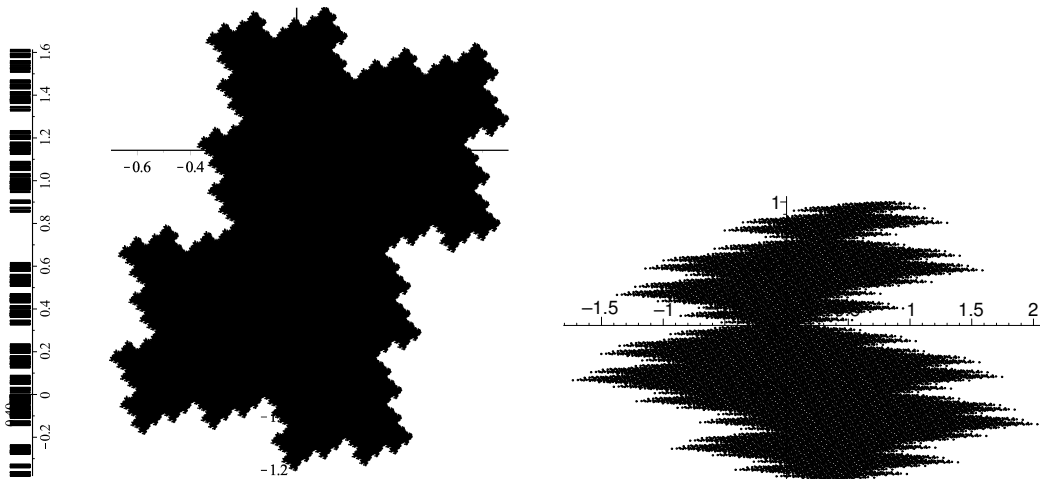


FIGURE 3. The window for $a \mapsto aab, b \mapsto ba$ (left) is one-dimensional. Here it is thickened for the sake of visualization. The windows for $a \mapsto abc, b \mapsto ab, c \mapsto b$ (middle) and for $a \mapsto abc, b \mapsto ab, c \mapsto a$ (right) are both BRS.

The connection between BRS and cqCPS established in Theorem 4.5 allows us to use Theorem 5.2 to construct several cqCPS (or CPS) that are bounded distance equivalent to some lattice $a\mathbb{Z}$. But it allows us also to go the opposite direction: the window of any CPS $\Lambda \subset \mathbb{R}$ with $\Lambda \stackrel{\text{bd}}{\sim} a\mathbb{Z}$ yields a BRS. Candidates for such CPS are provided by Theorem 3.5: discrete point sets Λ generated by Pisot substitutions are bounded distance equivalent to some $a\mathbb{Z}$. Hence the Pisot conjecture (or Pisot substitution conjecture) becomes important. In fact, the Pisot conjecture appears in several equivalent formulations, and in several levels of generalization [1]. The formulation needed here is the following.

Conjecture 5.3. *Let σ be a primitive substitution on $\mathcal{A} = \{1, \dots, n\}$, If σ is a Pisot substitution, and the characteristic polynomial of M_σ is irreducible over \mathbb{Q} , then the vertex sets of the tilings defined by σ (in the sense of Section 2) are CPS.*

In the case of two letters the conjecture is known to be true, as shown by Hollander and Solomyak [15].

Theorem 5.4. *Let σ be a primitive substitution on $\mathcal{A} = \{1, 2\}$. If σ is a Pisot substitution, and the characteristic polynomial of M_σ is irreducible, then the tilings defined by σ (in the sense of Section 2) are CPS.*

As a consequence of Theorems 3.5, 4.5, and 5.4, the window of the CPS of any Pisot substitution on two letters is a BRS, if the inflation factor is some quadratic irrational. Applied to the Fibonacci sequence this yields nothing new; compare Example 2.6: the window is an interval with length τ , hence by Kesten’s theorem this is a BRS. More interestingly, there are examples where the window has a boundary that is a fractal. Consider for instance the substitution $a \mapsto aab, b \mapsto ba$. The substitution matrix is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the eigenvalues are τ^2 and τ^{-2} . In particular, the inflation factor τ^2 is a Pisot number, and σ is a Pisot substitution. Hence by Theorem 3.5 for the vertex set Λ of the tilings of the line generated by σ holds: $\Lambda \stackrel{\text{bd}}{\sim} a\mathbb{Z}$. By Theorem 5.4 the tilings of the line generated by σ are CPS. Hence by Theorem 4.5 the window of this CPS is a BRS. The window for Λ is indicated in Figure 3 (left). This window is similar to a (fat) Cantor set: it has infinitely many connected components. Its total length is τ , hence by Theorem 5.2 it is not only a BRS, but equidecomposable to the interval $[0, \tau]$. The window W is the

(measurewise disjoint) union of a coupled iterated function system with two sets A and B :

$$A = (\tau^{-2}A) \cup (\tau^{-2}A + 1) \cup (\tau^{-2}B + 2), \quad B = (\tau^{-2}B) \cup (\tau^{-2}A + 2)$$

By a theorem of Hutchinson [11] such an iterated function system has a unique nonempty compact solution (A, B) . Here we have $W = A \cup B$.

In order to obtain higher dimensional BRS one can use Pisot substitutions with $d > 2$ letters. Generically this requires $H = \mathbb{R}^{d-1}$, yielding $(d-1)$ -dimensional windows that are candidates of BRS. Unfortunately we can not apply an analogue of Theorem 5.4: up to the knowledge of the authors the Pisot conjecture is still open for more than two letters. The fractal nature of many window sets for substitutions with more than two letters can make it tedious to prove that a given Pisot substitution actually yields a CPS. In [5] it is shown (on several pages) that the sequence generated by the Pisot substitution $a \rightarrow abc$, $b \mapsto ab$, $c \mapsto b$ indeed yields a CPS. Hence the window W for this CPS is a BRS. The slope for this example is $(\operatorname{Re}(\beta), \operatorname{Im}(\beta))^T$, where β is the complex (non-real) root of $x^3 - 2x^2 - 1$ with positive imaginary part. This set is shown in Figure 3 (middle). The Hausdorff dimension of its boundary is $1.2167\dots$ [5]. The corresponding iterated function system is given in [5] or [4, Example 7.5].

In such a way one gets a wealth of examples of BRS with fractal boundary. In [2] the authors proved that all three-letter Pisot substitutions where the trace of the substitution matrix is less than three fulfil the Pisot conjecture. They checked all 446683 Pisot substitutions with this properties using a computer and the so-called potential overlap algorithm [3]. One BRS arising from of these examples is shown in Figure 3 (right), obtained from the Pisot substitution $a \rightarrow abc$, $b \mapsto ab$, $c \mapsto a$; hence with inflation matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ [8]. It is a BRS with respect to the slope $\alpha' = (\alpha_1, \alpha_2)$, where $\alpha_1 = -0.8019\dots$ and $\alpha_2 = 0.5549\dots$ are the smaller roots of $x^3 - 2x^2 - x + 1$.

By Theorem 5.2 the last two examples are $(\mathbb{Z}^2 + (\operatorname{Re}(\beta), \operatorname{Im}(\beta))^T \mathbb{Z})$ -equidecomposable (respectively $(\mathbb{Z}^2 + \alpha' \mathbb{Z})$ -equidecomposable) to parallelograms spanned by vectors in $\mathbb{Z}^2 + (\operatorname{Re}(\beta), \operatorname{Im}(\beta))^T \mathbb{Z}$ (respectively in $\mathbb{Z}^2 + \alpha' \mathbb{Z}$). See also [4, Figure 7.3] or [23, Figure 2.2] for a particularly fuzzy example of a set that actually is a BRS.

6. BOUNDED DISTANCE EQUIVALENCE FOR TWO CUT-AND-PROJECT SETS

Jamie Walton [26] asked whether two CPS in the same hull are necessarily bounded distance equivalent. In this section we will show that there are two one-dimensional CPS in the same hull that are not bounded distance equivalent. Recall that the hull was defined for primitive substitutions above. More general, the hull of a discrete point set $\Lambda \subset \mathbb{R}$ is defined as the closure of $\{t + \Lambda \mid t \in \mathbb{R}\}$ under the local topology, see [4] for details. Roughly speaking, being in the same hull is the precise concept of Λ and Λ' being “locally indistinguishable”, i.e., Λ contains a translate of each finite pattern in Λ' and vice versa. This is literally true only if Λ and Λ' fulfil certain nice properties, see [4, Section 4.2].

For CPS this essentially boils down to the statement that two CPS are in the same hull if they can be generated with the same G, H, Γ , and the windows are translates of each other. (There are some subtleties to consider, but this is literally true for $G = \mathbb{R} = H$ and the windows being half-open intervals.) So the hull of a CPS can be obtained by considering all translations of the window W .

Theorem 6.1 and Lemmas 6.2 and 6.3 below are valid also in higher dimensions. For this purpose note that CPS can live as well in \mathbb{R}^e for some $e > 1$: just replace in Definition 2.4 of a CPS above $G = \mathbb{R}$ by $G = \mathbb{R}^e$.

Theorem 6.1. *Let Λ and Λ' be two CPS defined with the same lattice Γ and with the same projections π_1 and π_2 but with different windows W and W' respectively.*

If the windows W and W' are $\pi_2(\Gamma)$ -equidecomposable, then $\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$.

Proof. Let $W = W_1 \cup \dots \cup W_k$ be the decomposition given in Definition 5.1, and let $\mathbf{x}_i \in \Gamma$ be the vectors such that

$$W' = (W_1 + \pi_2(\mathbf{x}_1)) \cup \dots \cup (W_k + \pi_2(\mathbf{x}_k)).$$

In this case we can represent Λ as disjoint union $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$ where

$$\Lambda_i = \{\pi_1(x) \mid x \in \Gamma, \pi_2(x) \in W_i\}.$$

Therefore the set

$$\begin{aligned} \Lambda_i + \pi_1(\mathbf{x}_i) &= \{\pi_1(x + \mathbf{x}_i) \mid x \in \Gamma, \pi_2(x) \in W_i\} = \\ &= \{\pi_1(x + \mathbf{x}_i) \mid x \in \Gamma, \pi_2(x + \mathbf{x}_i) \in W_i + \pi_2(\mathbf{x}_i)\} = \{\pi_1(y) \mid y \in \Gamma, \pi_2(y) \in W_i + \pi_2(\mathbf{x}_i)\} \end{aligned}$$

is a subset of Λ' . Moreover, since the sets $W_i + \pi_2(\mathbf{x}_i)$ decompose W' we can decompose Λ' into

$$\Lambda' = (\Lambda_1 + \pi_1(\mathbf{x}_1)) \cup \dots \cup (\Lambda_k + \pi_1(\mathbf{x}_k)).$$

We define $f : \Lambda \rightarrow \Lambda'$ by $f(x) := x + \pi_1(\mathbf{x}_i)$ provided $x \in \Lambda_i$. This map is clearly a bijection and $\|f(x) - x\|$ is bounded by the length of the longest vector from \mathbf{x}_i . \square

In particular two $\mathbb{R}^d \times \mathbb{R}$ CPS with windows W and W' differ by a translation of a vector from $\pi_2(\Gamma)$ are bounded distance equivalent (actually they are translated copies since $k = 1$ in that case). Our next goal is to show that this condition can't be relaxed. Namely we will show that two halves of the Fibonacci sequence are not bounded distance equivalent although they can be constructed as $\mathbb{R} \times \mathbb{R}$ CPS with windows being intervals of equal length.

We are going to use the following Rado's theorem [22] (locally finite version of graph-theoretical Hall's marriage lemma).

Lemma 6.2. *Consider a bipartite graph $G = (V_1 \cup V_2, E)$ with countable vertex sets V_1 and V_2 such that any vertex has finite degree. Assume that for any finite subset X of V_1 there are at least $|X|$ vertices from V_2 that are connected with at least one vertex from X ; and the same holds for the roles of V_1 and V_2 exchanged. Then there exists a set of disjoint edges of G that covers all vertices of G .*

Lemma 6.3. *Consider a discrete point set A in \mathbb{R}^d such that $A \stackrel{\text{bd}}{\sim} \mathbb{Z}^d$. If A is represented as $A = M_1 \cup M_2 \cup \dots \cup M_n$ in such a way that $M_i \stackrel{\text{bd}}{\sim} M_j$ then $M_i \stackrel{\text{bd}}{\sim} n^{1/d} \mathbb{Z}^d$.*

The statement can be easily deduced from Laczkovich's criterion [19] but we would like to provide a purely combinatorial proof below.

Proof. Without loss of generality we can assume that $A = \mathbb{Z}^d$. It is enough to construct a bounded distance bijection between M_1 and the set of all integer points in $T := \{(nk_1, k_2, \dots, k_n) \mid k_i \in \mathbb{Z}\}$. Two lattices of the same density are bounded distance equivalent [6, Theorem 1], so we have $n^{1/d} \mathbb{Z}^d \stackrel{\text{bd}}{\sim} T$ and the claim of the lemma immediately follows from them. Let c_{ij} be a distance in the bounded distance bijection between M_i and M_j and $c = \max c_{ij}$.

Consider a bipartite graph with parts represented by points from M_1 and points from T . Two vertices are connected with an edge if and only if the distance between these vertices is at most $c + n$. We will show that this graph satisfies Lemma 6.2. For every k and for any k points from M_1 we have at least kn lattice points in A covered by c -balls centred at these points (we have at least k points from each M_i). For every such lattice point x we decrease the first coordinate to the closest integer divisible by n . In this way we will get a point $y \in T$ with distance at most n from x . The point y is at distance at

most $c + n$ from one of the k points from M_1 we used initially. For a fixed point $y \in T$ we reached y at most n times so we obtained at least k different points from T in $(c+n)$ -balls centred at given k points of M_1 .

The same is true for any k points from T : there are kn points in this $n \times 1 \times \dots \times 1$ bricks and at least k of them are from one M_i . Thus at least k points from M_1 are at distance c from these bricks and at distance $c + n$ from these points from T . So the conditions of Lemma 6.2 are fulfilled and we get the desired bounded distance bijection determined by the edges from Lemma 6.2. \square

Recall that the Fibonacci sequence Λ_F is the CPS with $d = 1$, $W = [-\frac{1}{\tau}, 1[$, lattice $\Gamma = \langle (\frac{1}{\tau}), (\frac{\tau}{1}) \rangle_{\mathbb{Z}}$, and π_1 and π_2 are orthogonal projections.

Let F_1 and F_2 be two CPS with the same initial data as the Fibonacci sequence but with $W_1 = [-\frac{1}{\tau}, \frac{1-\frac{1}{\tau}}{2}[$ and $W_2 = [\frac{1-\frac{1}{\tau}}{2}, 1[$; that is, F_1 and F_2 use disjoint halves of the window W . Therefore we call them *half-Fibonacci sequence(s)*. Note, that F_1 and F_2 use windows of the same length, so F_1 and F_2 are contained in the same hull, but these windows are not translated to each other by a vector from $\pi_2(\Gamma)$.

Theorem 6.4. *The Half-Fibonacci sequences F_1 and F_2 are not bounded distance equivalent.*

Proof. Assume $F_1 \stackrel{\text{bd}}{\sim} F_2$. The union $F = F_1 \cup F_2$ is the Fibonacci set and according to Kesten's theorem 4.1 is bounded distance equivalent to the lattice $\alpha\mathbb{Z}$ for $\alpha = \text{dens}^{-1}(F)$. Then according to Lemma 6.3 $F_1 \stackrel{\text{bd}}{\sim} F_2 \stackrel{\text{bd}}{\sim} 2\alpha\mathbb{Z}$.

However, according to Kesten's theorem 4.1 F_1 is not bounded distance equivalent to a lattice since the length of the window does not equal to the length of π_2 -projection of any vector of Γ . We've got a contradiction and hence F_1 and F_2 are not bounded distance equivalent. \square

In particular the last result raises further questions: Given some discrete point set (or tiling), generated by substitution or by some cut-and-project method, of how many equivalence classes (with respect to $\stackrel{\text{bd}}{\sim}$) does its hull consist? Theorem 3.2 yields a partial answer to this question for substitution sequences: if for any $\Lambda \in \mathbb{X}_\sigma$ holds $\Lambda \stackrel{\text{bd}}{\sim} a\mathbb{Z}$, then for all $\Lambda' \in \mathbb{X}_\sigma$ holds $\Lambda' \stackrel{\text{bd}}{\sim} a\mathbb{Z}$, hence there is only one equivalence class.

It would also be interesting to expand on Remark 6: For any primitive substitution σ with inflation factor $\lambda \in \mathbb{Z}$, can there be more than one equivalence class in the hull \mathbb{X}_σ ? Respectively, can there be a substitution sequence Λ in \mathbb{X}_σ that is not bounded distance equivalent to any $a\mathbb{Z}$?

On a different note it might be worth to study whether the results of this paper yield a BRS beyond the scope of Theorem 5.2. For instance it might be worth studying whether this construction yields some BRS with respect to a slope that is not completely irrational.

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