# SELF-DUAL TILINGS WITH RESPECT TO STAR-DUALITY

D. FRETTLÖH

ABSTRACT. The concept of star-duality is described for self-similar cut-and-project tilings in arbitrary dimensions. This generalises Thurston's concept of a Galois-dual tiling. The dual tilings of the Penrose tilings as well as the Ammann-Beenker tilings are calculated. Conditions for a tiling to be self-dual are obtained.

## 1. Introduction

Nonperiodic discrete structures like infinite words, tilings, or point sets, are a rich source of exciting objects studied within many fields of mathematics, including automata theory, combinatorics, discrete geometry, dynamical systems, group theory, mathematical physics and number theory. Many interesting examples of such structures can be produced by substitutions as well as by cut-and-project schemes. For such structures there are several concepts of 'dual' tilings (words, point sets) [17], [21], [24]. The present article describes how Thurston's concept of Galois-duality [23] generalises to arbitrary cut-and-project sets. This generalisation does not depend on Galois conjugacy but on the star map (see [15], or Definition 1.5 below) wherefore the term star-dual (resp. $\star$-dual) is used, rather than Galois dual. The benefits of star-duality are that it gives a purely algebraic approach to certain geometric problems; that it covers a pretty wide range of examples; and that it may help to friend the certain concepts of duality. Furthermore, and hopefully, it can be utilised to attack open problems in the theory of non-periodic structures.

The first section of the present paper collects necessary facts and definitions about substitutions and cut-and-project tilings, resp. model sets. Section 2 contains the definition of $\star$-duality, together with an example. The calculation of the $\star$-dual tilings of the famous Penrose tilings and Ammann-Beenker tilings [11] is carried out in Section 3. It turns out that the star-dual tilings of the Penrose tilings are the well-known Tübingen triangle tilings (and vice versa), which coincides with a related result in [2]. In contrast, the star-dual tilings of the Ammann-Beenker tilings are very similar to the Ammann-Beenker tilings themselves. In fact, they are mutually locally derivable [4] with the Ammann-Beenker tilings. This motivates the question whether there are self-dual tilings w.r.t. star-duality. Therefore, Section 4 is dedicated to self-duality in arbitrary dimensions. A necessary, and a sufficient condition for a tiling to be self-dual w.r.t. star-duality are obtained.

The concept of star-duality is not new, see for instance [23], [10], [24]. The relevant results of this paper are the following: First, Theorem 1.3 gives a useful technical result which is exploited in the next sections, and will be exploited further in future work. Second, the dual tilings of the Penrose tilings and the Ammann-Beenker tilings are obtained, where the latter is done for the first time here, up to knowledge of the author, where the former one is obtained in [2] with much more effort in a slightly different framework (but not in [10], as the title

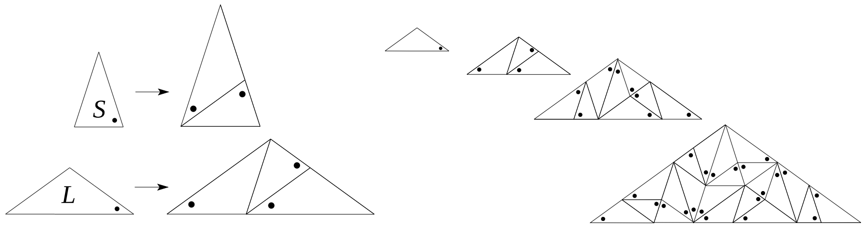


Figure 1. The substitution for the Penrose tiling in the version with triangles $S$ and $L$ (left), the first iterates of the substitution on the obtuse triangle $L$ (right). The substitution factor in this example is $\tau = \frac{\sqrt{5}+1}{2}$.

might suggest). Last, Lemma 4.2 and Corollary 4.3 are first steps towards a classification of self-dual tilings, in one dimension as well as in arbitrary dimension.

**Remark:** For convenience, we will exclude periodic structures in the following. Each occurring biinfinite sequence, tiling etc. is assumed to be nonperiodic. The set of positive integers is denoted by $\mathbb{N}$. The closure (interior) of a set $A$ is denoted by $\mathrm{cl}(A)$ ($\mathrm{int}(A)$).

1.1. **Substitution sequences.** Let $\mathcal{A}$ be an *alphabet*, i.e., a finite set of letters; let $\mathcal{A}^*$ be the set of all finite words over the alphabet $\mathcal{A}$, i.e., all words which are concatenations of letters of $\mathcal{A}$. Usually, the empty word $\epsilon$ (consisting of no letters) is required to be an element of $\mathcal{A}^*$. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^* \setminus \{\epsilon\}$. We will call such a substitution also *symbolic substitution* if we want to emphasise the difference to a tile-substitution. By setting $\zeta(a_1 \cdots a_m) = \zeta(a_1) \cdots \zeta(a_m)$, $\zeta$ acts also as a map from $\mathcal{A}^*$ to $\mathcal{A}^*$, from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{A}^{\mathbb{N}}$, and from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$. The family of all biinfinite words, where each finite subword is contained in some $\sigma^k(a)$ ($a \in \mathcal{A}$), is denoted $\mathcal{X}_\sigma$. There is a lot of literature devoted to symbolic substitutions, see for instance [16] and references therein for more details.

1.2. **Substitution tilings.** A tile is a nonempty compact subset of $\mathbb{R}^d$ which is the closure of its interior. A tiling of $\mathbb{R}^d$ is a set $\{T_j\}_{j \in \mathbb{N}}$ of tiles which is a covering of $\mathbb{R}^d$ (i.e., $\mathbb{R}^d = \bigcup_{j \in \mathbb{N}} T_j$) that is non-overlapping (i.e., $\mathrm{int}(T_j) \cap \mathrm{int}(T_i) = \varnothing$ for $j \neq i$). A simple way to generate tilings — periodic ones as well as nonperiodic ones — is via a tile-substitution. Roughly speaking, a tile-substitution is given by a finite set $\{T_1, \ldots, T_m\}$ of tiles, the *prototiles*, an expanding map $Q$, and a rule how to dissect each $QT_j$ into isometric copies of some prototiles $T_i$.

**Example:** A diagram of the tile-substitution for the Penrose tiling [11] is shown in Figure 1. Here, we use triangles as prototiles rather than rhombi or the celebrated kites and darts. Since the two prototiles are mirror-symmetric, but their substitutions are not, we have to respect the orientation of each tile. This is achieved by a black dot on each triangle. As indicated in the figure, iterating the substitution several times fills larger and larger portions of space, eventually resulting in a tiling of the plane. However, the proper definition of a tile-substitution and a substitution tiling is as follows.

**Definition 1.1.** *A (self-affine) tile-substitution is defined via a set of prototiles and a map $\sigma$. Let $T_1, T_2, \ldots T_m$ be nonempty compact sets — the* prototiles *— in $\mathbb{R}^d$, such that* $\mathrm{cl}(\mathrm{int}(T_j)) =$

*$T_j$ for each $i \le m$. Let $Q$ be an expanding linear map such that*

$$QT_j = \bigcup_{i=1}^{m} T_i + \mathcal{D}_{ij} \quad (1 \le j \le m), \tag{1}$$

*where the union is not overlapping (i.e., the interiors of the tiles in the union are pairwise disjoint); and each $\mathcal{D}_{ij}$ is a finite (possibly empty) subset of $\mathbb{R}^d$, called* digit set. *Then*

$$\sigma(T_j) := \{T_i + \mathcal{D}_{ij} \mid i = 1 \dots m\}$$

*is called a* (tile-)substitution. *By $\sigma(T_j + x) := \sigma(T_j) + Qx$ and $\sigma(\{T, T'\}) := \{\sigma(T), \sigma(T')\}$, $\sigma$ extends in a natural way to all sets $\{T_{i(k)} + x_k \mid k \in I\}$.*

The sum $T + \mathcal{D}_{ij} = T + \{d_{ij1}, \dots, d_{ijk}\}$ means $\{T + d_{ij1}, \dots, T + d_{ijk}\}$, with the convention $T + \varnothing = \varnothing$, if $\mathcal{D}_{ij}$ is empty. The set $I$ in the last line of the definition can be any index set. In particular, if $I = \mathbb{N}$, such a set can describe a tiling, given that the tiles do not overlap, and the union of all tiles is the entire space $\mathbb{R}^d$. Before stating the definition of a substitution *tiling*, let us make two remarks.

**Remark:** Often, a tile-substitution is defined without the requirement (1). To be precise, if the support of $\sigma(T_j)$ equals $QT_j$ — that is, if (1) holds —, the substitution is called *self-affine.* Moreover, if $Q$ is a homothety (i.e., just a multiplication by a scalar factor $\lambda$), the substitution is called *self-similar.* In this case, $\lambda$ is called the *substitution factor* of the substitution. E.g., the expanding linear map for the Penrose tiling is $Q = \left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau \end{smallmatrix}\right)$, where $\tau = (\sqrt{5}+1)/2$ is the golden mean. Thus, the substitution factor of the Penrose substitution in Figure 1 is $\tau$. Whenever the term 'self-similar tiling' ('self-affine tiling') is used in the sequel this is to be understood as 'self-similar substitution tiling' ('self-affine substitution tiling').

**Remark:** Every symbolic substitution (see Section 1.1) gives rise to a self-similar tile-substitution in $\mathbb{R}^1$ in a canonical way, just by replacing each letter by an interval. This can be done in such a way that the tile-substitution is self-similar. Namely, one uses the entries of a positive PF-eigenvector of the substitution matrix, see [16], [8].

**Definition 1.2.** *Let $\sigma$ be a tile-substitution. A tiling $\mathcal{T}$ is called a* substitution tiling *(with substitution $\sigma$), if each finite subset of $\mathcal{T}$ is a translate of some subset of some $\sigma^k(T_j)$, where $k \ge 0$, and $T_j$ some prototile. The family of all substitution tilings with substitution $\sigma$ is the* tiling space $\mathbb{X}_\sigma$.

This definition of a tiling space is equivalent to the usual one [22], whenever $\sigma$ is *primitive.* Primitive means that there is $k \ge 0$ such that a power of the *substitution matrix* (or 'incidence matrix') $M_\sigma = (|\mathcal{D}_{ij}|)_{1 \le i,j \le m}$ is strictly positive. Here, $|\mathcal{D}_{ij}|$ denotes the cardinality of $\mathcal{D}_{ij}$. It is also consistent with the usual definition of the associated dynamical system $(\mathcal{X}_\sigma, S)$ arising from $\sigma$, see [16], as well as with our definition of $\mathcal{X}_\sigma$ in Section 1.1.

1.3. **Iterated function systems.** The prototiles of a self-affine substitution can be derived from $Q$ and $(\mathcal{D}_{ij})_{1 \le i,j \le m}$ uniquely as follows: Multiplying the entire equation system (1) by $Q^{-1}$ yields the corresponding iterated function system (IFS) for the tiles of $\sigma$:

$$T_j = \bigcup_{i=1}^{m} Q^{-1}(T_i + \mathcal{D}_{ij}) \quad (1 \le j \le m) \tag{2}$$

Such an IFS is known to possess a unique nonempty compact solution (see for instance [5] and references therein). Since the prototiles of $\sigma$ fulfil (2), they are the unique solution of (2). In particular, by Definition 1.1, the IFS (2) is *non-overlapping*, that means, for each $j$, the interiors of the sets in the union are pairwise disjoint. Note that two parts $T_i, T_j$ of the solution are allowed to overlap. The point is that (2) induces a partition of each $T_j$ up to a set of measure 0.

Originally, the term IFS was coined for the case $m = 1$ only. An IFS with $m > 1$ components is often called 'graph-iterated function system' or 'multi-component IFS'. Here, we use the term IFS for all of them.

**Remark:** We just mentioned that each self-affine substitution gives rise to a non-overlapping IFS (2). Conversely, by multiplying (2) with $Q$, each non-overlapping IFS with a common contracting map $Q^{-1}$ defines a self-affine substitution as in Definition 1.1.

**Remark:** An IFS as above can be written as a finite state automaton (FSA). The states are the $T_j$. The input alphabet consists of all elements $x$ in $\bigcup_{1 \le i,j \le m} \mathcal{D}_{ij}$. The input of a digit $x$ changes the state from $T_j$ to $T_i$, if $x \in \mathcal{D}_{ij}$, otherwise the state is changed to FAIL. For an example, see Figure 3. For details, cf. [23], [10].

The following theorem ensures that it does not matter where the prototiles for a substitution $\sigma$ are located in space. The tiling space defined by $\sigma$ depends only on the way *how* tiles are dissected into smaller pieces, not *where* they are located. In particular, it does not matter whether the prototiles overlap or not. The important property is that the dissection of each prototile is non-overlapping. This theorem will be useful in the next sections, as well as for future work.

**Theorem 1.3.** *Let $\mathcal{D}$ be the digit set matrix for an IFS in $\mathbb{R}^d$ with a common contraction $Q$. If $(A_1, A_2, \ldots, A_m)$ is a solution for $\mathcal{D}$, then $(A_1 + t_1, A_2 + t_2, \ldots, A_m + t_m)$ is a solution for $\{\mathcal{D}_{ij} + (t_j - Qt_i)\}_{1 \le i,j \le m}$ for all $(t_1, t_2, \ldots, t_m) \in (\mathbb{R}^d)^m$.*

*Proof.* Throughout the proof, let $(A_1, \ldots, A_m)$ be a solution for $\mathcal{D}$, i.e.,

$$A_j = \bigcup_{i=1}^{m} Q^{-1}(A_i + \mathcal{D}_{ij}) \quad (1 \le j \le m).$$

First, let $A'_j = A_j + t$ for some common $t$ $(1 \le j \le m)$. Then the approach

$$\begin{array}{rcllll} A'_1 & = & QA'_1 + \mathcal{D}_{11} + x & \cup & \cdots & \cup \quad QA'_m + \mathcal{D}_{m1} + x \\ \vdots & & \vdots & & & \quad\ \vdots \\ A'_m & = & QA'_1 + \mathcal{D}_{1m} + x & \cup & \cdots & \cup \quad QA'_m + \mathcal{D}_{mm} + x \end{array}$$

yields

$$A_1 + t = QA_1 + Qt + \mathcal{D}_{11} + x \cup \cdots \cup QA_m + Qt + \mathcal{D}_{m1} + x,$$

and it follows $x = t - Qt$. It is straightforward to check that $(A'_1, \ldots, A'_m) = (A_1 + t, \ldots, A_m + t)$ is a solution for $\{\mathcal{D}_{ij} + (t - Qt)\}_{1 \le i,j \le m}$.

Next, let $t_1 = t_2 = \cdots = t_k = 0,\ t_{k+1} = \cdots = t_m = t$ and $A'_j = A_j + t_j$ for $k+1 \le j \le m$. Then the approach

$$\begin{array}{rcll} A_1 & = & \bigcup_{i=1}^k QA_i + \mathcal{D}_{i1} & \cup \ \bigcup_{i=j+1}^m QA'_i + \mathcal{D}_{i,1} + x \\ \vdots & & \vdots & \quad\vdots \\ A_k & = & \bigcup_{i=1}^k QA_i + \mathcal{D}_{ik} & \cup \ \bigcup_{i=k+1}^m QA'_i + \mathcal{D}_{i,k} + x \\ A'_{k+1} & = & \bigcup_{i=1}^k QA_i + \mathcal{D}_{i,k+1} + y & \cup \ \bigcup_{i=k+1}^m QA'_i + \mathcal{D}_{i,k+1} + z \\ \vdots & & \vdots & \quad\vdots \\ A'_m & = & \bigcup_{i=1}^k QA_i + \mathcal{D}_{i,m} + y & \cup \ \bigcup_{i=k+1}^m QA'_i + \mathcal{D}_{i,m} + z \end{array} \tag{3}$$

yields $y = t, x = -Qt$ and $z = t - Qt$. With these values, it is straightforward to check that $(A_1, \ldots, A_k, A_{k+1} + t, \ldots A_m + t)$ is a solution of (3).

By combining these two results, we obtain the digit matrix for $(A_1 + t_1, A_2 + t_2, \ldots, A_m + t_m)$ successively as follows.

$\mathcal{D}^{(1)} = \mathcal{D} + (t_1 - Qt_1) = \{\mathcal{D}_{ij} + (t_1 - Qt_1)\}_{1 \le i,j \le m}$ has the solution $(A_1 + t_1, A_2 + t_1, \ldots, A_m + t_1)$.

$$\mathcal{D}^{(2)} = \mathcal{D} + (t_1 - Qt_1) + \begin{pmatrix} 0 & t_2 - t_1 & \cdots & t_2 - t_1 \\ -Q(t_2 - t_1) & (E - Q)(t_2 - t_1) & \cdots & (E - Q)(t_2 - t_1) \\ \vdots & \vdots & & \vdots \\ -Q(t_2 - t_1) & (E - Q)(t_2 - t_1) & \cdots & (E - Q)(t_2 - t_1) \end{pmatrix}$$

has the solution $(A_1 + t_1, A_2 + t_1 + (t_2 - t_1), \ldots, A_m + t_1 + (t_2 - t_1))$ (where we use (3) with $y = t = t_2 - t_1$, and $E$ denotes the identity matrix), and so on, until

$$\mathcal{D}^{(m)} = \mathcal{D}^{(m-1)} + \begin{pmatrix} 0 & \cdots & 0 & t_m - t_{m-1} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & t_m - t_{m-1} \\ -Q(t_m - t_{m-1}) & \cdots & -Q(t_m - t_{m-1}) & (E - Q)(t_m - t_{m-1}) \end{pmatrix}$$

has the solution $(A_1 + t_1, A_2 + t_2, \ldots, A_{m-1} + t_{m-1}, A_m + t_{m-1} + (t_m - t_{m-1}))$.
Now, $\mathcal{D}^{(m)}$ is of the desired form, which proves the claim. $\square$

Regarding the action of a common regular linear map $T$ on the solution $(A_1, \ldots, A_m)$ with respect to $\mathcal{D}$ rather than the action of different translations is much simpler. The same is true if all translation vectors $t_i$ in the previous theorem had been equal.

**Lemma 1.4.** *Let $\mathcal{D}$ be the digit set matrix for an IFS in $\mathbb{R}^d$ with a common contraction $Q$. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be any invertible linear map. If $(A_1, A_2, \ldots, A_m)$ is a solution for $\mathcal{D}$, then $(TA_1, TA_2, \ldots, TA_m)$ is a solution for $T(\mathcal{D})$, where $T(\mathcal{D}) = \big(T(\mathcal{D}_{ij})\big)_{1 \le j,i \le m}$, $T(\mathcal{D}_{ij}) = \{Td_{ijk} \,|\, d_{ijk} \in \mathcal{D}_{ij}\}$.*

*Proof.*

$$A_j = \bigcup_{i=1}^m A_i + \mathcal{D}_{ij} \quad \Rightarrow \quad TA_j = T\Big(\bigcup_{i=1}^m A_i + \mathcal{D}_{ij}\Big) = \bigcup_{i=1}^m TA_i + T(\mathcal{D}_{ij}).$$

$\square$

1.4. **Model sets.** In the sequel we consider self-similar tilings which arise from *model sets*. Such tilings are known as cut-and-project tilings [1], [14], [15]; see also [16], where the cut-and-project method for one-dimensional tilings is called 'geometric realization'.

**Definition 1.5.** *A model set is defined via a* cut and project scheme*, i.e., a collection of spaces and mappings as follows.*

*Let $G, H$ be locally compact Abelian groups, $\Lambda$ be a lattice in $G \times H$ (that is, $\Lambda$ is a cocompact discrete subgroup of $G \times H$), $\pi_1 : G \times H \to G$, $\pi_2 : G \times H \to H$ be projections, such that $\pi_1|_\Lambda$ is injective, and $\pi_2(\Lambda)$ is dense in $H$. Let $W \subset H$ be a compact set — the* window *— such that the closure of* $\operatorname{int}(W)$ *equals $W$.*

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

*Then*

$$V := \{\pi_1(x) \,|\, x \in \Lambda,\ \pi_2(x) \in W\}$$

*is called a* model set.

*If $\mu(\partial(W)) = 0$, then $V$ is called* regular *model set.*

*The* star map *is the map $\star : \pi_1(\Lambda) \to H,\ x^\star = \pi_2(\pi_1^{-1}(x))$.*

We should mention that in many cases in the literature holds $G = \mathbb{R}^d$ and $H = \mathbb{R}^e$. One may assume this reading this article without loosing much information. However, the notion of star-duality is tailored to the general case, and this is where we leave the concept of Galois-duality. There are cut-and-project tilings (in fact, a lot of them) of the line or the plane, where $H$ is the field $\mathbb{Q}_p$ of $p$-adic integers, or a product of those with some Euclidean space $\mathbb{R}^e$, see [3], [18], [19], [20].

In the context of symbolic substitutions [16], the setting is very much the same, but unfortunately the terminology differs. For instance, the cut-and-project scheme is called geometric realization, the window is usually called (generalized) Rauzy fractal etc.

A model set is, by definition, a point set. There are several ways to assign a tiling to it. For instance, one may take the Voronoi cells of the point set as tiles. Whenever we can obtain a tiling $\mathcal{T}$ from a model set in such a way we call $\mathcal{T}$ a *cut-and-project tiling*. To be precise: If it is possible to assign a tiling $\mathcal{T}$ to a model set $V$ such that $V$ and $\mathcal{T}$ are mutually locally derivable (MLD) in the sense of [4], we call $\mathcal{T}$ a cut-and-project tiling. Roughly spoken, MLD means that one can obtain $V$ from $\mathcal{T}$ by local replacement rules and vice versa.

The self-similar cut-and-project tilings are the objects we will consider in the following. We should mention that there is a huge number of such tilings, including the Fibonacci tilings and the Tribonacci tilings [16], the Penrose tilings and the Ammann-Beenker tilings [11], the Chair tilings and the Sphinx tilings [3]. A collection of several additional examples is available online [9].

## 2. The $\star$-dual substitution

It is known that the window of a self-similar model set is the solution of an IFS, see for instance [12], [19], [21]. Since each non-overlapping IFS defines a tile-substitution (see the remark after (2)), several authors considered this 'dual substitution tiling' (or better, the substitution) induced by the IFS of the window. In [23], Thurston uses a different construction based on Galois conjugation in the field $\mathbb{Q}(\lambda)$, where $\lambda$ denotes the substitution factor of a self-similar tiling. One can utilise Theorem 1.3 to show that both approaches yield the same tiling spaces. Thurston called the tilings arising from this construction 'Galois duals' of the original tilings. We proceed by stating the definition of the $\star$-dual tiling of a self-similar cut-and-project tiling, which is a generalization of Galois-duality.

In general, the groups $G$ and $H$ in Definition 1.5 can be arbitrary locally compact Abelian groups. However, dealing with substitutions and linear maps, we require $G$ and $H$ to be equipped with a vector space structure in the sequel. This is no rigid restriction, since all examples in the literature fulfil this condition.

Furthermore, in this section we require the expanding linear map $Q : G \to G$ to be as follows: Let $T : G \times H \to G \times H$ be a hyperbolic linear map (i.e., there is no eigenvalue $e$ of $T$ with $|e| = 1$) such that $T|_G$ is an expansion and $T|_H$ is a contraction. Let $Q = T|_G$. Again, this is not really a restriction, the majority of well-studied tilings fit into this framework.

**Definition 2.1.** *Let $\mathcal{T}$ be a self-similar cut-and-project tiling with substitution $\sigma$ and expansion $Q$. Let $\mathcal{D} = (\mathcal{D}_{ij})_{1\le i,j\le m}$ be a corresponding digit matrix as in Definition 1.1 such that $\sigma$ is the substitution induced by $\mathcal{D}$. Furthermore, let $Q\pi_1(\Lambda) \subset \pi_1(\Lambda)$, and each $d \in \mathcal{D}_{ij}$ be in $\pi_1(\Lambda)$, with $\pi_1$ and $\Lambda$ as in Definition 1.5. Then $(\mathcal{D}^\star)^T$ defines a substitution $\sigma^\star$ with expansion $Q' := (T|_H)^{-1}$, which is called the* $\star$-dual substitution.

Here, $\mathcal{D}^\star = (\mathcal{D}^\star_{ij})_{1\le i,j\le m}$, and $\mathcal{D}^\star_{ij} = \{d^\star \mid d \in \mathcal{D}_{ij}\}$.

If the substitution is one-dimensional and self-similar with substitution factor $\lambda$, where $\lambda$ is a unimodular $PV$-number, then $Q' = \big(\mathrm{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)\big)^{-1}$, where $\lambda_2, \ldots \lambda_N$ are the Galois conjugates of $\lambda$, see [10], [8]. Therefore the name 'Galois duals' was coined in [23].

**Remark:** Note that the star-dual substitution also defines self-similar cut-and-project tilings, with the roles of $G$ and $H$ in Definition 1.5 reversed. In particular, a star-dual tiling of some tiling in $G$ is a tiling in $H$. There are tilings of the Euclidean plane (or the line) with $p$-adic internal spaces $H$, see [3], [20]. For instance, the star-dual tiling of a plane tiling may be a tiling in $\mathbb{Q}_p \times \mathbb{Q}_p$, where $\mathbb{Q}_p$ is the field of $p$-adic integers, which is a locally compact Abelian group as well.

**Remark:** It follows immediately from the definition that $(\sigma^\star)^\star = \sigma$. By Lemma 1.4, we can freely choose between applying the star map to $\mathcal{D}$ or $Q^{-1}\mathcal{D}$. The resulting substitutions are the same, up to a scaling of the prototiles.

**Example:** (Fibonacci squared) Consider the word substitution $\zeta$ on the alphabet $\{a, b\}$ given by $\zeta(a) = aab$, $\zeta(b) = ab$. The substitution matrix (or 'incidence matrix') is $M_\zeta = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. It is well known — and easy to see — that the PF-eigenvalue (the leading eigenvalue) of $M$ is the substitution factor for the tile-substitution. Moreover, the entries of a left eigenvector of $M$, corresponding to the PF-eigenvalue, contains the relative lengths of the prototiles for a self-affine tile-substitution [8], [16]. The leading eigenvalue in this case is $\tau^2 = \frac{\sqrt{5}+3}{2}$, which is the substitution factor of the desired substitution. Requiring the prototiles to be intervals, the lengths can be chosen as $\ell_a = 1$, $\ell_b = \tau^{-1}$. (Here is some freedom, we may choose any multiples of these lengths as well.) One appropriate tile-substitution is given by the digit set

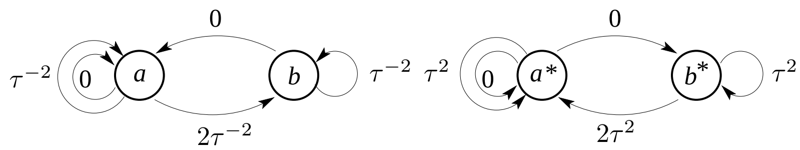


FIGURE 2. The automaton for the IFS for the squared Fibonacci substitution and its dual automaton.

matrix

$$\begin{pmatrix} \{0,1\} & \{0\} \\ \{2\} & \{1\} \end{pmatrix}.$$

The prototiles $T_a = [0,1]$ and $T_b = [0, \tau^{-1}]$ arise as the unique compact nonempty solution of the corresponding IFS

$$\begin{aligned} T_a &= \tau^{-2} T_a \cup \tau^{-2}(T_a + 1) \cup \tau^{-2}(T_b + 2), \\ T_b &= \tau^{-2} T_a \cup \tau^{-2}(T_b + 1). \end{aligned} \tag{4}$$

We proceed by considering the digit set of this IFS, $\mathcal{D} = \left(\begin{smallmatrix} \{0,\tau^{-2}\} & \{0\} \\ \{2\tau^{-2}\} & \{\tau^{-2}\} \end{smallmatrix}\right)$, in agreement with the last remark, and in order to illustrate the action of Galois conjugation. The IFS (4) can be represented by the automaton in Figure 2, left. (Vice versa, the automaton gives rise to the IFS (4)). The dual substitution $\zeta^\star$ arises from the IFS defined by $(\mathcal{D}^\star)^T$, or equivalently: from the dual automaton, where the orientation of each edge is reversed and the star map is applied to each edge label, see Figure 2, right. In this case, the star map is uniquely defined by $1^\star = 1$, $\tau^\star = -\tau^{-1}$ (each $\tau$ which occurs is replaced by its Galois conjugate $-\tau^{-1}$). Thus,

$$(\tau^{-2})^\star = (1 - \tau^{-1})^\star = (2 - \tau)^\star = 2 + \tau^{-1} = 1 + \tau = \tau^2.$$

The corresponding digit matrix, defining the dual substitution $\zeta^\star$, is

$$(\mathcal{D}^\star)^T = \begin{pmatrix} \{0, \tau^2\} & \{2\tau^2\} \\ \{0\} & \{\tau^2\} \end{pmatrix}.$$

The solution of the corresponding IFS — with digits $\tau^{-2} d_{ijk}$, where $d_{ijk}$ are the digits in $(\mathcal{D}^\star)^T$ — yields the prototiles $[0,\tau]$ and $[\tau, \tau+1]$ for $\zeta^\star$. Since this dual tile substitution is nice in the sense that the tiles are intervals — rather than some disconnected, fractally shaped sets — it can be written as a word substitution $a^\star \mapsto a^\star b^\star a^\star$, $b^\star \mapsto b^\star a^\star$. In particular, even though $\zeta \neq \zeta^\star$, they generate — up to a scaling by $\tau$ — the same tiling spaces (the same tilings), resp. the same biinfinite words.

## 3. DIMENSION 2

In dimension one there is a canonical way to associate a tiling to a model set, which is, by definition, not a tiling but a uniformly dense point set: The points define a partition of the line into intervals, these intervals are the tiles. Analogously, a tiling of $\mathbb{R}^1$ by intervals defines a point set by considering the set of all vertices, i.e., the set of all boundary points of the intervals. In particular, by assigning each interval to its left boundary point, there is a one-to-one correspondence between tiles of the tiling and points in the point set.

In two dimensions the situation becomes more difficult, at least if one wants to study the different types of tiles/points $x$ through their position $x^\star$ in the window $W$. It is not obvious which way to identify tilings with point sets (and vice versa) is the best one, or the most natural one.

On one hand, it seems natural to consider the vertices of a tiling $\mathcal{T}$. In fact, this point of view is taken in many cases, see e.g. [2], [12]. Then, in general, there is no natural one-to-one correspondence between tiles and points. In particular, the positions $x^\star$ in $W$, where $x$ is some vertex, may give us no relevant information about the tiles in $\mathcal{T}$. One simple reason is that, in general, the number of tiles per unit area differs from the number of vertices per unit area.

On the other hand, in order to assign a point to each tile, one can use control points [22]. Then there is a one-to-one correspondence between tiles and points. But there are many ways to choose such control points, thus one may end up with different windows for different choices.

In the sequel we avoid all the problems mentioned above by computing some $\star$-dual substitutions of prominent two-dimensional tilings. These computations follow the methods in [10] closely. In the framework of Definition 1.1, the number of prototiles of a two-dimensional substitution is considerably large, since we identify prototiles only up to translation, not up to isometries. For instance, the Penrose tiling uses only two different triangles (up to isometries) as prototiles, see Figure 1. But each of those occurs in 10 different orientations in the tilings. Moreover, we need to distinct a tile and its mirror image (see the dots in Figure 1), thus there are 40 prototiles altogether with respect to translations. Rather than working with a $40 \times 40$ digit matrix, we will slightly change our framework: Instead of digits representing translations, we go over to 'digits' which are isometries. The entries in the according matrices now are functions rather than translation vectors. Each function is of the form $x \mapsto QRx + d$, where $Q$ is — as usual — an expanding linear map which is the same for all functions, $R$ is some rotation or reflection, and $d$ is a translation vector, as above. In other words: We consider *matrix function systems* instead of digit matrices, cf. [13]. Since we are considering plane tilings, we will identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Then, a rotation about the origin is just a multiplication with some complex number $z$ with $|z| = 1$. Each reflection in the complex plane can be expressed as complex conjugation, followed by some rotation. The expanding map $Q$ will always be a homothety in the remainder of this section. Our goal is to express all translations and rotations, and thus all maps $f_i$ occurring in the IFS under consideration, in the ring $\mathbb{Z}[\xi]$, where $\xi$ is some root of unity. To obtain the dual substitution, we consider the inverse maps $f_i^{-1}$ and apply a primitive element of the Galois group $Gal(\mathbb{Z}[\xi]/\mathbb{Z})$ to them. In plain words: we replace each $\xi$ occurring by an appropriate Galois conjugate $\xi'$ in order to obtain the desired maps $(f_i^{-1})^\star$. This is in fact how the star map acts in the following cases, compare [2]. To simplify notation, we denote $(f^{-1})^\star$ by $f^\sharp$.

3.1. **The $\star$-dual of the Penrose tiling.** Consider the prototiles of the Penrose tiling as in Figure 1. It is known that the vertices of a Penrose tiling are closely related to the cyclotomic field $\mathbb{Q}(\xi_5)$, where $\xi := \xi_5 = e^{\frac{2\pi i}{5}}$. E.g., all vertices, and all edge lengths, can be expressed in the ring $\mathbb{Z}[\xi]$ of integers in $\mathbb{Q}(\xi)$. Thus, let the small triangle $S$ have vertices 0, $-1$ and $e^{\frac{2\pi i 3}{10}} = -\xi^4$, and the large triangle have vertices 0, $\tau$ and $e^{\frac{2\pi i}{10}} = -\xi^3$. Then, the IFS for these tiles — arising from the substitution — uses the common contracting factor $\tau^{-1} = \xi + \xi^4$,

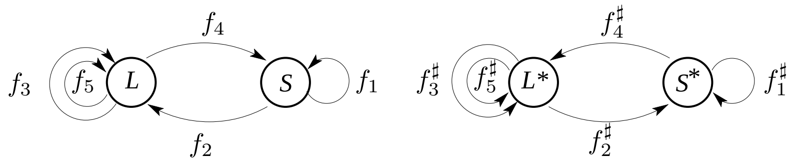


FIGURE 3. The automaton of the IFS of the Penrose tiles (left), compare (5), and its dual automaton (right).

and the IFS reads:

$$S = f_1(S) \,\cup\, f_2(L), \quad L = f_3(L) \,\cup\, f_4(S) \,\cup\, f_5(L), \tag{5}$$

where

$$\begin{aligned} f_1(x) &= (\xi+\xi^4)(-\xi^4)x - 1 - \xi^3, \\ f_2(x) &= (\xi+\xi^4)(-\xi)x - 1 - \xi^3 - \xi^4, \\ f_3(x) &= (\xi+\xi^4)\xi^3 x + 1 + \xi, \\ f_4(x) &= (\xi+\xi^4)(-\xi^2)\overline{x} + 1 - \xi^3, \\ f_5(x) &= (\xi+\xi^4)(-1)\overline{x} + 1 - \xi^2 - \xi^3. \end{aligned}$$

The corresponding automaton is shown in Figure 3 (left). Note, that in [10] the maps are different and do *not* define the Penrose substitution. Thus, the author of [10] did not compute the $\star$-dual (or 'Galois-dual') of the Penrose substitution, but of a different one, using the same triangles $S$ and $L$ as prototiles.

In order to calculate the $\star$-dual substitution, consider the dual automaton (see Figure 3, right), resp. the dual IFS. Let $\varphi \in Gal(\mathbb{Z}[\xi])$ with $\varphi(\xi) = \xi^3$. Note that $\varphi$ is uniquely defined by this requirement. The $f_i^\sharp$ are obtained by applying $\varphi$ to $f_i^{-1}$:

$$\begin{aligned} f_1^\sharp(x) &= (\xi+\xi^4)(\xi^3 x + \xi^3 + \xi^2) \\ f_2^\sharp(x) &= (\xi+\xi^4)(\xi^2 x + \xi + \xi^2 + \xi^4) \\ f_3^\sharp(x) &= (\xi+\xi^4)(-\xi x + \xi + \xi^4) \\ f_4^\sharp(x) &= (\xi+\xi^4)(\xi\overline{x} - \xi + \xi^2) \\ f_5^\sharp(x) &= (\xi+\xi^4)(\overline{x} - 1 + \xi + \xi^4) \end{aligned}$$

The corresponding dual IFS reads:

$$S^\star = f_1^\sharp(S^\star) \,\cup\, f_4^\sharp(L^\star), \qquad L^\star = f_2^\sharp(S^\star) \,\cup\, f_3^\sharp(L^\star) \,\cup\, f_5^\sharp(L^\star). \tag{6}$$

A numerical computation of the solution yields the image in Figure 4 (left). In the figure, the two components of the solution are torn apart; actually, the solutions do overlap. This fact is not relevant, due to Theorem 1.3. The important point is that the dissection of each prototile is non-overlapping.

It is reasonable to assume that the solution consists of two triangles similar to the Penrose triangles. In order to make this precise we list the coordinates of the triangles:

$$L^\star : 1+\xi^2+\xi^4,\, 1+\xi+\xi^3,\, 1+\xi^2+\xi^3, \quad S^\star : \xi^2, \xi^3, 2\xi^2 - \xi + \xi^4 - 2\xi^3.$$

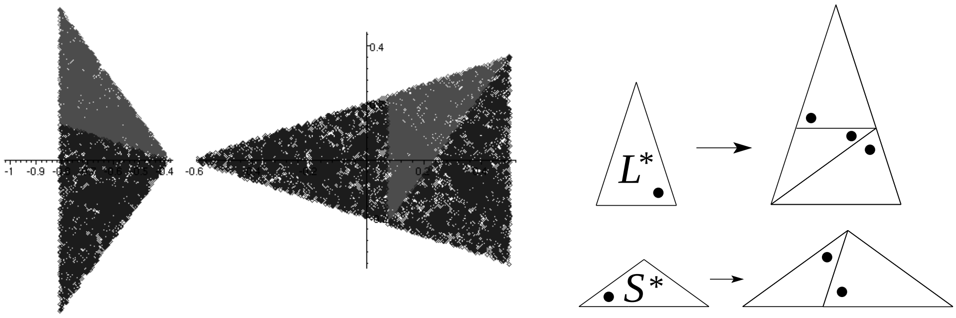


FIGURE 4. A numerical computation of the solution of the dual IFS of the Penrose substitution (left), and the Tübingen triangle substitution (right). One may guess from the images that this is the $\star$-dual of the Penrose substitution, which turns out to be true.

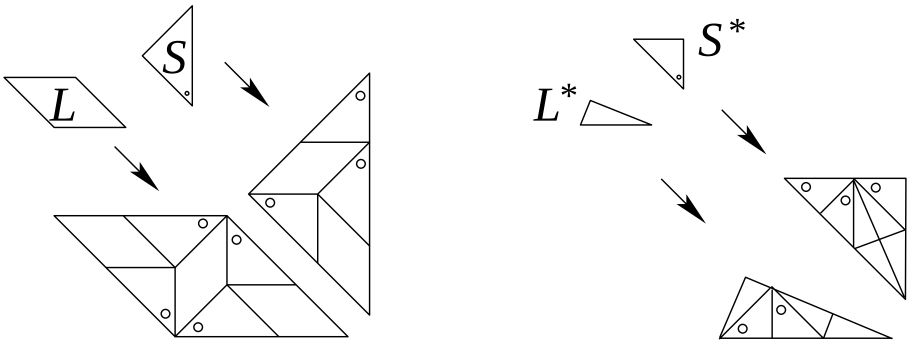


FIGURE 5. The substitution rule for the Ammann-Beenker tiling (left), with prototiles $S, L$, and the star-dual substitution (right), with prototiles $S^\star, L^\star$. Even though the two substitutions look different on a first glance, they define very similar tilings, see Figure 6.

All this information provided, it is an easy — but pretty lengthy — exercise to check that these triangles are in fact a solution of (6). The tile-substitution arising from (6) is well-known: It is the Tübingen Triangle tile-substitution (see Figure 4, right). See [9] for more details and images of this substitution, as well as several others. Altogether we have established:

**Theorem 3.1.** *The $\star$-dual of the Penrose substitution is the Tübingen triangle substitution, and vice versa.*

3.2. **The $\star$-dual of the Ammann-Beenker tiling.** In a very similar fashion, one can compute the star-dual of the Ammann-Beenker tiling [11]. The substitution rule of the Ammann-Beenker tiling is shown in Figure 5 (left). A part of an Ammann-Beenker tiling is shown in Figure 6 (left). The Ammann-Beenker tiling is closely related to the cyclotomic field $\mathbb{Q}(\xi_8)$, where $\xi := \xi_8 = e^{\frac{2\pi i}{8}}$. Its $\star$-dual can be calculated in a very similar fashion as in the case of the Penrose tiling. The substitution factor is $\lambda = 1 + \sqrt{2} = 1 + \xi + \xi^7 = 1 + \xi + \xi^{-1}$, its inverse $\lambda^{-1} = -1 + \xi + \xi^{-1}$. Let $S$ be the triangular prototile, with vertices $0, i, 1+i$, and let $L$ the rhombic prototile with vertices $0, 1, 1+\xi, \xi$. The IFS for these prototiles is

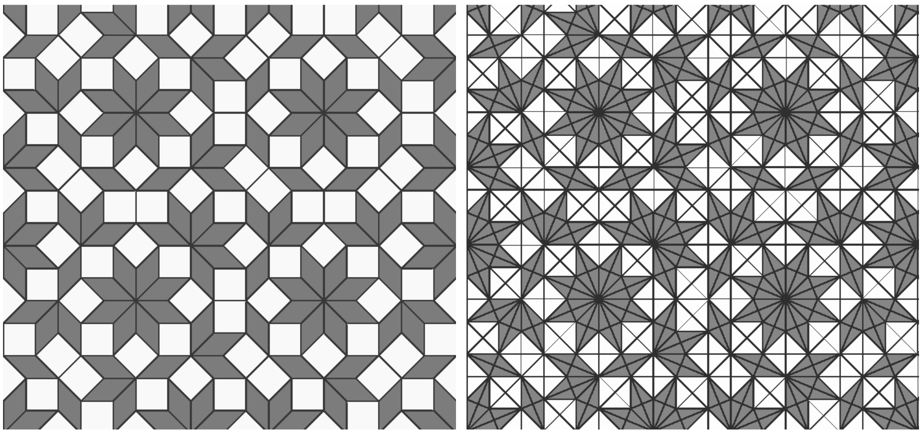

FIGURE 6. Patches of the Ammann-Beenker tiling (left) and its star-dual (right). The patches are equivalent in the sense, that one obtains the right patch by dividing each tile in the left patch into four tiles.

$$L = f_1(L) \cup f_2(L) \cup f_3(L) \cup f_6(S) \cup f_7(S) \cup f_8(S) \cup f_9(S),$$
$$S = f_4(L) \cup f_5(L) \cup f_{10}(S) \cup f_{11}(S) \cup f_{12}(S),$$

where

$$\begin{aligned} f_1(x) &= \lambda^{-1}\xi^4 x + 1 + \xi, & f_7(x) &= \lambda^{-1}\xi^3 x + 1 + 2\xi + \xi^2 + \xi^7, \\ f_2(x) &= \lambda^{-1}\xi\overline{x} + 1 + \xi + \xi^2 + \xi^7, & f_8(x) &= \lambda^{-1}\xi^6\overline{x} + 2 + \xi + \xi^2 + \xi^7, \\ f_3(x) &= \lambda^{-1}\xi^7\overline{x} + 1 + \xi + \xi^2, & f_9(x) &= \lambda^{-1}\xi^7 x + 1, \\ f_4(x) &= \lambda^{-1}x + \xi + \xi^2, & f_{10}(x) &= \lambda^{-1}\xi^5 x + \xi + \xi^3, \\ f_5(x) &= \lambda^{-1}\xi^2 x + \xi, & f_{11}(x) &= \lambda^{-1}\xi^6\overline{x} + 1 + \xi + \xi^2, \\ f_6(x) &= \lambda^{-1}\xi^2\overline{x} + \xi, & f_{12}(x) &= \lambda^{-1}\xi^3 x + 2\xi + \xi^2. \end{aligned}$$

The functions $f_i^\sharp$ of the dual IFS are easily obtained by considering the inverse functions $f_i^{-1}$, and replacing each $\xi$ occurring by its Galois conjugate $\xi^3$ (or $\xi^5$ or $\xi^7$, the resulting dual tiling is always the same). The dual IFS reads:

$$L^\star = f_1^\sharp(L^\star) \cup f_2^\sharp(L^\star) \cup f_3^\sharp(L^\star) \cup f_4^\sharp(S^\star) \cup f_5^\sharp(S^\star),$$
$$S^\star = f_6^\sharp(L^\star) \cup f_7^\sharp(L^\star) \cup f_8^\sharp(L^\star) \cup f_9^\sharp(L^\star) \cup f_{10}^\sharp(S^\star) \cup f_{11}^\sharp(S^\star) \cup f_{12}^\sharp(S^\star).$$

Its solution are the two tiles shown in Figure 5 (right): $S^\star$ is an isosceles orthogonal triangle, and $L^\star$ is a thin orthogonal triangle. Note that the original triangular prototile $S$ can be dissected into two copies of $S^\star$, and the rhombic prototile $L$ can be dissected into four copies of $L^\star$. In fact, unlike in the Penrose case, the $\star$-dual tiling of the Ammann-Beenker tiling is closely related to the Ammann-Beenker tiling itself, compare Figure 6: dissecting each $L$ into four tiles, and each $S$ into two tiles, yields the $\star$-dual tiling of the Ammann-Beenker tiling. Altogether we have established:

**Theorem 3.2.** *The $\star$-dual of the Ammann-Beenker tiling is mutually locally derivable with the Ammann-Beenker tiling.*

## 4. Self-duality

The last example, the Ammann-Beenker tiling and its $\star$-dual, motivates the question whether there are substitutions which are strictly self-dual.

**Definition 4.1.** *Let $\sigma$ be a tile-substitution such that $\mathbb{X}_\sigma$ consists of cut-and-project tilings. If $\mathbb{X}_\sigma = \mathbb{X}_{\sigma^\star}$, then $\sigma$ is called* self-dual (with respect to $\star$-duality).

Thus, the example in Section 2, Fibonacci squared, is self-dual. Since the star-dual tiling of a tiling in $G$ is a tiling in $H$ (cf. Definition 1.5), a trivial necessary condition for a tiling to be self-dual is $G = H$. Hence, if the substitution factor $\lambda$ of a self-dual substitution is a unimodular PV-number, then $\lambda$ must be a quadratic algebraic number. A refined necessary condition is obtained by considering the substitution matrix $M_\sigma$.

**Lemma 4.2.** *If $\sigma$ is a self-dual substitution in $\mathbb{R}^d$, then the following property holds for the substitution matrix $M_\sigma = (|\mathcal{D}_{ij}|)$:*

$$(M_\sigma)^T = P M_\sigma P^{-1}, \tag{7}$$

*where $P$ is some permutation matrix.*

*Proof.* The result follows immediately from $M_{\sigma^\star} = \big(|(\mathcal{D}_{ij})^\star|\big)^T$, by definition of $\sigma^\star$. The tiling spaces $\mathbb{X}_\sigma$ defined by a substitution are unique up to enumeration of the tiles, thus the permutations enter. □

In fact, permutations has to be taken into account. There are self-dual substitutions where $(M_\sigma)^T \neq M_\sigma$; in other words: where $M_\sigma$ is not symmetric. For instance, the dual substitution of $\sigma: \; a \to ab, \; b \to aab$ is $\sigma^\star: \; a^\star \to b^\star b^\star a^\star, \; b^\star \to b^\star a^\star$. By identifying the first tile — $a$ — of $\sigma$ with the second tile — $b^\star$ — of the dual substitution $\sigma^\star$, one obtains that the two substitutions define the same tiling spaces. Hence $\sigma$ is self-dual. The substitution matrix $M_\sigma = \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right)$ is not symmetric, but

$$P^{-1} M_\sigma P = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\2&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&2\\1&1\end{pmatrix} = (M_\sigma)^T,$$

thus (7) holds.

A sufficient condition for a substitution to be self-dual is an immediate consequence of Theorem 1.3.

**Corollary 4.3.** *Let $\sigma$ be a substitution with digit matrix $\mathcal{D}$ and $\sigma^\star$ its dual substitution, with digit matrix $\mathcal{D}' = (\mathcal{D}^\star)^T$. If there are vectors $t_1, t_2, \ldots, t_m \in \mathbb{R}^d$, $A := (t_j - Qt_i)_{1 \le i,j \le m}$ and a permutation matrix $P$ such that*

$$P\mathcal{D}'P^{-1} = \mathcal{D} + A,$$

*then $\sigma$ is self-dual.* □

This result gives a purely algebraic test for self-duality.

## 5. Outlook

The notion of star-duality is a concept tailored to arbitrary model sets in any locally compact Abelian group. Restricted to the unimodular case, it coincides with other notions of duality, in particular the 'natural decomposition method' in [21], and the dual tilings considered by P. Arnoux, S. Ito and others, see for instance [7], [17]. This is the reason why we were lazy at some occasions, using the term 'self-dual' instead of 'self-$\star$-dual' or 'self-dual w.r.t. star-duality'. Theorem 1.3 can be utilised to show the equivalence of the concepts mentioned.

The benefit of star-duality seems to be its generality, and its accessibility by purely algebraic methods. This paper tries to make first steps in this direction. For future work, it should be promising to apply star-duality to the case of word substitutions on two or three letters. In the case of Sturmian substitutions (symbolic substitutions on two letters, where the corresponding biinfinite words contain exactly $n+1$ subwords of length $n$, for all $n \geq 0$, see [16]) the results in [6] can be used to give a complete characterisation of self-dual (Sturmian) substitutions. For details we refer to future work.

A further question is: are there self-dual substitutions in dimension $d \geq 2$? The Ammann-Beenker tiling is very close to be self-dual, but is is not exactly self-dual. In dimension one, there are several self-dual substitutions. Trivially, these examples generalise to higher dimensions, by using the Cartesian product (for instance, Fibonacci times Fibonacci). But it remains to find a two-dimensional self-dual substitution apart from that.

## Acknowledgements

It is a pleasure to thank V. Sirvent for cooperation and helpful comments. The author is indebted to the referee for many valuable remarks. This work was supported by the German Research Council (DFG), within the CRC 701.

## References

[1] M. Baake: A guide to mathematical quasicrystals, in *Quasicrystals*, eds J.-B. Suck, M. Schreiber and P. Häussler, Springer, Berlin (2002), pp. 17-48; `math-ph/9901014`.

[2] M. Baake, P. Kramer, M. Schlottmann, D. Zeidler: Planar patterns with fivefold symmetry as sections of periodic structures in 4-space, *Int. J. Mod. Phys.* **B 4**, (1990) 2217-2268.

[3] M. Baake, R.V. Moody and M. Schlottmann: Limit-(quasi)periodic point sets as quasicrystals with $p$-adic internal spaces, *J. Phys. A: Math. Gen.* **31** (1998) 5755-5765; `math-ph/9901008`.

[4] M. Baake, M. Schlottmann and P.D. Jarvis: Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability, *J. Phys. A: Math. Gen.* **24** (1991) 4637-4654.

[5] C. Bandt: Self-similar tilings and patterns described by mappings, in: *The mathematics of long-range aperiodic order*, ed.: R.V. Moody, NATO ASI Ser. C, Math. Phys. Sci. 489 (1997) pp. 45-83.

[6] V. Berthé, A. de Luca, C. Reutenauer: On an involution of Christoffel words and Sturmian morphisms, Rapport de recherche 06044, preprint, LIRMM (2006).

[7] H. Ei: Some properties of invertible substitutions of rank $d$, and higher dimensional substitutions. *Osaka J. Math.* **40** (2003) 543-562.

[8] D. Frettlöh: Duality of model sets generated by substitutions, *Rev. Roumaine Math. Pures Appl.* **50** (2005) 619-639; `math.MG/0601064`.

[9] D. Frettlöh, E. Harriss: Tilings encyclopedia, available online at: `http://tilings.math.uni-bielefeld.de`.

[10] G. Gelbrich: Fractal Penrose tiles II: Tiles with fractal boundary as duals of Penrose triangles, *Aequationes Math.* **54** (1997) 108-116.

[11] B. Grünbaum, G.C. Shephard: *Tilings and patterns*, Freeman, New York (1987).

[12] E. O. Harriss: *On Canonical Substitution Tilings*, PhD Thesis, Imperial College London (2003).
[13] J.-Y. Lee, R.V. Moody and B. Solomyak: Pure point dynamical and diffraction spectra, *Ann. Henri Poincaré* **3** (2002) 1003-1018; `mp_arc/02-39`.
[14] Y. Meyer: *Algebraic numbers and harmonic analysis*, North-Holland Math. Lib. **2** North-Holland, Amsterdam (1972).
[15] R.V. Moody: Model sets: A survey, in: *From Quasicrystals to More Complex Systems*, eds. F. Axel, F. Dénoyer, J.P. Gazeau, EDP Sciences, Les Ulis, and Springer, Berlin (2000), pp. 145-166; `math.MG/0002020`.
[16] N. Pytheas Fogg: *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Mathematics **1794**, Springer, Berlin (2002).
[17] Y. Sano, P. Arnoux, S. Ito: Higher dimensional extensions of substitutions and their dual maps, *J. Anal. Math.* **83** (2001) 183-206.
[18] A. Siegel: Reprsèentation des systm̀es dynamiques substitutifs non unimodulaires, *Ergodic Theory and Dynam. Systems* **23** (2003) 1247-1273.
[19] B. Sing: Iterated function systems in mixed Euclidean and p-adic spaces, in: *Complexus Mundi - Emergent Patterns in Nature*, ed. M.M. Novak, World Scientific, Singapore (2006), pp. 267-276.
[20] B. Sing: Kolakoski-(2m,2n) are limit-periodic model sets, *J. Math. Phys.* **44**, (2003) 899-912.
[21] V.F. Sirvent, Y. Wang: Self-affine tiling via substitution dynamical systems and Rauzy fractals, *Pacific J. Math.* **206** (2002) 465-485.
[22] B. Solomyak: Dynamics of self-similar tilings, *Ergodic Theory Dynam. Systems* **17** (1997) 695-738.
[23] W. Thurston: Groups, tilings and finite state automata, Lecture notes AMS 1989.
[24] A. Vince: Digit tiling of Euclidean space, in *Directions in mathematical quasicrystals*, eds. M. Baake and R.V. Moody, CRM Monograph Series Vol.13, AMS, Providence, RI (2000) pp 329-370.

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

*E-mail address*: `dirk.frettloeh@math.uni-bielefeld.de`

*URL*: `http://www.math.uni-bielefeld.de/baake/frettloe`