Dual substitution patterns arising from model sets

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Summer School in Aperiodic Order

August 8–13, 2005

University of Victoria

Substitutions

Symbolic substitution: $\mathcal{A}$ alphabet, $\mathcal{A}^\star$ all finite words.

$$\sigma : \mathcal{A} \to \mathcal{A}^\star$$

Set $\sigma(ab) := \sigma(a)\sigma(b)$, then σ extends to $\mathcal{A}^\star$ and $\mathcal{A}^{\mathbb{Z}}$.

Ex.: $\mathcal{A} = \{S, L\}, \quad \sigma(S) = L, \quad \sigma(L) = LS.$

$$S \stackrel{\sigma}{\to} L \stackrel{\sigma}{\to} SL \stackrel{\sigma}{\to} LSL \stackrel{\sigma}{\to} LSLLS \stackrel{\sigma}{\to} LSLLSLSL \stackrel{\sigma}{\to} \cdots$$

Geometric substitution:

T_i *prototiles*, Q expanding linear map,

$$QT_i = (T_{i_1} + x_{i_1}) \cup (T_{i_2} + x_{i_2}) \cup \cdots \cup (T_{i_{n(i)}} + x_{i_{n(i)}})$$

(nonoverlapping). Then

$$\sigma(T_i) := \{T_{i_1} + x_{i_1}, T_{i_2} + x_{i_2}, \ldots, T_{i_{n(i)}} + x_{i_{n(i)}}\}.$$

σ extends to all sets $\{T_j + x_i \mid T_j \text{ prototile}, i \in I\}$.

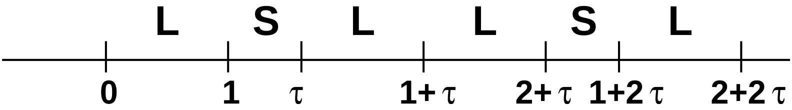
Ex.:

$L = [0,1],\ S = [1,\tau] \quad (\tau = \frac{1+\sqrt{5}}{2} \approx .1.618,$ 'golden mean').

$$\tau L = L \cup S, \quad \tau S = L + \tau$$

Thus

$$\sigma(L) = \{L, S\}, \quad \sigma(S) = \{L + \tau\}$$



Model Sets

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

Model Sets

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^{d+e} & \xrightarrow{\pi_2} & \mathbb{R}^e \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

- Λ a *lattice* in $\mathbb{R}^{d+e}$
- π_1, π_2 *projections*
  - $\pi_1|_\Lambda$ injective
  - $\pi_2(\Lambda)$ dense
- W *compact*
  - cl(int(W))= W
  - $\mu(\partial(W)) = 0$

Then $V = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *model set*.

The *star map* : $\quad * : \pi_1(\Lambda) \to \mathbb{R}^e, \; x^* = \pi_2 \circ {\pi_1}^{-1}(x)$

Definition

An algebraic integer λ is called *PV–number*, if $|\lambda| > 1$, and for all its algebraic conjugates λ_i holds: $|\lambda_i| < 1$.
(...Salem–number... if $|\lambda_i| \le 1$)

Theorem [Meyer, '95]

If $\mathcal{T}$ is a (sufficiently nice) *substitution* point set *and* a *model set*, then the substitution
factor is a *PV–number* (or a Salem–number).

Fact [e.g. Pleasants, '00]

For a given substitution on m prototiles, where the factor λ is a *PV–number* of degree m, and a *unit* in $\mathbb{Z}[\lambda]$, there are standard constructions of the lattice Λ in $\mathbb{R}^{d+e}$.

Fact [Schlottmann, '98]

$$\mathrm{dens}(V) = \mu(W)/\det\Lambda$$

($\mathrm{dens}(V) = \lim_{r\to\infty} \frac{V \cap r\mathbb{B}^d}{\mathrm{vol}(r\mathbb{B}^d)}$, average number of points per unit cell.)

Duality – The Example

Consider the 1-dim substitution

$$\boxed{\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML}$$

Substitution matrix: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

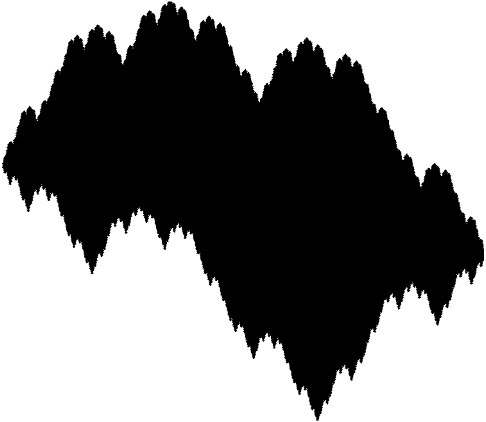
Characteristic Polynomial: $x^3 - 3x^2 + 1$

Eigenvalues: $\lambda = \lambda_1, \qquad \lambda_2, \qquad \lambda_3$

$$\frac{\sin(\frac{4\pi}{9})}{\sin(\frac{\pi}{9})}, \; -\frac{\sin(\frac{\pi}{9})}{\sin(\frac{2\pi}{9})}, \; \frac{\sin(\frac{2\pi}{9})}{\sin(\frac{4\pi}{9})}$$

$$\approx 2.879, \quad -0.532, \quad 0.6527$$

Here: $d = 1, \; e = 2,$

$W =$ 

$$\boxed{\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML}$$

Geometric realization:

$$S: \ \text{interval of length} \frac{\sin(\frac{2\pi}{9})}{\sin(\frac{\pi}{9})} = \lambda - 1$$

$$M: \ \text{interval of length} \frac{\sin(\frac{3\pi}{9})}{\sin(\frac{\pi}{9})} = \lambda^2 - 2\lambda$$

$$L: \ \text{interval of length} \frac{\sin(\frac{4\pi}{9})}{\sin(\frac{\pi}{9})} = \lambda$$

V : set of (right) endpoints of these intervals.

Let $\sigma(V) = V$, e.g.:

$$\ldots SMLLML | LMLSMLLMLMLSMLLMLLMLSMLL \ldots$$

The star map: $\quad * : \pi_1(\Lambda) \to \mathbb{R}^e, \qquad x^* = \pi_2 \circ \pi_1^{-1}(x)$

Here: If $x = k + \ell\lambda + m\lambda^2 \quad \in \mathbb{Z}[\lambda]$,

then $$x^* = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \ell \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} + m \begin{pmatrix} {\lambda_2}^2 \\ {\lambda_3}^2 \end{pmatrix}.$$

$$(\lambda x)^* = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} x^* =: Qx^*$$

$$W = \overline{V^*}$$

$$\boxed{\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML}$$

$$V = \sigma(V) = ...SMLLML|LMLSMLLMLMLSMLLMLLMLSMLL...$$

$$\begin{array}{rcl} V &=& V_L \quad \cup \quad V_M \quad \cup \quad V_S \\ V_L &=& \lambda V \quad \cup \quad \lambda V_L - \lambda^2 + \lambda \\ V_M &=& \lambda V - \lambda \\ V_S &=& \lambda V_M - \lambda^2 + \lambda \end{array}$$

$$
\begin{aligned}
V^* &= V_L^* \cup V_M^* \cup V_S^* \\
V_L^* &= QV^* \cup QV_L^* - \begin{pmatrix} {\lambda_2}^2 \\ {\lambda_3}^2 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} \\
V_M^* &= QV^* - \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} \\
V_S^* &= QV_M^* - \begin{pmatrix} {\lambda_2}^2 \\ {\lambda_3}^2 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}
\end{aligned}
$$

IFS:

A pair $(\mathbb{R}^d, F)$, where $F = \{f_0, f_1, \ldots, f_n\}$ is a set of contractive maps.

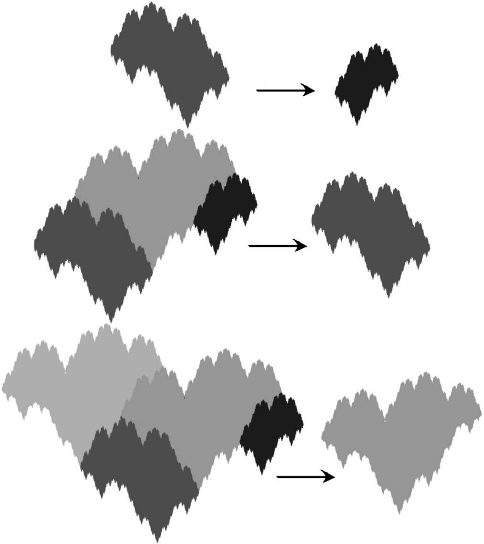
Theorem [Hutchinson, '81]

For any IFS exists a unique compact set K such that $K = \bigcup_{f_i \in F} f_i(K)$.

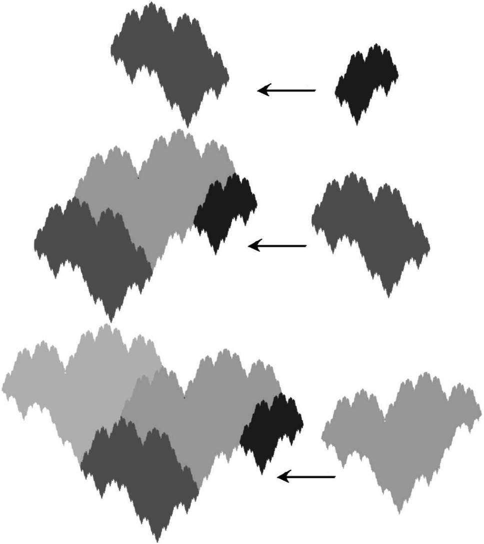
The same is true for multi-component IFS:

$$K = \bigcup_{i=1}^{n} K_i, \quad K_i = f_{ij}(K_{i(j)})$$

The Dual Substitution



The Dual Substitution



Choose control points on tiles $\rightsquigarrow$ $V' = V'_S \cup V'_M \cup V'_L$.

Dual substitution σ', choose V' s.t. $\sigma(V') = V'$.

Equation system for V':

$$\begin{array}{ccccccc}
V' & = & V'_L & \cup & V'_M & \cup & V'_S \\
V'_L & = & Q^{-1}V'_L & \cup & Q^{-1}V'_M & \cup & Q^{-1}V'_L - \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
V'_M & = & Q^{-1}V'_S & \cup & Q^{-1}V'_M - \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} & \cup & Q^{-1}V'_L - \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} \\
V'_S & = & Q^{-1}V'_L + \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} - 2\begin{pmatrix} {\lambda_2}^2 \\ {\lambda_3}^2 \end{pmatrix} & \cup & Q^{-1}V'_M + \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} - 2\begin{pmatrix} {\lambda_2}^2 \\ {\lambda_3}^2 \end{pmatrix} & &
\end{array}$$

Apply the new star map $\quad x^\star = \pi_1(\pi_2^{-1}(x)) \quad \rightsquigarrow$ IFS.

Solution of this new IFS:

$$L = [-\lambda, 0], \quad M = [\lambda - \lambda^2, -\lambda], \quad S = [-2\lambda^2 + 1, -2\lambda^2 + \lambda]$$

Claim: V' is a model set (up to a set of zero density).

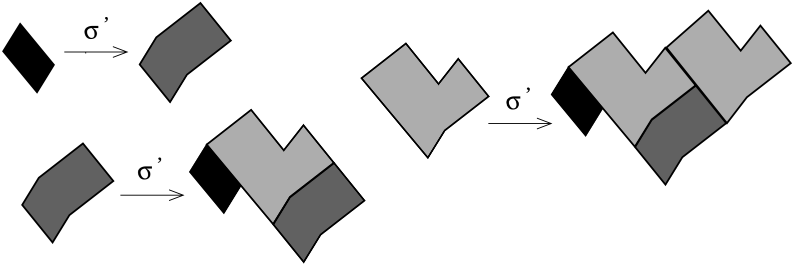
- $\mu(W') = \lambda - 1 > 0$ ✓
- $\mu(\partial W') = 0$ ✓
- $\mathrm{dens}(V') = \mu(W') / \det \Lambda$ **?**

det **Λ** :

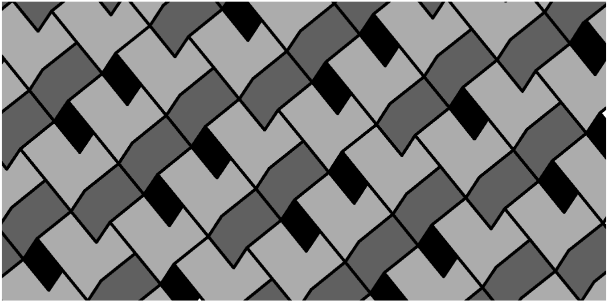
$$(\lambda_2 - \lambda)(\lambda_3 - \lambda)(\lambda_3 - \lambda_2) = \ldots = \frac{27}{64 {s_1}^2 {s_2}^2 {s_4}^2} = 9$$

dens V′:

The new substitution works also with *polygons*:



(not longer shape preserving), and yields a tiling.



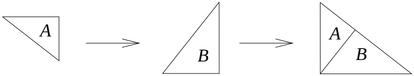
The *areas* of these tiles can be computed exactly.

The relative *frequencies* of the prototiles are known.

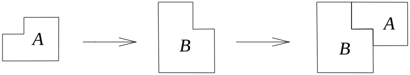
$$\rightsquigarrow \mathrm{dens}(V') = \ldots = (\lambda - 1)/9 = \mu(W')/\det \Lambda$$

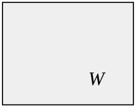
Thus V' is a model set (up to a set of zero density).

a.)

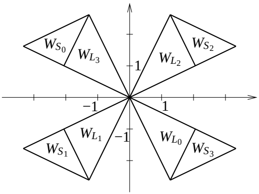

A
B
A
B

b.)

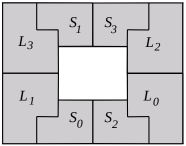

A
B
A
B


W

a.)


W_{S_0}
W_{L_3}
W_{L_2}
W_{S_2}
1
−1
1
W_{L_1}
−1
W_{L_0}
W_{S_1}
W_{S_3}

b.)


S_1
S_3
L_3
L_2
L_1
L_0
S_0
S_2

Resume

Done:

- How to compute certain window sets
- Duality of selfsimilar model sets explained
- All dual tilings shown here are model sets

Todo: E.g.

- Which substitutions are self–dual?
- Duals of Ammann–Beenker, Penrose, . . . ?