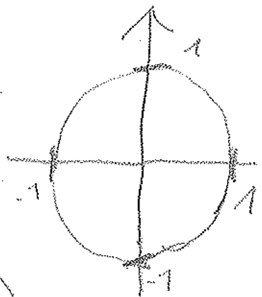
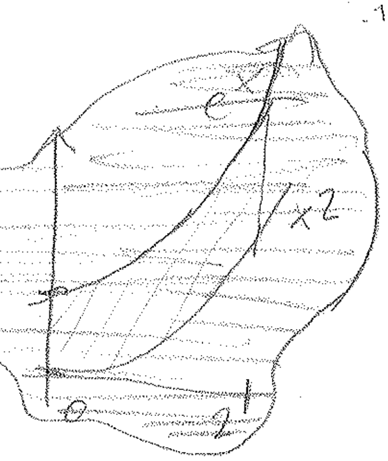
3. Mehrdimensionale Integrale („Integration über Normalbereiche“)

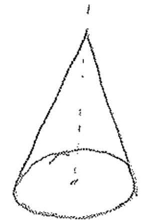
Fragen: 1. Fläche Kreis:



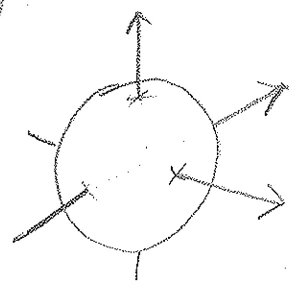
2. Fläche



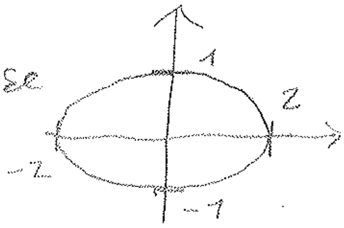
2. Vol Kegel:

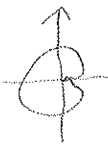
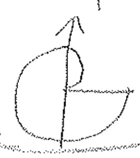


3. Vol Kugel



4. Fläche Ellipse



; (Kardioide  ; Archim. Spirale )

Antwort:

Satz: Ist $G \subseteq \mathbb{R}^2$; so ist Fläche(G)

(3.1) $= Fl(G) = \int_G 1 \, d(x,y)$.

Ist $G \subseteq \mathbb{R}^3$; so ist $Vol(G) = \int_G 1 \, d(x,y,z)$

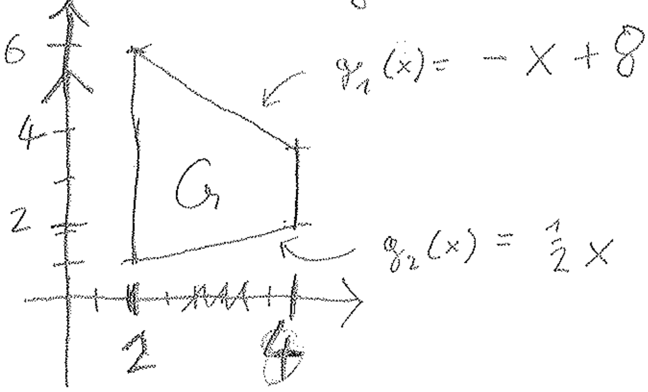
Jetzt müssen wir nur noch „$\int_G f(x,y) d(x,y)$“ verstehen / erklären. Das ist im Allg. sehr schwierig [viele Kapitel in Ana II]. Es ist aber in dem Fall einfach:

(3.2) **SATZ** Ist $f: \mathbb{R}^2 \to \mathbb{R}$ stetig; ist $G = \left\{ \binom{x}{y} \in \mathbb{R}^2 \;\middle|\; a \le x \le b;\ g_1(x) \le y \le g_2(x) \right\}$ so ist $\int_G f(x,y) \, d(x,y) = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \right) dx$. Analog

$G = \left\{ \binom{x}{y} \;\middle|\; a \le y \le b;\ g_1(y) \le x \le g_2(y) \right\}$: $\int_G f(x,y) \, d(x,y) = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y) \, dx \, dy$

<u>Bemerk</u> Die Darstellung für das G zu finden ist oft das Schwierige. Rest ist Routine.

<u>Bsp 0.:</u>



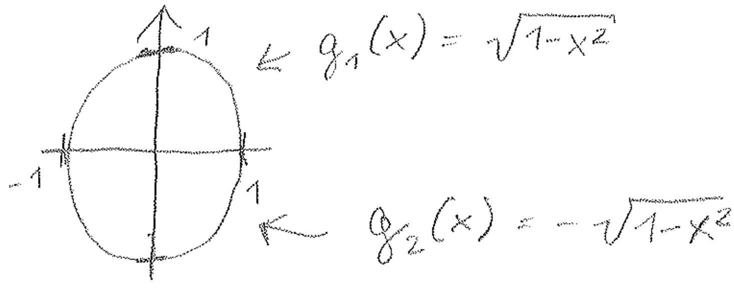
$$Fl(G) = \int_2^4 \int_{\frac{1}{2}x}^{-x+8} 1 \, dy \, dx = \int_2^4 \left(y \Big|_{y=\frac{1}{2}x}^{-x+8} \right) dx$$

$$= \int_2^4 -x+8-\tfrac{1}{2}x \, dx = \int_2^4 -\tfrac{3}{2}x+8 \, dx$$

$$= \left(-\frac{3}{4}x^2 + 8x \Big|_{x=2}^{4} \right) = -\frac{3}{4}\cdot 4^2 + 8\cdot 4 - \left(-\tfrac{3}{4}2^2 + 8\cdot 2\right)$$

$$= -12+32+3-16 = \underline{\underline{7}}$$

<u>Bsp 1:</u>



$$Fl(G) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dy \, dx = \int_{-1}^{1} \left(y \Big|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \right) dx$$

$$= \int_{-1}^{1} \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) dx = \int_{-1}^{1} 2\sqrt{1-x^2} \, dx \qquad x = \sin u$$

$$u = \arcsin x$$

$$\int 2\sqrt{1-x^2}\, dx = \int 2 \cdot \sqrt{1-(\sin u)^2} \cdot \sqrt{1-(\sin u)^2} \, du \qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$= \int 2\left(1-(\sin u)^2\right) du = \int 2\cos^2 u \, du \qquad \Rightarrow dx = \sqrt{1-x^2}\, du$$

$$\int 2\cos^2 u\,du = \int \underbrace{2\cos u}_{f} \cdot \underbrace{\cos u}_{g'}\,du = +2\cos u \sin u + \int 2\sin^2 u\,du$$

$$= 2\cos u \sin u + \int 2(1-\cos^2 u)\,du$$

$$= 2\cos u \sin u + \int 2\,du - 2\int \cos^2 u\,du$$

$$\Rightarrow 2\int 2\cos^2 u\,du = 2\cos u \sin u + 2u \qquad |:2$$

$$\Rightarrow \int \ldots = \int 2\cos^2 u\,du = \cos(\arcsin x)\cdot \sin(\arcsin x) + \arcsin x$$

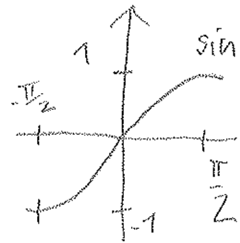
$$= \sqrt{1-\sin^2(\arcsin x)} + \arcsin x$$

$$= \sqrt{1-x^2} + \arcsin x$$

Also $\int_{-1}^{1} 2\sqrt{1-x^2}\,dx = \left(\sqrt{1-x^2} + \arcsin x \Big|_{x=-1}^{1}\right)$

$$= 0 + \frac{\pi}{2} - \left(0 - \frac{\pi}{2}\right)$$

$$= \pi$$



$\sin\frac{\pi}{2} = 1$

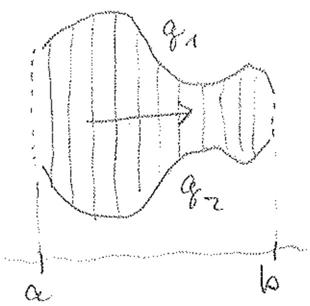
$\Rightarrow \arcsin(\sin\frac{\pi}{2}) = \frac{\pi}{2} =$
$= \arcsin 1$

Ebenso: Fl () $= r^2 \cdot \pi$

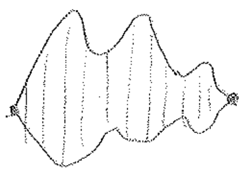
Bsp 2,3,4 analog. (Kardioide, Spirale: später)

Bemerk. So ein Gebiet G wie in 3.2 heißt „Normalbereich

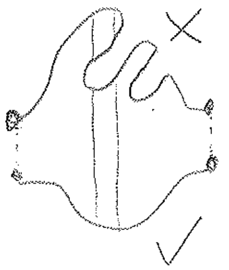
• Idee:



auch



nicht



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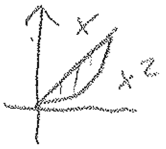
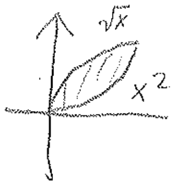
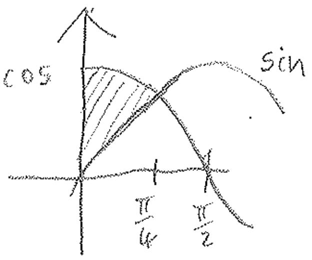
a) 

$G = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; 0 \le x \le 1 ; \; x^2 \le y \le x \right\}$

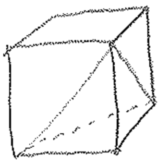
b) 

$G = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; 0 \le x \le 1 ; \; x^2 \le y \le \sqrt{x} \right\}$

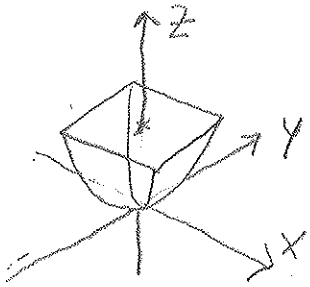
c) 

$G = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; 0 \le x \le \frac{\pi}{4} ; \; \sin x \le y \le \cos x \right\}$

• $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

d) 

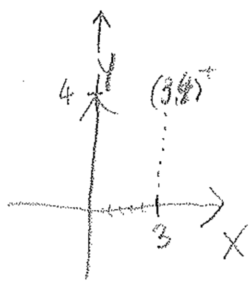
$G = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 0 \le x \le 1 ; \; 0 \le y \le 1 - x ; \; 0 \le z \le 1 - x - y \right\}$

e) Igluzelt. 

$G = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 0 \le z \le 1 ; \; -\sqrt{z} \le x \le \sqrt{z} \right.$
$-\sqrt{z} \le y \le \sqrt{z}$

Jeweils Fläche bzw. Vol. berechnen.

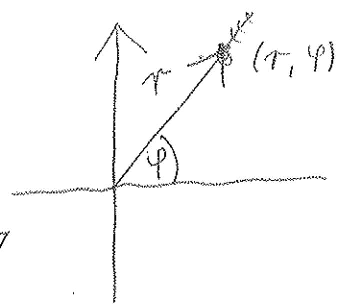
Polarkoordinaten



← (Ost-West & Nord-Süd)

vs.

(Richtung & Entfernung) →



$x, y \in \mathbb{R}$

$r \in \mathbb{R}^+$; $\varphi \in [0, 2\pi[$

Bogenmaß, s.o.

Umrechnung:

- $r = \sqrt{x^2+y^2}$
- $\frac{y}{x} = \tan\varphi$ auflösen, aufpassen:
  - bei $x = 0$ (dann $\varphi \in \{\frac{\pi}{2}; \frac{3\pi}{2}\}$)
  - dass $\varphi \in [0, 2\pi[$; bei Bedarf $+\pi$; $+2\pi$

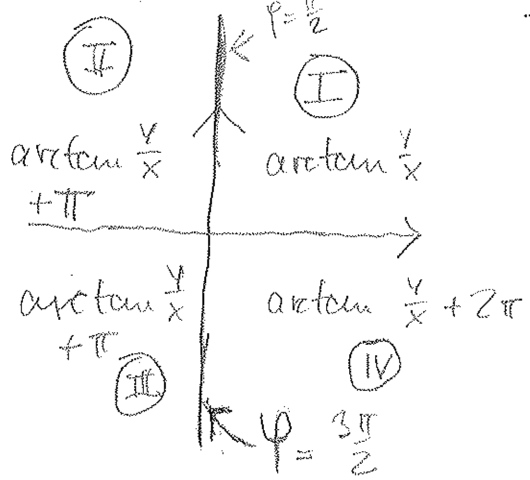
sowie

- $x = r\cos\varphi$; $y = r \cdot \sin\varphi$.

z.B. • $(3,4)^T$ in Kartesischen (also x-y-) Koordinaten

wird zu $r = \sqrt{3^2+4^2} = \sqrt{25} = 5$;

$\frac{4}{3} = \tan\varphi \Rightarrow \varphi = \arctan\frac{4}{3} = 0{,}927\ldots$



• $(\sqrt{2}, \frac{\pi}{4})$ wird zu

$x = \sqrt{2} \cdot \cos\frac{\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{2} = 1$

$y = \sqrt{2} \cdot \sin\frac{\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1.$

Bogenmaß: („Rad")

$2\pi \mathrel{\hat{=}} 360°$

$\pi \mathrel{\hat{=}} 180°$

$\frac{\pi}{2} \mathrel{\hat{=}} 90°$

Umrechnung

$$\text{Rad} \underset{\cdot\frac{2\pi}{360}}{\overset{\cdot\frac{360}{2\pi}}{\rightleftarrows}} \text{Grad}$$

(bzw. $\frac{\pi}{180}$)

In Polarkoordinaten ist ein Kreis (Radius: b) auch ein Normalgebiet: $G = \{(r,\varphi) \mid 0 \le r \le b;\ 0 \le \varphi \le 2\pi\}$

Satz 3.4: Ist $G \subseteq \mathbb{R}^2$ in Polarkoordinaten gegeben;

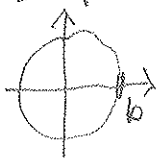
$$G = \{(r,\varphi) \mid a \le \varphi \le b;\ g_1(\varphi) \le r \le g_2(\varphi)\}$$

(bzw. $\mid a \le r \le b;\ g_1(r) \le \varphi \le g_2(r)\}$)

so ist $\quad Fl(G) = \int_a^b \int_{g_1(\varphi)}^{g_2(\varphi)} r \cdot dr\, d\varphi = \int_G r\, d(r,\varphi)$

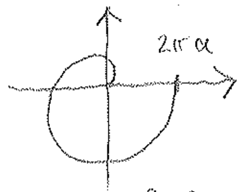
(bzw. $\int_a^b \int_{g_1(r)}^{g_2(r)} r \cdot d\varphi\, dr$)

Bsp 1.) Fl. Kreis: $\int_0^b \int_0^{2\pi} r\, d\varphi\, dr = \int_0^b \left(r\varphi \Big|_{0=\varphi}^{2\pi} \right) dr$



$$= \int_0^b 2\pi r - 0\, dr = \pi r^2 \Big|_{r=0}^{b} = \pi \cdot b^2 \quad \text{(Ja!)}$$

2.) Fl. Archimedische Spirale:



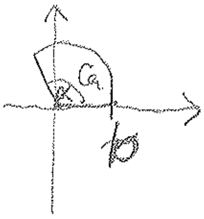
$$G = \{(r,\varphi) \mid 0 \le \varphi \le 2\pi;\ 0 \le r \le a \cdot \varphi\}$$

$$Fl(G) = \int_G r\, d(r,\varphi) = \int_0^{2\pi} \int_0^{a\varphi} r\, dr\, d\varphi = \int_0^{2\pi} \left(\tfrac{1}{2} r^2 \Big|_0^{a\varphi} \right) d\varphi$$

$$= \int_0^{2\pi} \tfrac{1}{2} a^2 \varphi^2\, d\varphi = \left(\tfrac{1}{2} a^2 \cdot \tfrac{1}{3} \varphi^3 \Big|_0^{2\pi} \right) = \tfrac{1}{6} \cdot a^2 \cdot 8\pi^3$$

$$= \tfrac{4}{3} a^2 \pi^3$$

Aufg: a)

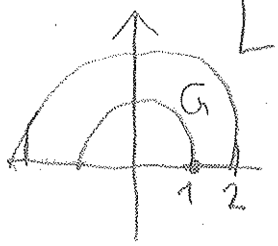


$Fl(G) = ?$

$G = \{(r, \varphi) \mid 0 \le \varphi \le \alpha ;\ 0 \le r \le b\}$

$$\left[\int_0^b \int_0^\alpha r \, d\varphi \, dr = \int_0^b \alpha r \, dr = \left(\tfrac{1}{2}\alpha r^2\right)\Big|_0^b = \tfrac{1}{2}\alpha b^2\right]$$

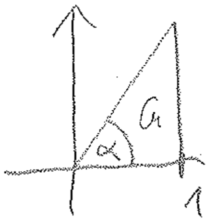
b)



$Fl(G) = ?$ $\quad G = \{(r, \varphi) \mid 0 \le \varphi \le \pi ;\ 1 \le r \le 2\}$

(F2 S. 167)

c)



$Fl(G) = ?$

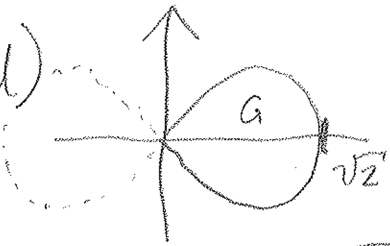
$G = \{(r, \varphi) \mid 0 \le \varphi \le \alpha ;\ 0 \le r \le \frac{1}{\cos\varphi}\}$

- Tipp: $\int \frac{1}{(\cos(x))^2} dx = \frac{\sin x}{\cos x}$

$$\left[\int_0^\alpha \int_0^{\frac{1}{\cos\varphi}} r \, dr \, d\varphi = \int_0^\alpha \left(\tfrac{1}{2} r^2 \Big|_{0=r}^{\frac{1}{\cos\varphi}}\right) d\varphi = \int_0^\alpha \tfrac{1}{2} \frac{1}{(\cos(\varphi))^2} d\varphi\right.$$

$$\left. = \frac{1}{2}\left(\frac{\sin\varphi}{\cos\varphi}\Big|_0^\alpha\right) = \frac{1}{2}\left(\frac{\sin\alpha}{\cos\alpha}\right)\right]$$

d)



$Fl(G) = ?$ $\quad G = \{(r, \varphi) \mid -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4} ;\ 0 \le r \le \sqrt{2\cos(2\varphi)}\}$

„Lemniskate"

$$\left[\int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2\cos(2\varphi)}} r \, dr \, d\varphi = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tfrac{1}{2} 2\cos(2\varphi) \, d\varphi = \left(\tfrac{1}{2}\sin(2\varphi)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right.\right.$$

$$\left. = \tfrac{1}{2}\left(\sin\tfrac{\pi}{2} - \sin\left(\tfrac{-\pi}{2}\right)\right) = \tfrac{1}{2}(1 - (-1)) = \underline{\underline{1}}\right]$$

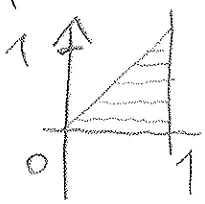
$$\cancel{= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \cos 2\varphi \, d\varphi + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\varphi \, d\varphi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\varphi \, d\varphi}$$

$$\cancel{= \tfrac{1}{2}\left(\sin 2\varphi \Big|_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} + \sin 2\varphi \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \sin 2\varphi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right) = \tfrac{1}{2}(-1 - 0 + 1 - (-1) + 0 - 1)}$$

- Schwerpunkte von 43 a) – e) (c): knifflig)

Ganz analog gehen allgemeinere Integrale.

<u>Bsp</u> $\int_G x^2 y \, d(x,y)$ mit $G = \{(x,y) \mid 0 \le y \le 1;\ y \le x \le$



$$\int_0^1 \int_y^1 x^2 y \, dx \, dy = \int_0^1 \frac{1}{3} x^3 y \Big|_{x=y}^{1} dy$$

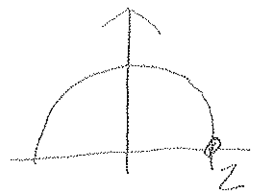
$$= \int_0^1 \frac{1}{3} y - \frac{1}{3} y^4 \, dy = \left(\frac{1}{6} y^2 - \frac{1}{15} y^5 \Big|_{y=0}^{1} \right) = \frac{1}{6} - \frac{1}{15} - 0$$

$$= \frac{3}{30} = \underline{\underline{\frac{1}{10}}}$$

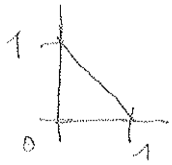
(Weitere Aufg.: Ana II Eichhorn Bl. 13; Heuser 2 S. 472 7-11.)

s.u.

e) $\int_G x^2 y \, d(x,y)$; $G =$

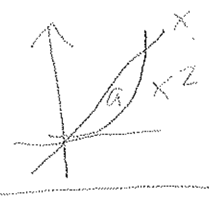


f) $\int_G (x+y^2) \, d(x,y)$ mit $G =$



g) $\int_G xy \, d(x,y)$

G:



h) $\int_G x^2 + y^2 \, d(x,y)$

G: