Stabilization due to Additive Noise

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Stabilization due to Noise

Well known phenomenon due to Multiplicative Noise.

1. By Itô noise, due to Itô-Stratonovic correction, or Stratonovic noise due to averaging over stable and unstable directions
   - **For SDE:** [Arnold, Crauel, Wihstutz '83], [Pardoux, Wihstutz '88 '92].....
   - **For SPDE:** [Kwiecinska '99],[Caraballo, Mao et.al. '01], [Cerrai '05], [Caraballo, Kloeden, Schmallfuß '06]....
2. **By Rotation:**[Baxendale et.al.'93], [Crauel et.al.'07].....

Consider here:

- **Degenerate additive noise**
- Effect of noise transported by the nonlinearity
- Stabilization effect on dominating behaviour

Introduction

- SPDEs of Burgers-type near a change of stability
- Dominant modes evolve on a slow time-scale
- Stable modes decay on a fast time-scale
- Evolution of dominant modes given by Amplitude eq.
- Formal derivation well known [Cross, Hohenberg, '93]

AIM:

- Rigorous error estimates for Amplitude equations
- Understand interplay between noise and nonlinearity

Examples

1. **Burgers equation**

$$\partial_t u = \partial_x^2 u + \nu u + u\partial_x u + \sigma\xi$$

2. **Surface Growth**

$$\partial_t h = -\partial_x^4 h - \nu\partial_x^2 h - \partial_x^2|\partial_x h|^2 + \sigma\xi$$

3. **Rayleigh Bénard Convection**
   3D-Navier-Stokes coupled to a heat equation

Some Related Multiscale Results

1. **[Majda, Timofeyev, Vanden-Eijnden '01, '02, '03]**
   - truncated Burgers system
   - formal expansion
   - rigorous via Kurtz theorem
   - no error estimates
2. **[Roberts '03]**
   - formal expansion using computer algebra
   - numerical examples for stabilization
   - no error estimates
3. **[A. Hutt '08]**
   - Similar effects for different models
   - formal calculation and numerical results

Numerical Example

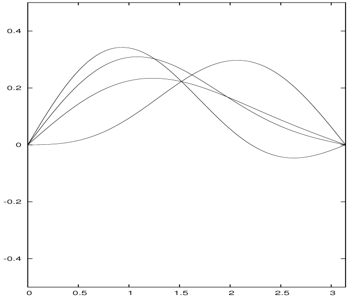
Example: Burgers-type equation

$$\partial_t u = (\partial_x^2 + 1)u + \tfrac{1}{100}u + u\partial_x u + \tfrac{\sigma}{10}\xi \qquad \text{(B)}$$

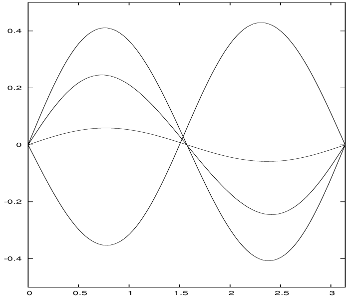
- $u(t,x) \in \mathbb{R}$, $t > 0$, $x \in [0,\pi]$
- Dirichlet boundary conditions $(u(t,0) = u(t,\pi) = 0)$
- $\xi(t,x) = \partial_t\beta(t)\sin(2x)$ – highly degenerate noise

Snapshots of solutions

$u(t,x)$ over x for different values of t



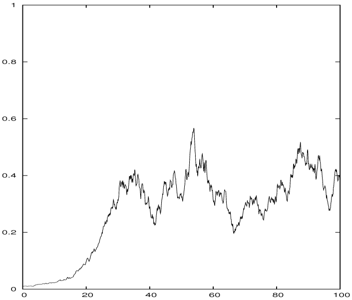
$\sigma = 2$



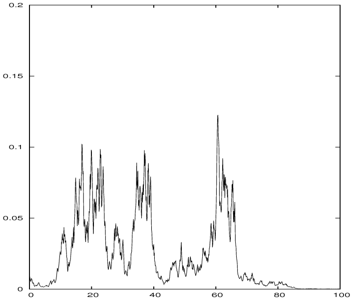
$\sigma = 10$

First Fouriermode

Stabilization of first mode due to larger noise.



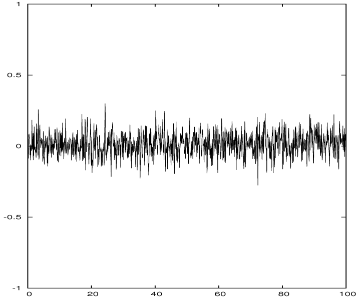
$\sigma = 2$



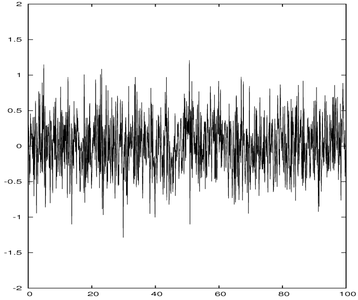
$\sigma = 10$

Second Fouriermode

There is only noise on the second mode.



$\sigma = 2$



$\sigma = 10$

Numerical Observations

Observation:

- 0 is is stabilized (sin destabilized) by large noise (see [Roberts '03])
- Large noise acting on $\sin(2x)$

An Equation of Burgers type

For simplicity only a scalar Burgers equation in this talk.

Equation of Burgers type

$$\partial_t u = (\partial_x^2 + 1)u + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \xi \qquad \text{(B)}$$

- $u(t,x) \in \mathbb{R}$, $t > 0$, $x \in [0,\pi]$
- Dirichlet boundary conditions $(u(t,0) = u(t,\pi) = 0))$
- $\nu\epsilon^2 u$ linear (in)stability
- $|\nu\epsilon^2| \ll 1$ distance from bifurcation
- $\xi(t,x)$ Gaussian white noise

The Linear Operator

The Linear Operator:

$$L = -\partial_x^2 - 1 \text{ Dirichlet b. c. on } [0, \pi]$$

- Orthonormal system generated by $\sin(kx)$, $k = 1, 2, \ldots$
- Eigenvalues: $\lambda_k = k^2 - 1, \quad k = 1, 2, \ldots$

$$0 = \lambda_1 < \omega < \lambda_2 < \ldots < \lambda_k \to \infty$$

- The dominant mode

$$\mathcal{N} = \mathrm{span}\{\sin\} \text{ -- the kernel of } L$$

The Noise

Two cases of noise:

- First:
  White noise acting directly on $\mathcal{N}$
- Later:
  Degenerate noise not acting on $\mathcal{N}$

Wiener Process

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \xi \qquad \text{(B)}$$

Noise: $\xi(t,x) = \partial_t W(t,x)$

$$W(t,x) = \sum_{k=1}^{\infty} \sigma_k \beta_k(t) \sin(kx)$$

- $\sigma_k \in \mathbb{R}, \qquad |\sigma_k| \leq C$
- $\{\beta_k\}_{k\in\mathbb{N}}$ i.i.d. Brownian motions

Remark: For space-time white noise $\sigma_k = 1 \ \forall k$.

Question:

How does noise affects the dynamics of dominant modes in $\mathcal{N}$?

The Amplitude Equation

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \partial_t W \qquad \text{(B)}$$

Ansatz:

$$u(t,x) = \epsilon a(\epsilon^2 t)\sin(x) + \mathcal{O}(\epsilon^2)$$

Result: Amplitude Equation

$$\partial_T a = \nu a - \tfrac{1}{12}a^3 + \partial_T \beta, \qquad \text{(A)}$$

where $\beta(T) = \epsilon\sigma_1\beta_1(\epsilon^{-2}T)$ rescaled noise in $\mathcal{N}$.

Interesting fact:

Nonlinearity $B(u,v) = \frac{1}{2}\partial_x(uv)$ does not map $\mathcal{N}$ to $\mathcal{N}$!
Higher order modes are involved!

Formal Calculation

$$\partial_t u = -Lu + \nu\epsilon^2 u + B(u,u) + \epsilon^2 \partial_t W \qquad \text{(B)}$$

Ansatz:

$$u(t,x) = \epsilon \underbrace{A(\epsilon^2 t)}_{\in\mathcal{N}} + \epsilon^2 \underbrace{\psi(\epsilon^2 t)}_{\perp\mathcal{N}} + \ldots$$

Thus ($T = \epsilon^2 t$, P_c Projection onto $\mathcal{N}$, $P_s = I - P_c$)
as $P_c B(A,A) = 0$

$$\partial_T A = \nu A + 2P_c B(A,\psi) + \partial_T P_c \tilde{W} + \mathcal{O}(\epsilon)$$

and

$$\epsilon^2 \partial_T \psi = -L\psi + P_s B(A,A) + \epsilon \partial_T P_s \tilde{W} + \mathcal{O}(\epsilon)\ ,$$

where $\tilde{W}(T) = \epsilon W(\epsilon^{-2} T)$.

Formal Calculation II

Neglecting all small terms leads to

$$\partial_T A = \nu A + 2P_c B(A, \psi) + \partial_T P_c \tilde{W}$$

with

$$\psi = L^{-1} P_s B(A, A) \ .$$

Using $A(T) = a(T) \sin$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta, \tag{A}$$

where $-\frac{1}{12} = 2P_c B(\sin, L^{-1} P_s B(\sin, \sin))$.

The Theorem

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon^2 \partial_t W \qquad \text{(B)}$$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta \qquad \text{(A)}$$

Theorem – Approximation [B. '07] [B., Elhaddad '08]

u is solution of (B) – a is solution of (A)
$u(0) = \epsilon a(0) \sin + \epsilon^2 \psi_0$ with $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$.

Then for κ, T_0, $p > 0$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0\epsilon^{-2}]} \| u(t) - \epsilon a(t\epsilon^2) \sin \| > \epsilon^{2-\kappa} \Big) < C\epsilon^p.$$

Impact of the Noise

Recall:

Dominant modes driven only by noise acting on $\mathcal{N}$.

No impact of $\beta_2,\ \beta_3,\ \ldots$

$$\partial_T a = \nu a - \tfrac{1}{12} a^3 + \partial_T \beta\ , \qquad \text{(A)}$$

where $\beta(T) = \epsilon\sigma_1\beta_1(\epsilon^{-2}T)$ rescaled noise in $\mathcal{N}$.

Stabilization

Can degenerate noise have an effect on the dominant mode?

Does this lead to the Stabilization?

Stabilisation due to Additive Noise – Setting

Assumption:

No noise on the dominant mode – highly degenerate noise

Question: How does noise interact with the nonlinearity?

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \sigma_\epsilon \phi$$

- Dirichlet boundary conditions on $[0, \pi]$
- $\mathcal{N} = \text{span}\{\sin\}$ – One dominating mode
- $\phi(t, x) = \partial_t \beta_2(t) \sin(2x)$ – Noise only on 2^{nd} mode

Previous Result

Previous Approximation Result:

If $\sigma_\epsilon = \sigma\epsilon^2$, then for $t \in [0, T_0\epsilon^{-2}]$

$$u(t) = \epsilon a(\epsilon^2 t)\sin + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \partial_T a = \nu a - \tfrac{1}{12}a^3$$

No impact of Noise!

Need larger Noise!

Stabilisation due to Additive Noise – Result

Consider larger noise (i.e., $\sigma_\epsilon = \sigma\epsilon$)

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \sigma\epsilon\phi \qquad \text{(B2)}$$

Amplitude Equation [DB, Hairer, Pavliotis, 07]

$$da = (\nu - \tfrac{\sigma^2}{88})a\,dT - \tfrac{1}{12}a^3 dT + \tfrac{\sigma}{6} a \circ d\tilde{\beta}_2 \qquad \text{(A2)}$$

in Stratonovic sense, with $\tilde{\beta}_2(T) = \epsilon\beta_2(\epsilon^{-2}T)$.

- For $\nu \in (0, \sigma^2/88)$
  Stabilisation of 0 $\longleftrightarrow$ Destabilisation of sin
- Technical problem:
  $$u(t) - \epsilon a(\epsilon^2 t)\sin \approx \frac{\epsilon^2}{\lambda_1} \underbrace{\partial_T \tilde{\beta}_2(T)}_{\text{white noise}} \sin(2\cdot) + \mathcal{O}(\epsilon^2)$$

Formal Motivation

$$da = (\nu - \tfrac{\sigma^2}{88})adT - \tfrac{1}{12}a^3dT + \tfrac{\sigma}{6}a \circ d\tilde{\beta}_2 \qquad \text{(A2)}$$

Stabilization effect

Itô to Stratonovic correction is $-\frac{\sigma^2}{72}a$

Where does the other term comes from?

Consider slow time: $(u(t) = \epsilon\psi(\epsilon^2 t))$

$$\partial_T \psi = -\epsilon^{-2}L\psi + \nu\psi + \epsilon^{-1}B(\psi,\psi) + \epsilon^{-1}\partial_T\tilde{\Phi} \qquad \text{(B2')}$$

Formal Calculation

$$\partial_T \psi = -\epsilon^{-2} L\psi + \nu\psi + \epsilon^{-1} B(\psi,\psi) + \epsilon^{-1}\partial_T \tilde{\Phi}_2$$

Ansatz with $\psi_k \in \text{span}(\sin(kx))$:

$$\psi(T) = \psi_1(T) + \psi_2(T) + \epsilon\psi_3(T) + \mathcal{O}(\epsilon)$$

1st mode: (using $B_n(\psi_k, \psi_l) = 0$ for $n \notin \{|k-l|, k+l\}$)

$$\partial_T \psi_1 = \nu\psi_1 + 2\epsilon^{-1} B_1(\psi_2, \psi_1) + 2B_1(\psi_2, \psi_3) + \mathcal{O}(\epsilon)$$

2nd mode: $L\psi_2 = \epsilon B_2(\psi_1, \psi_1) + \epsilon\partial_T \tilde{\Phi}_2 + \mathcal{O}(\epsilon^2)$

3rd mode: $L\psi_3 = 2B_3(\psi_2, \psi_1) + \mathcal{O}(\epsilon)$

New contribution to 1st mode:

$$4\epsilon^2 B_1(L^{-1}\partial_T\tilde{\Phi}_2, L^{-1}B_3(\partial_T\tilde{\Phi}_2, \psi_1))$$

Formal Motivation

New contribution to 1st mode ($\psi_1 = a \sin$):

$$4\epsilon^2 B_1(L^{-1}\partial_T \tilde{\Phi}_2, L^{-1} B_3(\partial_T \tilde{\Phi}_2, \psi_1)) = c(\epsilon \partial_T \tilde{\beta}_2)^2 a$$

What is noise^2?

Instead of $\epsilon \partial_T \tilde{\beta}_2$ we use $Z_\epsilon(T) = \epsilon^{-1} \int_0^T e^{-3(T-s)\epsilon^{-2}} d\tilde{\beta}_2(s)$.

Lemma [B,Hairer,Pavliotis '07] Averaging with error bounds

Some assumptions on Hölder-Quotients of a, then

$$\int_0^T a(s) Z_\epsilon(s)^2 ds = \tfrac{1}{6} \int_0^T a(s) ds + r_\epsilon(T)$$

where $\mathbb{E} \sup_{[0,T_0]} |r_\epsilon|^p \leq C_{T_0,\kappa,p} \epsilon^{\frac{p}{2}-\kappa}$.

Stabilisation due to Additive noise – Theorem

Theorem [B, Hairer, Pavliotis '07]

Let u be a continuous $H_0^1([0,\pi])$-valued solution of (B2) with $u(0) = \epsilon a(0)\sin + \epsilon\psi_0$,

where $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$.

Let a be a solution of (A2) and define

$$R(t) = e^{-Lt}\psi_0 + \sigma\left(\int_0^t e^{-3(t-s)}d\beta_2(s)\right)\sin(2\,\cdot),$$

then for all $\kappa, p, T_0 > 0$ there is a constant C such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t)\sin - \epsilon R(t)\|_{H^1} > \epsilon^{3/2-\kappa}\Big) \le C\epsilon^p.$$

More Noise – Near White Noise

What about more noise?

More Noise – Near White Noise

$$\partial_t u = -Lu + \nu\epsilon^2 u + \tfrac{1}{2}\partial_x u^2 + \epsilon\partial_t W \qquad \text{(B3)}$$

with $W(t,x) = \sum_{k=2}^{\infty} \beta_k(t)\sin(kx)$ (near white noise)

Amplitude Equation

There is a Brownian motion B and constants $(\nu_0,\ \sigma_a,\ \sigma_b)$ s. t.

$$da = \nu_0 a\, dT - \tfrac{1}{12} a^3 dT + \sqrt{\sigma_a a^2 + \sigma_b}\, dB\ . \qquad \text{(A3)}$$

Multiplicative AND Additive Noise!

Additive noise arises from noise^2 times independent noise.

Relies on martingale approximation result (one-dimensional)
Error estimate depends on estimate for quadratic variations.

More Noise – Near White Noise

Lemma [B, Hairer, Pavliotis, '07]

$M(t)$ continuous martingale with quadratic variation f
g arbitrary adapted increasing process with $g(0) = 0$.
Then, with respect to an enlarged filtration, there exists a continuous martingale $\tilde{M}(t)$ with quadratic variation g such that, for every $\gamma < 1/2$ there exists a constant C with

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} |M(t) - \tilde{M}(t)|^p \\ &\leq C\left(\mathbb{E}g(T)^{2p}\right)^{1/4}\left(\mathbb{E}\sup_{t\in[0,T]}|f(t)-g(t)|^p\right)^\gamma \\ &\quad + C\mathbb{E}\sup_{t\in[0,T]}|f(t)-g(t)|^{p/2} . \end{aligned}$$

More Noise – Theorem

[B, Hairer, Pavliotis, 07

For $\alpha \in [0, \frac{1}{2})$ let u be a cont. $H_0^\alpha([0,\pi])$-valued sol. of (B3) with $u(0) = \epsilon a(0) \sin + \epsilon\psi_0$,
where $\psi_0 \perp \sin$ and $a(0), \psi_0 = \mathcal{O}(1)$.
Let a be a solution of (A3) and define

$$R(t) = e^{-tL}\psi_0 + \int_0^t e^{-(t-s)L} dW(s) .$$

Then for all $\kappa, p, T_0 > 0$ there is a constant $C > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T_0\epsilon^{-2}]} \|u(t) - \epsilon a(\epsilon^2 t) \sin - \epsilon R(t)\|_{H^\alpha} > \epsilon^{\frac{5}{4}-\kappa} \right) \le C\epsilon^p .$$

Summary

- SPDEs of Burgers type near a change of stability
- Approximation of transient dynamics via amplitude equations
- Stabilisation due to additive noise
- Effect of noise on dominant modes
- Noise transported by nonlinearity between Fourier-modes

Further results:

- Attractivity results
- Approximation of moments
- Approximation of invariant measures