Local Shape of Random Invariant Manifolds

Dirk Blömker

Universität Augsburg
Mathematisch Naturwissenschaftliche Fakultät

November 20, 2009

joint work with : Wei Wang (Nanjing / Adelaide)

Local Shape of Random Invariant Manifolds

Dirk Blömker

Introduction

SPDE

RDS

LRIM

Main Results

Flow on M

Idea of Proof

Conclusions

Consider here:

- Equation of Burgers type (quadratic nonlinearity)
- perturbed by simple multiplicative noise
- deterministic fixed point 0
- local random invariant manifolds near 0 using a cut-off
- structure of the manifold near 0

Contents

- SPDE of Burgers type
- RDS – Random Dynamical Systems
- RIM – Random Invariant Manifolds
- Main results on LRIM (local RIM)
- Flow on the manifold
- Some ideas of proofs

An Equation of Burgers type

Equation of Burgers type

$$du = [Lu + \nu u + B(u,u)]dt + \sigma u \circ d\omega \qquad \text{(B)}$$

- L - non positive differential operator on Hilbert-space H
  Exp. $L = \partial_x^2 + 1$ on $[0, \pi]$ Dirichlet b.c.
- Kernel $\mathcal{N} = N(L)$, finite dimensional
- Bilinear operator $B : H \times H \to D((1-L)^{-\alpha})$, $\alpha \in [0,1)$
  Exp. $B(u,v) = \partial_x(uv)$
- $\{\omega(t)\}_{t\geq 0}$ – standard two-sided Brownian motion in $\mathbb{R}$
- σ – noise strength
- ν – distance from bifurcation

An Equation of Burgers type

It is known:

(B) generates a Random Dynamical System on H.

- How?
- Definition of RDS?

Wiener space and Shift

Consider the probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P})$, where

$$\Omega_0 = \{\omega \in C^0(\mathbb{R}, \mathbb{R}) \ : \ \omega(0) = 0\} \ .$$

On Ω_0 let $\mathbb{P}$ be the the two-sided Wiener measure.
The identity on Ω_0 is a Brownian motion.

Define the Shift $\theta_\tau : \Omega_0 \to \Omega_0$

$$\theta_\tau \omega(t) = \omega(t + \tau) - \omega(\tau) \ ,$$

which is measure preserving/ergodic with respect to $\mathbb{P}$.

Random Dynamical System (RDS)

[L. Arnold, Crauel, Schmalfuß, Flandoli, Scheutzow, Chueshov, Duan, Caraballo, Kloeden, Robinson,....]

A RDS on H over the shift θ_t on $(\Omega_0, \mathcal{F}_0, \mathbb{P})$ is a measurable map

$$\begin{aligned} \varphi : \mathbb{R}^+ \times \Omega_0 \times H &\to H \\ (t, \omega, u) &\mapsto \varphi(t, \omega)u \end{aligned}$$

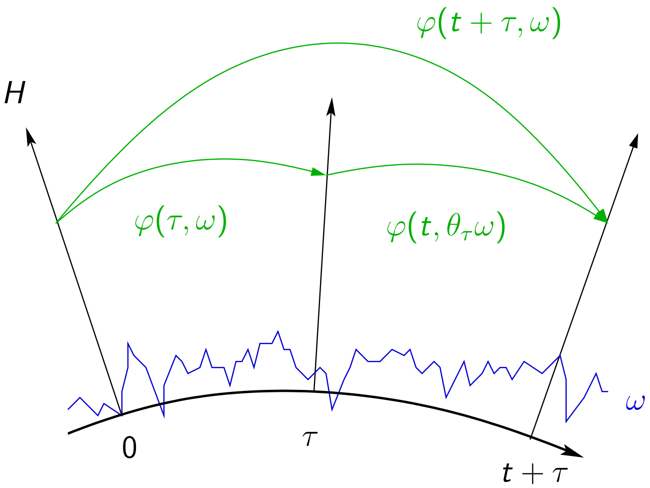
with the cocycle property

$$\varphi(0, \omega) = Id, \qquad \varphi(t, \theta_\tau \omega)\varphi(\tau, \omega) = \varphi(t + \tau, \omega)$$

for all $t, \tau \in \mathbb{R}^+$ and $\omega \in \Omega_0$.

Remark: Usually, $\varphi(t, \omega)u$ is continuous in t and in u.

Cocycle Property $\varphi(t,\theta_\tau\omega)\varphi(\tau,\omega)=\varphi(t+\tau,\omega)$



Ornstein-Uhlenbeck process

Definition (OU-process)

Define on Ω_0 ,

$$z(\omega) = -\sigma \int_{-\infty}^{0} e^{s}\omega(s)ds\,.$$

and

$$z(t) = z(\theta_t\omega) = -\sigma \int_{-\infty}^{t} e^{s-t}\omega(s)ds + \sigma\omega(t)\ .$$

$t \mapsto z(\theta_t\omega)$ is continuous and solves

$$dz = -zdt + \sigma d\omega.$$

Remark: $z(t)$ is a stationary OU-process on the Wiener space.

Transformation

Theorem

The solution to (B) generates a RDS.

Using the standard transformation

$$v(t) = e^{-z(t)}u(t)$$

Equation (B) becomes:

$$\partial_t v = Lv + zv + \nu v + e^z B(v, v),$$

The solution defines a RDS, which by the transformation defines the RDS φ for (B).

Random Invariant Manifold (RIM)

Definition (Random Invariant Manifold)

A random set $M(\omega)$ is positive invariant for the RDS φ, if

$$\varphi(t,\omega)M(\omega) \subset M(\theta_t\omega) \text{ for all } t \geq 0.$$

If

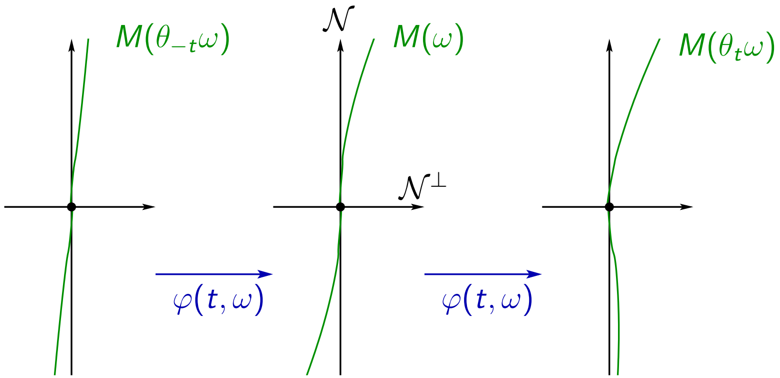
$$M(\omega) = \{u + \psi(\omega, u) | u \in \mathcal{N}\}$$

is the graph of a random Lipschitz mapping

$$\psi(\omega, \cdot) : \mathcal{N} \to \mathcal{N}^\perp ,$$

then $M(\omega)$ is called a Lipschitz invariant manifold (RIM).

RIM are moving in time!



Lipschitz Condition

Theorem

If the nonlinearity is globally Lipschitz with sufficiently small Lipschitz-constant, then there exists a RIM.
The RIM is pull-back attracting.

See for example:
[Duan,Lu, Schmalfuß '03, '04]
[Duan, Wang '07]
[Mohammed, Zhang, Zhao, 08]

Based on Fixed-Point arguments / Ljapunov-Perron method

Local Random Invariant Manifold (LRIM)

compare [Lu, Schmalfuß, 07]

A random set $M^R(\omega)$ is a LRIM with radius $R > 0$ of (B), if it is graph of a random function $\psi(\omega, \cdot) : \mathcal{N} \to \mathcal{N}^\perp$ such that for all bounded sets $B \subset B_R(0) \subset H$

$$\varphi(t, \omega)[M^R(\omega) \cap B] \subset M^R(\theta_t \omega)$$

for all $t \in [0, \tau_e(\omega))$ with

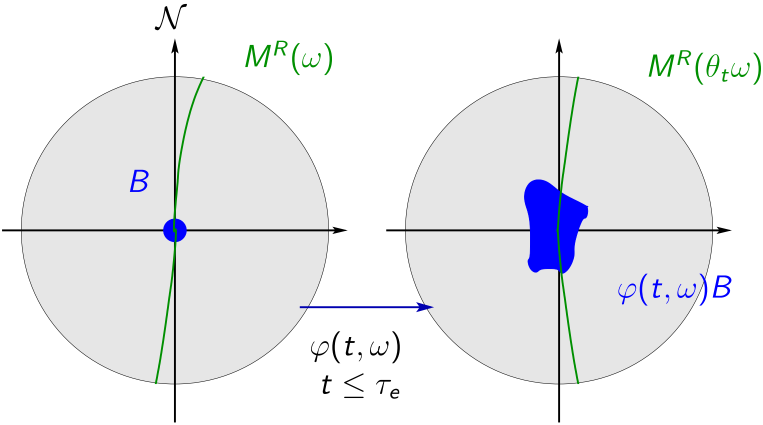
$$\tau_e(\omega) = \inf\{t \geq 0 : \varphi(t, \omega)[M^R(\omega) \cap B] \not\subset B_R(0)\}.$$

$B \subset B_R(0)$ might also be random.

Key Idea

Take a cut-off at radius $R > 0$ for (B) such that the nonlinearity is Lipschitz with small constant.

A Sketch of a LRIM



Existence

Theorem (DB, Wang '09)

The RDS φ defined by (B) has a LRIM $M^R(\omega)$ for sufficiently small $R > 0$.
It is given as the graph of a random Lipschitz map defined by $h(\omega, \cdot) : \mathcal{N} \to \mathcal{N}^\perp$:

$$M^R(\omega) = \Big\{ \Big(\xi, e^{z(\omega)} h(\omega, e^{-z(\omega)}\xi) \Big) \in B_R(0) \ : \ \xi \in \mathcal{N} \Big\} \ .$$

Remark: The LRIM is locally exponentially attracting in the pullback sense. (compare [Duan, Wang '07] for RIM)

LRIM = RIM

Is the LRIM a RIM for (B)?

The precise relation between LRIM and a RIM is not yet settled.

Problem: Both are moving in time and parts might always leave the ball $B_R(0)$.

Solutions:

- If $\nu < 0$ it is straightforward to show that a small random neighboorhood of 0 does not leave $B_R(0)$.
- Take random radius R?

Local Shape

Denote by P_s the projection onto $\mathcal{N}^\perp$
and define $B_s = P_s B$ and $L_s = P_s L$.

Theorem (DB, Wang '09)

Suppose $\sigma > 0$, $|\nu| < \sigma$ and $R \leq 1$, and let h be the LRIM given by the previous theorem.
Then

$$\|e^{z(\omega)} h(\omega, e^{-z(\omega)}\xi) - L_s^{-1} B_s(\xi,\xi)\| \leq C(\|\xi\| + R^2 + \sqrt{\sigma}) \cdot \|\xi\|^2$$

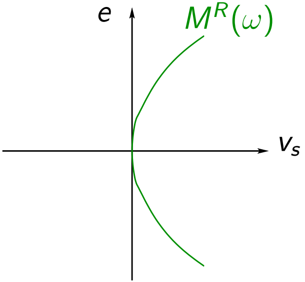
holds for all $\|\xi\| \leq \frac{1}{2}R$
with probability larger than $1 - C\exp\{-1/\sqrt{\sigma}\}$.

Remark: It is possible to extend the bound for $M^R(\omega)$ to bounds for $M^R(\theta_t\omega)$ on some time-intervals.

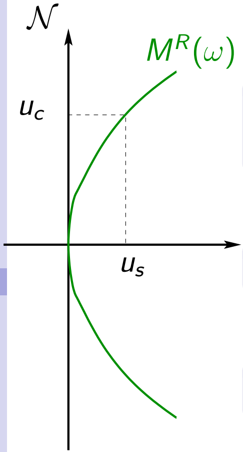
Example

Suppose $\mathcal{N} = \text{span}\{e\}$ and fix $\xi = \alpha \cdot e \in \mathcal{N}$.
The LRIM $\mathcal{M}^R(\omega)$ is given (with high probability) as the graph of

$$L_s^{-1} B_s(\xi, \xi) = \alpha^2 L_s^{-1} B_s(e, e) =: \alpha^2 v_s \perp \mathcal{N}.$$



Flow along the Manifold



With high probability:

$$u = u_c + u_s \qquad u_c \in \mathcal{N}, \; u_s \perp \mathcal{N}$$
$$\text{with } u_s = e^z h(e^{-z} u_c) \approx L_s^{-1} B_s(u_c, u_c) \; .$$

Thus projecting (B)

$$du_c = [\nu u_c + B_c(u_c + u_s, u_c + u_s)]dt + \sigma u_c \circ d\omega$$

For our example $a \sin = u_c$ and

$$da = [\nu a - \tfrac{1}{12} a^3]dt + \sigma a \circ d\omega$$

Flow along the Manifold

Problem:

The equation for the flow on $M(\omega)$ holds true at a single time t with high probability.

If one has bounds on $M^R(\theta_t\omega)$ on time-intervals, then it is possible to extend this result.

Amplitude Equations

The flow along the manifold
has a relation to amplitude equations.

Consider the special scaling:

$$du = [Lu + \nu_0\epsilon^2 u + B(u,u)]dt + \epsilon u \circ d\omega \qquad \text{(B)}$$

For simplicity only the example:

- $L = \partial_x^2 + 1$ and Dirichlet b.c on $[0, \pi]$
- $B(u, v) = \partial_x(uv)$
- $\mathcal{N} = \text{span}(\sin)$

Amplitude Equations

Theorem [DB '07]

Under the previous assumptions,
suppose that $P_c u(0) = \mathcal{O}(\epsilon)$, $P_s u(0) = \mathcal{O}(\epsilon^2)$.

Then with high probability

$$u(t) = \epsilon A(\epsilon^2 t) \sin + \mathcal{O}(\epsilon^2) \quad \text{for all } t \in [0, T_0 \epsilon^{-2}]$$

where A solves

$$dA = [\nu A - \tfrac{1}{12} A^3] dt + A \circ d\tilde{\omega}$$

where $\tilde{\omega}(T) = \epsilon \omega(t\epsilon^{-2})$ is a rescaled Brownian motion.

Some ideas of Proof

Cut Off

compare [Caraballo, Langa, Robinson '01], [Lu, Schmalfuß, 07]

Let $\chi : H \to \mathbb{R}$ be a bounded smooth function such that $\chi(u) = 1$ if $\|u\| \leq 1$ and $\chi(u) = 0$ if $\|u\| \geq 2$.
For all $R > 0$ define

$$\chi_R(u) = \chi(u/R) \text{ for all } u \in H$$

$$B^{(R)}(u) = \chi_R(u) B(u, u).$$

Now $B^{(R)}$ is globally Lipschitz-continuous with constant

$$\text{Lip}(B^{(R)}) = C_B C_\chi R \to 0 \text{ for } R \to 0\,.$$

Consider the following cut-off equation

$$du = [Lu + \nu u + B^{(R)}(u)]dt + \sigma u \circ d\omega\,, \qquad u(0) = u_0\,.$$

Cut off

Transformation $v = ue^{-z}$ yields

$$\partial_t v = Lv + zv + \nu v + e^{-z} B^{(R)}(e^z v)\,, \qquad v(0) = u_0 e^{-z(0)}\,.$$

In order to obtain a RIM for the RDS $\varphi^R(t,\omega)$ of the cut-off equation, we consider the RIM of the transformed equation above.

We use the Ljapunov-Perron Method.

The fixed point space

Let $-\lambda_* < 0$ be the largest non-zero eigenvalue of L.
For $-\nu < \eta < \lambda_* - \nu$ define the Banach space

$$C_\eta^- = \Big\{ v \in C^0((-\infty, 0], H) : \|v\|_{C_\eta^-} < \infty \Big\}$$

with norm

$$\|v\|_{C_\eta^-} = \sup_{t \le 0} \Big\{ e^{\eta t - \int_0^t z(\tau) d\tau} \|v(t)\| \Big\} < \infty.$$

Ljapunov-Perron Operator

Define the evolution operator:

$$S(t,\tau) = e^{(L+\nu)(t-\tau)+\int_\tau^t z(r)dr} .$$

The nonlinearity: $\mathcal{B}(\tau) = e^{-z(\tau)} \cdot B^{(R)}\left(v(\tau)e^{z(\tau)}\right)$

Given $\xi \in \mathcal{N}$, define the nonlinear operator $\mathcal{T}$ on C_η^- by

$$\mathcal{T}(v)(t) = S(t,0)\xi + \int_0^t S(t,\tau)P_c\mathcal{B}(\tau)d\tau + \int_{-\infty}^t S(t,\tau)P_s\mathcal{B}(\tau)d\tau$$

with $v \in C_\eta^-$, $\omega \in \Omega_0$, and $t \leq 0$.

Fixed Point

The operator $\mathcal{T}$ has a unique fixed point $v^* = v^*(\omega, \xi) \in C_\eta^-$.

Define $h(\omega, \xi) = P_s v^*(0, \omega; \xi)$, then

$$\mathcal{M}(\omega) = \{(\xi, h(\omega, \xi)) : \xi \in \mathcal{N}\}$$

is a RIM for the transformed cut-off equation.

Manifold

Define a Lipschitz mapping ψ by

$$\begin{aligned}\psi(\omega,\cdot):\mathcal{N} &\rightarrow \mathcal{N}^{\perp}\,,\\ \xi &\mapsto \psi(\omega,\xi)=e^{z(\omega)}h(\omega,e^{-z(\omega)}\xi)\,.\end{aligned}$$

Then

$$\mathcal{M}^R_{cut}(\omega)=\{(\xi,\psi(\omega,\xi)):\xi\in\mathcal{N}\}$$

is a RIM for the RDS φ^R of the cut-off equation, and

$$\mathcal{M}^R(\omega)=\mathcal{M}^R_{cut}(\omega)\cap B_R(0)$$

defines a LRIM of the RDS $\varphi(t,\omega)$.

Local Shape

By Definition of $\mathcal{T}$ and h, as $v^* = \mathcal{T}(v^*)$

$$\begin{aligned} h(\xi) &= P_s v^*(0) \\ &= P_s \mathcal{T}(v^*)(0) \\ &= \int_{-\infty}^{0} S(0,\tau) e^{-z(\tau)} P_s B^{(R)}(v^*(\tau,\xi) e^{z(\tau)}) d\tau. \end{aligned}$$

This allows for estimates on h and thus on ψ, where estimates on v^* in C_η^- are necessary.

Conclusions

Results:

- Existence of LRIM using a cut-off
- LRIM is locally a parabola
- Flow on the manifold (cf. Amplitude equations)

To do:

- Relation of RIM of (B) to LRIM?
- Is M^R a RIM in a small (random) neighboorhood of 0?
- Formulation of LRIM without cut-off?