Linear stability analysis for stochastic Theta-methods applied to systems of SODEs

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joint work with Cónall Kelly and Thorsten Sickenberger

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Outline

Stochastic differential equations

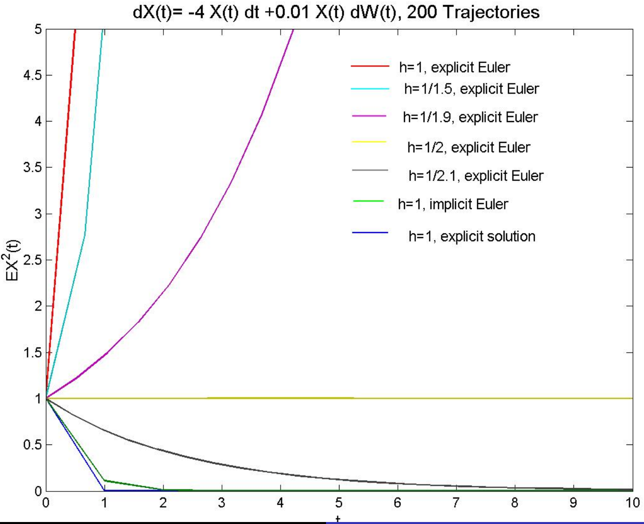
Itô stochastic ordinary differential equations (SODEs) on $\mathcal{J} := [0, \infty)$

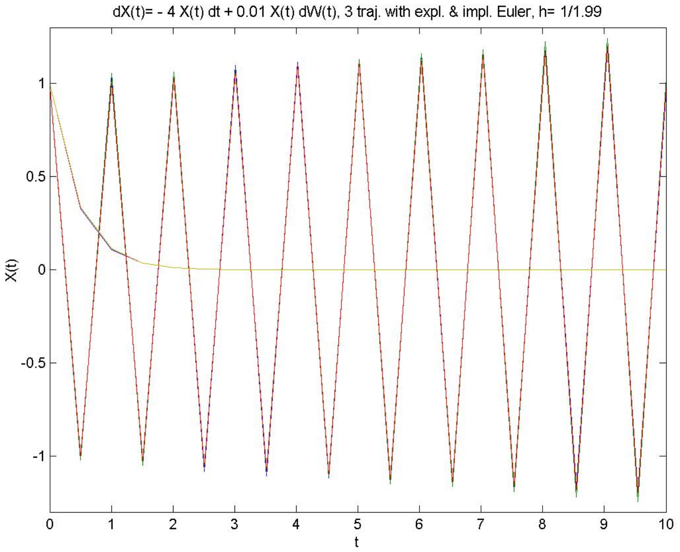
$$X(s)\Big|_0^t = \int_0^t f(s, X(s))\,\mathrm{d}s + \sum_{r=1}^{m} \int_0^t g_r(s, X(s))\,\mathrm{d}W_r(s), \;\; X(0) = X_0$$

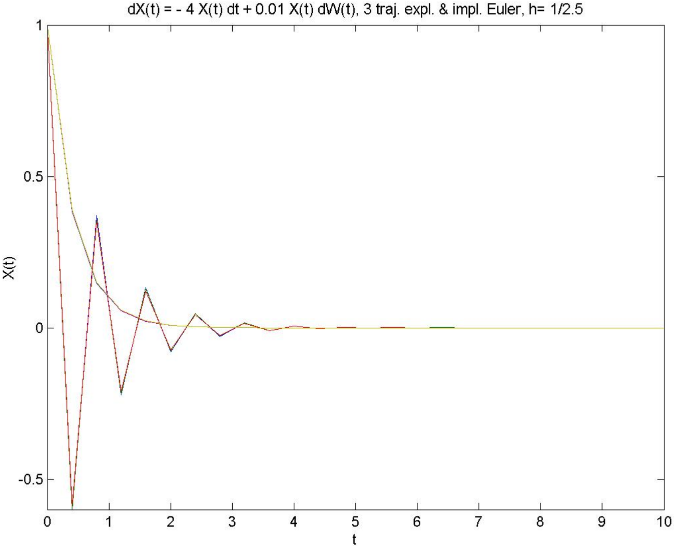
- m scalar Wiener processes: $W_r = \{W_r(t, \omega), t \in J, \omega \in \Omega\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in J}, \mathbb{P})$.
- coefficients: (globally Lipschitz) $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $G = (g_1, \ldots, g_m) : J \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$;
- initial data: $X(0)$ is a given $\mathcal{F}_0$-measurable initial value, independent of the Wiener process and with finite second moment.

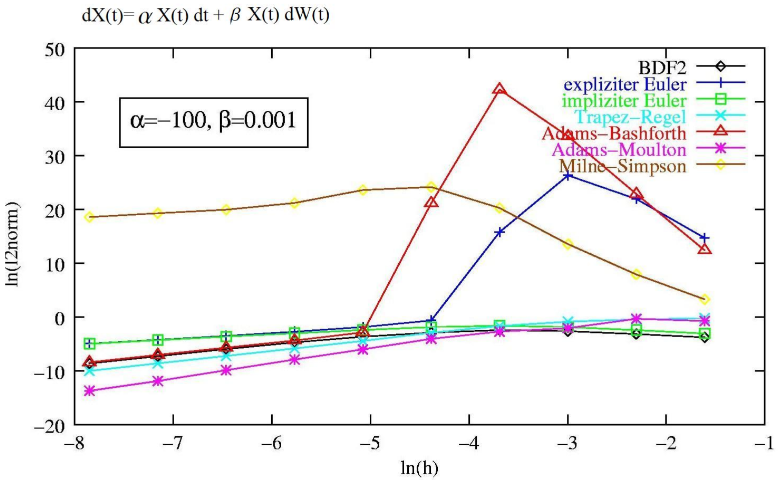
We assume that there exists a path-wise unique strong solution $X(\cdot)$ of the above equation.

Stability behaviour




dX(t)= - 4 X(t) dt + 0.01 X(t) dW(t), 3 traj. with expl. & impl. Euler, h= 1/1.99
X(t)
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
10
t


dX(t) = - 4 X(t) dt + 0.01 X(t) dW(t), 3 traj. expl. & impl. Euler, h= 1/2.5
1
0.5
0
-0.5
X(t)
0
1
2
3
4
5
6
7
8
9
10
t


dX(t)= α X(t) dt + β X(t) dW(t)
α=−100, β=0.001
BDF2
expliziter Euler
impliziter Euler
Trapez−Regel
Adams−Bashforth
Adams−Moulton
Milne−Simpson
ln(l2norm)
ln(h)
50
40
30
20
10
0
−10
−20
−8
−7
−6
−5
−4
−3
−2
−1

Linear stability analysis for ODEs

- Question: given an ODE $x'(t) = f(x(t))$ and a numerical method, does the (convergent) method share the qualitative properties of the ODE and if so, under which restrictions on the step-size?
- (Usually) first step: linear stability analysis, using the test equation $x'(t) = \lambda x(t)$, $\lambda \in \mathbb{C}$.
- Based on: linearisation and centering of nonlinear ODE around an equilibrium, the resulting linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ (A the Jacobian of f evaluated at equilibrium) is then diagonalised and the system thus decoupled, justifying the use of the scalar test equation.

Linear stability analysis for SODEs

- Question: given an SODE and a numerical method, does the (convergent) method share the qualitative properties of the SODE and if so, under which restrictions on the step-size?
- (Usually) first step: linear stability analysis, now with which test equation?
- Further questions: Stability in which sense, i.e. in the a.s. sense or in mean-square? What effect does the m-dim noise have?
- Still holding: linearisation and centering of nonlinear SODE around an equilibrium, the resulting linear system is now $\mathrm{d}X(t) = (AX(t))\mathrm{d}t + \sum_{r=1}^{m} B_r X(t)\mathrm{d}W_r(t)$ (A, B_r the Jacobians of f, g_r evaluated at equilibrium). Simultaneously diagonalisable?

Linear stability analysis for SODEs

- Most existing results for scalar $\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W_1(t)$, e.g., Mitsui & Saito, Higham, Debrabant & Rößler, B. & Horvath-Bokor & Winkler, mean-square stability, various methods, strong and weak convergence;
- Higham: results for scalar $\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W_1(t)$, stochastic θ-method, a.s. sense;
- Saito & Mitsui: analysis for 2-dim systems, 1 WP, Euler-Maruyama method, mean-square sense wrt a certain logarithmic matrix norm;
- Rathinasamy & Balachandran: analysis of weak second-order Runge-Kutta methods for systems with 1 and several noises, mean-square sense wrt a certain logarithmic matrix norm.

Linear stability analysis for SODEs

Goal:

Develop a systematic stability analysis of numerical methods, justifying the choice of test equations/systems, gaining insight into deterministic/stochastic features relevant for stability issues, identifying benchmark problems, develop appropriate analytical techniques

Linear stability analysis for SODEs

Definition

1. The zero solution of an SDE is mean-square stable/a.s. stable if and only if, for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that

$$\mathbb{E}|X(t)|^p < \epsilon, \quad t \geq 0, \quad / \quad |X(t)| < \epsilon, \quad t \geq 0, \quad a.s.$$

whenever $\mathbb{E}|X(0)|^p < \delta$ / $|X(0)| < \delta$;

2. The equilibrium is asymptotically mean-square stable/a.s. stable if and only if it is mean-square stable/a.s. stable, and for all $X(0) \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}|X(t)|^p = 0 \quad / \quad \lim_{t\to\infty} X(t) = 0 \quad a.s.$$

Part 1: A scalar test equation with m Wiener processes

Part 1: Consider simultaneously diagonalisable drift and diffusion matrices and m Wiener processes and the θ-Maruyama and θ-Milstein method wrt mean-square stability.

The test equation and the methods

We consider $$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \sum_{r=1}^{m} \mu_r X(t)\mathrm{d}W_r(t), \quad \lambda,\, \mu_r \in \mathbb{C}, \tag{1}$$

the θ-Maruyama method with $W_r(t_i + h) - W_r(t_i) \sim \sqrt{h}\,\xi_{r,i}$ and $\xi_{r,i}$ is $\mathcal{N}(0,1)$

$$X_{i+1} = X_i + h\,(\theta\lambda X_{i+1} + (1-\theta)\lambda X_i) + \sqrt{h}\sum_{r=1}^{m} \mu_r X_i\,\xi_{r,i}\,, \quad i = 0,\, 1, \ldots\,, \tag{2}$$

and the θ-Milstein method with $\int_t^{t+h}\int_t^s W_{r_1}(u)\mathrm{d}W_{r_2}(s)$ as $h(\xi_{r,i}^2 - 1)$

$$\begin{aligned} X_{i+1} \quad &= \quad X_i + (h\,(\theta\lambda X_{i+1} + (1-\theta)\lambda X_i) + \sqrt{h}\sum_{r=1}^{m} \mu_r X_i\,\xi_{r,i} \\ &\quad + \frac{1}{2}\,h\sum_{r=1}^{m} \mu_r^2 X_i\,(\xi_{r,i}^2 - 1)\,, \quad i = 0,\, 1, \ldots \end{aligned} \tag{3}$$

Mean-square stability analysis, θ-Maruyama method

Rewrite $\quad X_{i+1} = X_i + h\,(\theta\lambda X_{i+1} + (1-\theta)\lambda X_i) + \sqrt{h}\sum_{r=1}^m \mu_r X_i\,\xi_{r,i}$

as recurrence $\quad X_{i+1} = \left(a + \sum_{r=1}^m b_r\,\xi_{r,i}\right) X_i$

with $\quad a := 1 + \frac{h\lambda}{1-\theta h\lambda}, \quad b_r := \frac{\sqrt{h}\mu_r}{1-\theta h\lambda}$

Then squaring and taking expectation yields a recurrence for $\mathbb{E}|X_i|^2$:
$\mathbb{E}|X_{i+1}|^2 = \left(|a|^2 + \sum_{r=1}^m |b_r|^2\right)\mathbb{E}|X_i|^2\,.$

Result: the zero solution of the above recurrence is asymptotically mean-square stable if and only if $|a|^2 + \sum_{r=1}^m |b_r|^2 < 1\,.$

Mean-square stability analysis, θ-Milstein method

Rewrite $X_{i+1} =$
$X_i + h\,(\theta\lambda X_{i+1} + (1-\theta)\lambda X_i) + \sqrt{h}\sum_{r=1}^m \mu_r X_i\,\xi_{r,i} + \frac{1}{2}\,h\sum_{r=1}^m \mu_r^2 X_i\,(\xi_{r,i}^2 - 1)$

as recurrence $X_{i+1} = \left(\hat{a} + \sum_{r=1}^m b_r\,\xi_{r,i} + \sum_{r=1}^m c_r\,\xi_{r,i}^2\right) X_i$
with $\hat{a} := a - \sum_{r=1}^m c_r, \quad b_r := \frac{\sqrt{h}\mu_r}{1-\theta h\lambda}, \quad c_r = \frac{\frac{1}{2}h\mu_r^2}{1-\theta h\lambda}$

Then squaring and taking expectation yields a recurrence for $\mathbb{E}|X_i|^2$:
$\mathbb{E}|X_{i+1}|^2 = \left(|a|^2 + \sum_{r=1}^m |b_r|^2 + 2\sum_{r=1}^m |c_r|^2\right) \mathbb{E}|X_i|^2 \,.$

Result: the zero solution of the above recurrence is asymptotically mean-square stable if and only if $|a|^2 + \sum_{r=1}^m |b_r|^2 + 2\sum_{r=1}^m |c_r|^2 < 1\,.$

Comparison of stability conditions with original parameters

In terms of λ, μ_r, θ, h we have that the zero solution of the test equation is asymp. ms-stable iff.

$$\Re(\lambda) + \frac{1}{2}\sum_{r=1}^{m} |\mu_r|^2 < 0\,,$$

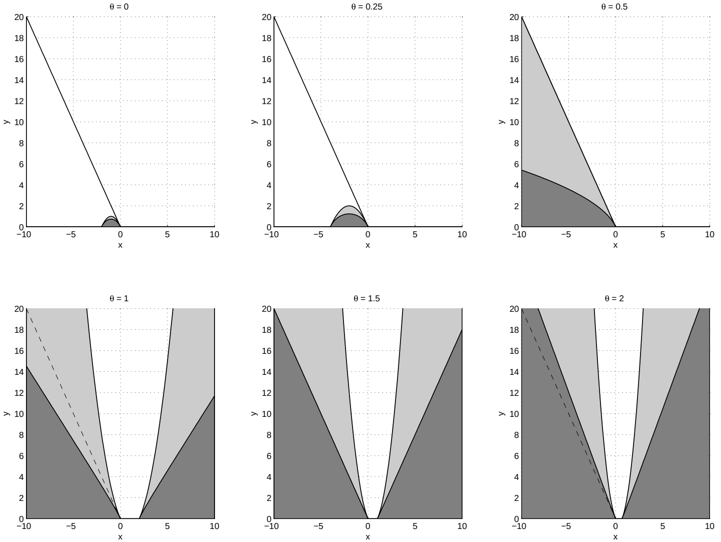
the θ-Maruyama method is asymp. ms-stable iff.

$$\Re(\lambda) + \frac{1}{2}\sum_{r=1}^{m} |\mu_r|^2 + \frac{1}{2}h(1-2\theta)|\lambda|^2 < 0\,,$$

the θ-Milstein method is asymp. ms-stable iff.

$$\Re(\lambda) + \frac{1}{2}\sum_{r=1}^{m} |\mu_r|^2 + \frac{1}{2}h(1-2\theta)|\lambda|^2 + \frac{1}{4}h\sum_{r=1}^{m} |\mu_r|^4 < 0\,,$$

Comparison of stability regions for $m = 1$, $x = h\lambda$, $y = h\mu_1^2$, λ, μ_1 real



Part 2: Linear Systems of SODEs

Part 2: Consider d-dimensional linear systems of SODEs, that is
$\mathrm{d}X(t) = (AX(t))\mathrm{d}t + \sum_{r=1}^{m} B_r X(t)\mathrm{d}W_r(t)$
where A, B_r are $d \times d$-dimensional matrices.

Obvious: full systems have too many parameters.

Derive simple test systems of SODEs based on stochastic stabilisation and destabilisation and analyse the θ-Maruyama method wrt mean-square and a.s. stability.

General idea starting with Mao, 1994

Compares solutions of d-dim. ODE $x'(t) = f(x(t))$
with those of d-dim. SODE $\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t)$, W r-dim. WP

Stabilisation: find a diffusion g, given a drift f s.t. solutions of stoch. system satisfy

$$\lim_{t\to\infty} X(t) = 0, \quad a.s.,$$

Destabilisation find a diffusion g, given a drift f, s.t. solutions of stoch. system satisfy

$$\liminf_{t\uparrow\tau_e} |X(t)| > 0, \quad a.s.,$$

Solutions of a scalar equation may be stabilised by state-dependent Wiener perturbations, independent stochastic perturbations can destabilize solutions when the number of dimensions increases to two and higher.

Nonlinear theory Appleby, Mao, Rodkina

Theorem

If there exists $\varphi \in (0,1)$ *such that for all* $x \in \mathbb{R}^d$,
$|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2-\varphi)|x^T g(x)|^2 \leq 0$,
and for every $L > 0$, $\min_{|x|=L} |x^T g(x)| > 0$, *then*
$\lim_{t\to\infty} X(t) = 0, \quad a.s.$

Theorem

If there exists $\varphi \in (0,1)$ *such that for all* $x \in \mathbb{R}^d$,
$|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2+\varphi)|x^T g(x)|^2 \geq 0$,
then $\liminf_{t\uparrow\tau_e^\xi} |X(t)| > 0, \quad a.s.$

Linear examples (based on Nonlinear theory by Appleby, Mao, Rodkina)

$$(1)\ d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \sum_{r=1}^{m} \begin{pmatrix} \frac{\sigma}{\sqrt{m}} & 0 \\ 0 & \frac{\sigma}{\sqrt{m}} \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dW_r(t)$$

$$(2)\ d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \sum_{r=1}^{m} \begin{pmatrix} 0 & -\frac{\varepsilon}{\sqrt{m}} \\ \frac{\varepsilon}{\sqrt{m}} & 0 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dW_r(t).$$

$$(3)\ d\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} dt + \varepsilon \begin{pmatrix} X_2 dW_1(t) \\ X_3 dW_2(t) \\ X_1 dW_3(t) \end{pmatrix}.$$

Stability results

SODE	MS-stab. (cont.)	a.s.-stab. (cont., (3) suff.)
Eq. (1) (stabilising)	$\lambda + \frac{1}{2}\sigma^2 < 0$	$\lambda - \frac{1}{2}\sigma^2 < 0$
Eq. (2) (destabilising)	$\lambda + \frac{1}{2}\varepsilon^2 < 0$	$\lambda + \frac{1}{2}\varepsilon^2 < 0$
Eq. (3) (destabilising)	$\lambda + \frac{1}{2}\varepsilon^2 < 0$	$\lambda + \frac{1}{2}\varepsilon^2 < 0$
scalar case: Eq. (1), $d = m = 1$	$\lambda + \frac{1}{2}\sigma^2 < 0$	$\lambda - \frac{1}{2}\sigma^2 < 0$

Test equations

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dW_1(t) + \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dW_2(t), \quad t > 0, \tag{4}$$

and

$$d\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} dt + \varepsilon \begin{pmatrix} X_2 dW_1(t) \\ X_3 dW_2(t) \\ X_1 dW_3(t) \end{pmatrix}, \quad t > 0, \tag{5}$$

Θ-Maruyama methods

$$\begin{pmatrix} X_{1,n+1} \\ X_{2,n+1} \end{pmatrix} = \begin{pmatrix} \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} + \frac{\sqrt{h}\sigma\xi_{1,n+1}}{1-\theta h\lambda} & \frac{-\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda} \\ \frac{\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda} & \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} + \frac{\sqrt{h}\sigma\xi_{1,n+1}}{1-\theta h\lambda} \end{pmatrix} \begin{pmatrix} X_{1,n} \\ X_{2,n} \end{pmatrix} \tag{6}$$

and

$$\begin{pmatrix} X_{1,n+1} \\ X_{2,n+1} \\ X_{3,n+1} \end{pmatrix} = \begin{pmatrix} \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} & \frac{\sqrt{h}\varepsilon\xi_{1,n+1}}{1-\theta h\lambda} & 0 \\ 0 & \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} & \frac{\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda} \\ \frac{\sqrt{h}\varepsilon\xi_{3,n+1}}{1-\theta h\lambda} & 0 & \frac{1+(1-\theta)h\lambda}{1-\theta h\lambda} \end{pmatrix} \begin{pmatrix} X_{1,n} \\ X_{2,n} \\ X_{3,n} \end{pmatrix}, \tag{7}$$

Mean-square stability results

(based on Bellmann result on product of random matrices for MS stability)

Theorem

The equilibrium solution of (6) is asymptotically mean-square stable iff

$$\lambda + \frac{1}{2}(\sigma^2 + \varepsilon^2) + \frac{1}{2}\,h\,(1 - 2\,\theta)\lambda^2 < 0,$$

The equilibrium solution of (7) is asymptotically mean-square stable iff

$$\lambda + \frac{1}{2}\varepsilon^2 + \frac{1}{2}\,h\,(1 - 2\,\theta)\lambda^2 < 0,$$

a.s.stability results

(based on discrete Martingale convergence theorems)

Theorem

The equilibrium solution of (6) is a.s. asymptotically stable if

$$\lambda + \frac{1}{2}(\sigma^2 + \varepsilon^2) + \frac{1}{2}\, h\,(1 - 2\,\theta)\lambda^2 < 0,$$

The equilibrium solution of (7) is a.s. asymptotically stable if

$$\lambda + \frac{1}{2}\varepsilon^2 + \frac{1}{2}\, h\,(1 - 2\,\theta)\lambda^2 < 0,$$

Summary and Work in Progress

- We suggest several multi-dimensional test equations to perform a linear stability analysis of numerical methods for systems of SODEs. The main points are:
- Test equations for this type of analysis require some justification and some thought.
- Multi-dimensional noise and/or systems affect the stability behaviour of the methods.
- Using stochastic perturbation structures from the theory of stochastic stabilisation and destabilisation appear to yield useful test systems.
- We have carried further the mean-square and a.s. stability analysis of θ-methods.
- Analysis of further methods.
- Analysis for 'more pathological' deterministic behaviour, e.g., nonnormality, stiffness.

$$\mathrm{d}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} \lambda & b & 0 \\ 0 & \lambda & b \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix}\mathrm{d}t + \sum_{r=1}^{3}\begin{pmatrix} \frac{\sigma}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{\sigma}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{\sigma}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix}\mathrm{d}W_r(t) \tag{8}$$

and

$$\mathrm{d}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} \lambda & b & 0 \\ 0 & \lambda & b \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix}\mathrm{d}t + \varepsilon\begin{pmatrix} X_2\,\mathrm{d}W_1(t) \\ X_3\,\mathrm{d}W_2(t) \\ X_1\,\mathrm{d}W_3(t) \end{pmatrix}, \tag{9}$$

with non-random initial values $(X_1(0), X_2(0), X_3(0))^T$ and $\lambda = -1,\ b = 10$ and $\sigma = \varepsilon = 0.05$.

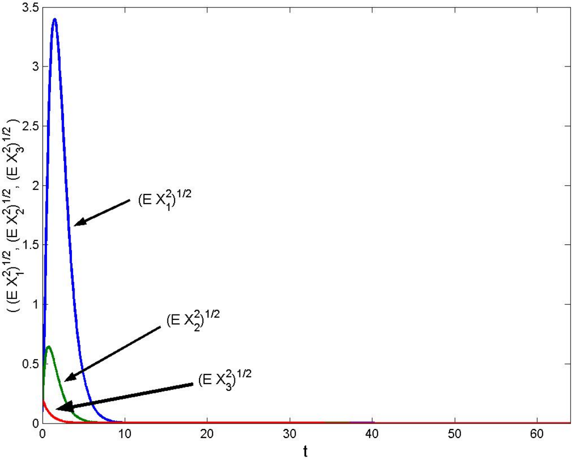


Figure 2. Simulations for Eq. (8) with $\theta = 0.5$ and $h = 0.03125$.

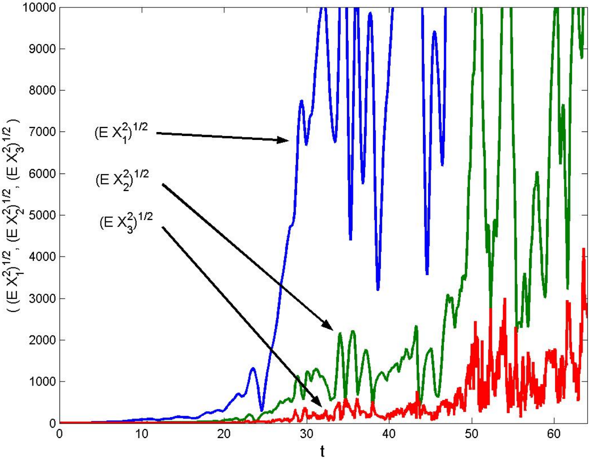


Figure 3. Simulations for Eq. (9) with $\theta = 0.5$ and $h = 0.03125$.

Thank you for your attention