SPDES driven by Lévy processes

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Outline

- Some remarks about semigroup theory
- Lévy processes - Poisson Random Measure
- Stochastic Integration in Banach spaces
- SPDEs driven by Lévy processes
- SPDEs of Reaction Diffusion Type driven by Lévy processes (If the time allows)

A typical Example

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary.

The Equation:

$$
(\star) \quad \begin{cases} \frac{du(t,\xi)}{dt} & = & \sum_{i=1}^d \frac{\partial^2}{\partial \xi_i^2} u(t,\xi) + \alpha \nabla u(t,\xi) + g(u(t,\xi)) \dot{L}(t,\xi) \\ & & + f(u(t,\xi)), \qquad \xi \in \mathcal{O},\ t > 0; \\ u(0,\xi) & = & u_0(\xi) \quad \xi \in \mathcal{O}; \\ u(t,\xi) & = & u(t,\xi) = 0, \quad t \geq 0, \xi \in \partial\mathcal{O}; \end{cases}
$$

where $u_0 \in L^p(\mathcal{O})$, $p \geq 1$, g a certain mapping and $L = \{L(t,\xi)\}_{\substack{0 \leq t < \infty \\ \xi \in \mathcal{O}}}$ is a space time Lévy noise.

Problem: To find a process

$$
u : [0,\infty) \times \mathcal{O} \longrightarrow \mathbb{R}
$$

solving Equation $(\star)$ in some certain sense.

The Abstract Cauchy Problem

Linear evolution equations, as parabolic, hyperbolic or delay equations, can often be formulated as an evolution equation in a Banach space E:

Given:

- E Banach space,
- the pair $(A, \text{dom}(A))$, where $\text{dom}(A)$ is a dense linear subspace of E and $A : \text{dom}(A) \to E$ a linear operator;
- initial value $u_0 \in E$;

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- initial value $u_0 \in E$;

Problem: The solution to the following initial valued problem:

$$(*) \quad \begin{cases} u'(t) &= A\, u(t), \quad t \geq 0, \\ u(0) &= u_0 \in E. \end{cases}$$

The Laplace Operator

Example 1 *In one of the first slides we had the following example: Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary.*

$$(\star) \quad \begin{cases} \frac{du(t,\xi)}{dt} &= \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} u(t,\xi), \quad t > 0,\ \xi \in \mathcal{O}; \\ u(0,\xi) &= u_0(\xi), \quad \xi \in \mathcal{O}; \\ u(t,\xi) &= 0, \quad t \geq 0;\ \xi \in \partial\mathcal{O} \end{cases}$$

Formulated in semigroup theory, $(\star)$ gives the following Cauchy problem:

$$\begin{aligned} E &:= L^p(\mathcal{O}), \quad 1 < p < \infty, \\ A &= \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}, \quad u(0) = u_0; \\ \text{dom}(A) &:= W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O}). \end{aligned}$$

The Abstract Cauchy problem:

We assume that A is a generator of a C_0–semigroup on E. Then the solution of the problem $(*)$ can be defined as

$$e^{-tA}u_0 = u(t, u_0), \quad \forall u_0 \in E, \ \forall t \geq 0.$$

The Abstract Cauchy problem:

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Let $f \in L^1([0,\infty); E)$. The solution of a the perturbed problem

$$(\bullet) \quad \begin{cases} u'(t) & = & Au(t) + f(t), \quad t \geq 0, \\ u(0) & = & u_0 \in E. \end{cases}$$

is given by the mild solution

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\, ds, \quad t \in (0, T].$$

Analytic Semigroups

Definition 1 *A family of operators $\{e^{-z\,A}\}_{z\in\Sigma_\delta\cup\{0\}} \subset L(X)$ is called an analytic semigroup if*

- $e^{-0A} = I$ *and* $e^{-(z_1+z_2)A} = e^{-z_1 A}e^{-z_2 A}$ *for all* $z_1, z_2 \in \Sigma_\delta$*;*
- *the map* $z \mapsto e^{-zA}$ *is analytic in* Σ_δ*;*
- $\lim_{z\to 0, z\in\Sigma_{\delta'}} e^{-zA}x = x$ *and* $0 < \delta' < \delta$*.*

A be a sectorial and densely defined operator in E

$$\Longleftrightarrow$$

the semigroup $\{e^{-tA}\}_{t\geq 0}$ generated by A on E is analytic;

$$\Longrightarrow$$

$$\left|Ae^{-tA}x\right| \leq \frac{M}{t}\,|x| \text{ for all } x \in E,\ t \in (0,T].$$

Analytic Semigroups

$$\begin{aligned} E &:= L^p(\mathcal{O}), \qquad 1 < p < \infty, \\ A &= \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}, \qquad u(0) = u_0; \\ \text{dom}(A) &:= W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O}). \end{aligned}$$

$$\Longrightarrow$$

$$\left|e^{-tA}x\right|_{W^{2,p}(\mathcal{O})} \leq \frac{M}{t} \; |x|_{L^p(\mathcal{O})} \text{ for all } x \in L^p(\mathcal{O}),\ t \in (0,T].$$

A typical Example

The Equation:

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(\star) \quad \begin{cases} \frac{du(t,\xi)}{dt} & = \sum_{i=1}^d \frac{\partial^2}{\partial \xi_i^2} u(t,\xi) + \alpha \nabla u(t,\xi) + g(u(t,\xi)) \dot{L}(t,\xi) \\ & \quad + f(u(t,\xi)), \qquad \xi \in \mathcal{O},\ t > 0; \\ u(0,\xi) & = u_0(\xi) \quad \xi \in \mathcal{O}; \\ u(t,\xi) & = u(t,\xi) = 0, \quad t \ge 0, \xi \in \partial\mathcal{O}; \end{cases}
$$

where $u_0 \in L^p(\mathcal{O})$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ a certain function and L is a space time Lévy noise specified later.

Problem: To find a process

$$
u : [0,\infty) \times \mathcal{O} \longrightarrow \mathbb{R}
$$

solving Equation ($\star$).

A Lévy Process

Definition 2 *Let E be a Banach space. An E–valued stochastic process $L = \{L(t), 0 \le t < \infty\}$ is a Lévy process over $(\Omega; \mathcal{F}; \mathbb{P})$ iff*

- $L(0) = 0$;
- *L has independent and stationary increments;*
- *L is stochastically continuous, i.e. for any $A \in \mathcal{B}(E)$ the function $[0, \infty) \ni t \mapsto \mathbb{E}1_A(L(t)) \in \mathbb{R}$ is continuous;*
- *L has a.s. càdlàg[a] paths;*

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[a] càdlàg = continue à droite, limitée à gauche.

Lévy - Khinchin - Formula

E denotes a separable Banach space and E' the dual on E.
If L is an E-valued Lévy process, then there exist (see e.g. Linde (1986))

- $a \in E'$,
- a positive operator $Q : E' \to E$,
- and a Lévy measure $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ (called usually the characteristic measure of L).

such that following formula holds for all $y \in E'$

$$\mathbb{E}\, e^{i\langle L(1),y\rangle} = \exp\left\{ i\langle a,y\rangle\lambda - \frac{1}{2}\langle Qy,y\rangle + \int_E \left(e^{i\lambda\langle y,a\rangle} - 1 - i\lambda y 1_{\{|y|\le 1\}} \right) \nu(dy) \right\}.$$

A Lévy Process

In what follows E denotes a separable Banach space, $\mathcal{B}(E)$ denotes the Borel-σ algebra on E and E' the dual on E.

Definition 3 *(see Linde (1986), Section 5.4) A symmetric*[a] *σ–finite, Borel-measure $\nu : \mathcal{B}(E) \to \mathbb{R}^+$ is called a Lévy measure if $\nu(\{0\}) = 0$ and the function*

$$E' \ni a \mapsto \exp\left(\int_E (\cos(\langle x, a\rangle) - 1)\,\nu(dx)\right) \in \mathbb{C}$$

is a characteristic function of a certain Radon measure on E.

An arbitrary σ-finite Borel measure ν is a Lèvy measure if its symmetrization $\nu + \nu^-$ is a symmetric Lévy measure.

[a] $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(E)$

Poisson Random Measure

Remark 1 *Let L be a Lévy process over $(\Omega, \mathcal{F}, \mathbb{P})$. Defining the so–called counting measure for $A \in \mathcal{B}(E)$*

$$N(t, A) = \sharp \{s \in (0, t] : \Delta L(s) = L(s) - L(s-) \in A\} \in \mathbb{N} \cup \{\infty\}$$

one can show that

- $N(t, A)$ *is a random variable over* $(\Omega; \mathcal{F}; \mathbb{P})$*;*
- $N(t, A) \sim$ *Poisson* $(t\nu(A))$ *and* $N(t, \emptyset) = 0$*;*
- *For any disjoint sets* $A_1, \ldots, A_n$*, the random variables* $N(t, A_1), \ldots, N(t, A_n)$ *are independent;* *(independently scattered)*

Poisson Random Measure

Definition 4 *Let $(Z, \mathcal{Z})$ be a measurable space and $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. A Poisson random measure on $(Z, \mathcal{Z})$ is a measurable function*

$$\eta : (\Omega, \mathcal{F}) \to (M_I(Z), \mathcal{M}_I(Z))^{a}$$

such that

- $\eta(\centerdot, \emptyset) = 0$ *a.s.*
- η *is a.s.* σ*–additive.*
- η *is a.s. independently scattered.*
- *for each $A \in \mathcal{Z}$ such that $\mathbb{E}\,\eta(\cdot, A)$ is finite, $\eta(\cdot, A)$ is a Poisson random variable with parameter $\mathbb{E}\,\eta(\cdot, A)$.*

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[a] $M_I(Z)$ denotes the set of all integer valued measures from $\mathcal{Z}$ into $\mathbb{N}$ and $\mathcal{M}_I(Z)$ is the σ-field on $M_I(Z)$ generated by functions $i_B : M(Z) \ni \mu \mapsto \mu(B) \in \mathbb{N}$, $B \in \mathcal{Z}$.

Poisson Random Measure

Let $(S, \mathcal{S})$ be a measurable space and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space.

Definition 5 *(see Ikeda Watanabe - 1981) A time homogeneous Poisson random measure* η *on* $(S, \mathcal{S})$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$*, is a measurable function*

$$\eta : (\Omega, \mathcal{F}) \to (M_I(S \times \mathbb{R}_+), \mathcal{M}_I(S \times \mathbb{R}_+)),$$

such that

(i) for each $B \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+)$*,* $\eta(B) := i_B \circ \eta : \Omega \to \bar{\mathbb{N}}$ *is a Poisson random variable with parameter* $\mathbb{E}\eta(B)$[a]*;*

(ii) η *is independently scattered;*

(iii) for each $U \in \mathcal{S}$*, the* $\bar{\mathbb{N}}$*-valued process* $(N(t, U))_{t\geq 0}$ *defined by*

$$N(t, U) := \eta(U \times (0, t]), \quad t \geq 0$$

is $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted and its increments are independent of the past, i.e. if* $t > s \geq 0$*, then* $N(t, U) - N(s, U) = \eta(U \times (s, t])$ *is independent of* $\mathcal{F}_s$*.*

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[a] If $\mathbb{E}\eta(B) = \infty$, then obviously $\eta(B) = \infty$ a.s..

Poisson Random Measure

Example 2 *Let E be of M type p, η be a time homogeneous Poisson random measure on E with intensity ν, where ν is a p-integrable symmetric Lévy measure. Then, the stochastic process (Dettweiler 1984)*

$$[0,\infty) \ni t \mapsto \hat{L}(t) := \int_0^t \int_E z\, \tilde{\eta}^{a}(dz,dt)$$

is a Lévy process with characteristic measure ν.

[a] Give a Poisson random measure $\eta : \mathcal{B}(E) \times \mathcal{B}([0,\infty)) \longrightarrow \mathbb{N}_0$ we denote the compensated Poisson random measure by $\tilde{\eta}$.

Poisson Random Measure

Definition 6 *Let*

$$\eta : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

be a Poisson random measure on E over $(\Omega; \mathcal{F}; \mathbb{P})$ and $\{\mathcal{F}_t, 0 \leq t < \infty\}$ the filtration induced by η. Then the predictable measure

$$\gamma : \Omega \times \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$$

is called compensator of η, if for any $A \in \mathcal{B}(E)$ the process

$$\tilde{\eta}(A \times (0,t]) := \eta(A \times (0,t]) - \gamma(A \times [0,t])$$

is a local martingale over $(\Omega; \mathcal{F}; \mathbb{P})$.

Remark 2 *The compensator is unique up to a $\mathbb{P}$-zero set and in case of a time homogeneous Poisson random measure given by*

$$\gamma(A \times [0,t]) = t\,\nu(A), \quad A \in \mathcal{B}(E).$$

Space - Time - White - Noise

Let us recall the Definition of a Gaussian white noise (Dalang 2003):

Definition 7 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* $(S, \mathcal{S}, \sigma)$ *a measure space. Then a Gaussian white noise on* S *based on* σ *is a measurable mapping*

$$W : (\Omega, \mathcal{F}) \to (M(S), \mathcal{M}(S))^{a}$$

- *For* $A \in \mathcal{S}$ *,* $W(A)$ *is a real valued Gaussian random variable with mean* 0 *and variance* $\sigma(A)$*, provided* $\sigma(A) < \infty$*;*
- *if* A *and* $B \in \mathcal{S}$ *are disjoint, then the random variables* $W(A)$ *and* $W(B)$ *are independent and* $W(A \cup B) = W(A) + W(B)$*.*

[a] $M(S)$ denotes the set of all measures from $\mathcal{S}$ into $\mathbb{R}$, i.e. $M(S) := \{\mu : \mathcal{S} \to \mathbb{R}\}$ and $\mathcal{M}(S)$ is the σ-field on $M(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{S}$.

Space - Time - White - Noise

Put

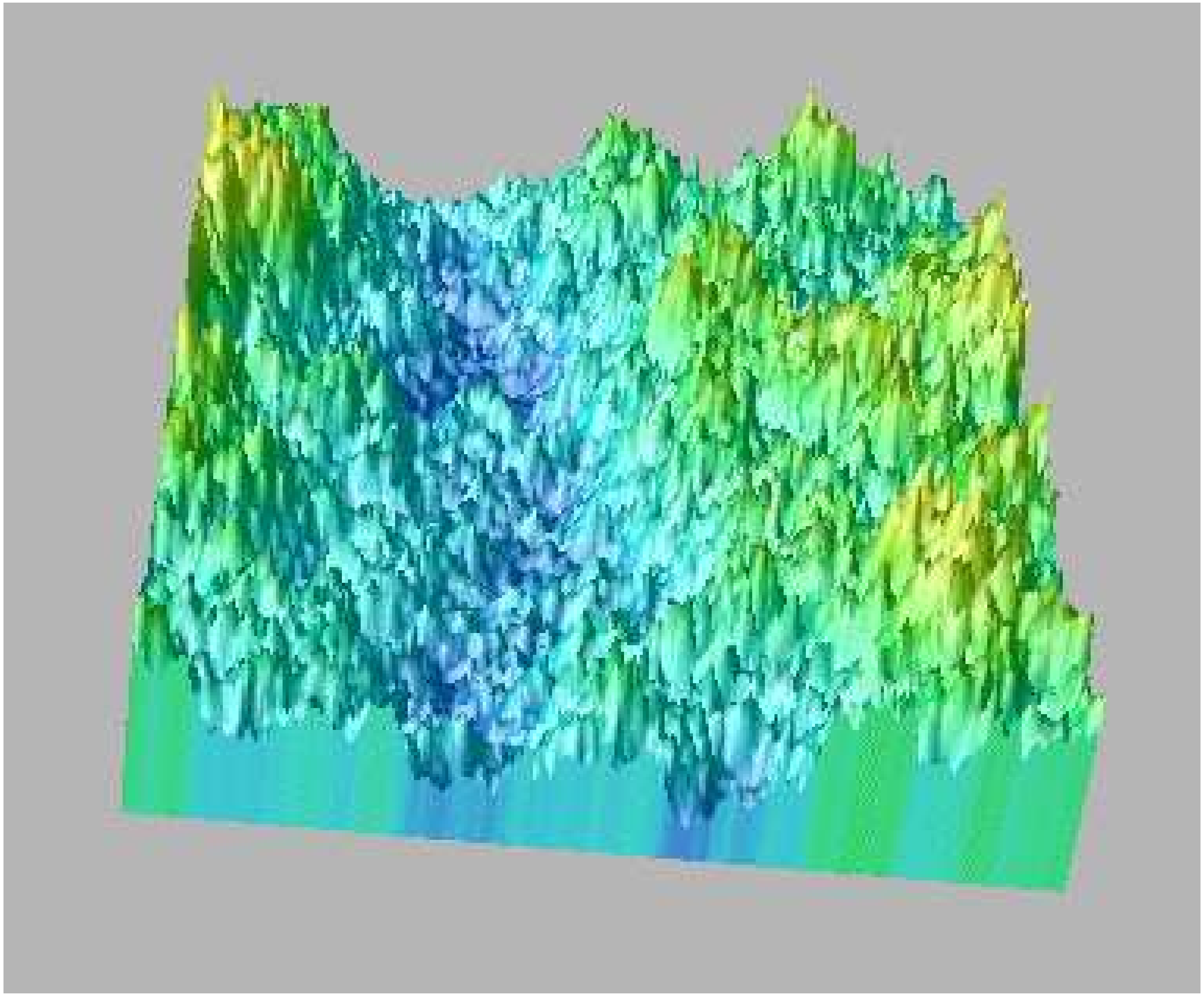
- $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
- $S = \mathcal{O} \times [0, \infty)$,
- $\mathcal{S} = \mathcal{B}(\mathcal{O}) \times \mathcal{B}([0, \infty))$
- $\sigma = \lambda_{d+1}$[a].

Then, by definition, the space time Gaussian white noise is the measure valued process process

$$t \mapsto W(\cdot \times [0, t)).$$

[a] λ_{d+1} denotes the Lebesgue measure in $\mathbb{R}^d$.

Space - Time - White - Noise



Space - Time - White - Noise

Definition 8 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $(S, \mathcal{S}, \sigma)$ be a measurable space. Then the Lévy white noise on S based on σ with characteristic jump size measure $\nu \in \mathcal{L}(\mathbb{R})$ is a measurable mapping*

$$L : (\Omega, \mathcal{F}) \to (M(S), \mathcal{M}(S))^a$$

such that

- *For $A \in \mathcal{S}$, $L(A)$ is a real valued infinite divisible random variables with characteristic exponent*
$$e^{i\theta L(A)} = \exp\left(\sigma(A) \int_{\mathbb{R}} \left(1 - e^{i\theta x} - i\sin(\theta x)\right) \nu(dx)\right),$$
provided $\sigma(A) < \infty$.

- *if A and $B \in \mathcal{S}$ are disjoint, then the random variables $L(A)$ and $L(B)$ are independent and $L(A \cup B) = L(A) + L(B)$.*

---

[a] $M(E)$ denotes the set of all measures from $\mathcal{E}$ into $\mathbb{R}$, i.e. $M(S) := \{\mu : \mathcal{S} \to \mathbb{R}\}$ and $\mathcal{M}(S)$ is the σ-field on $M(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{S}$.

Space - Time - White - Noise

Again put

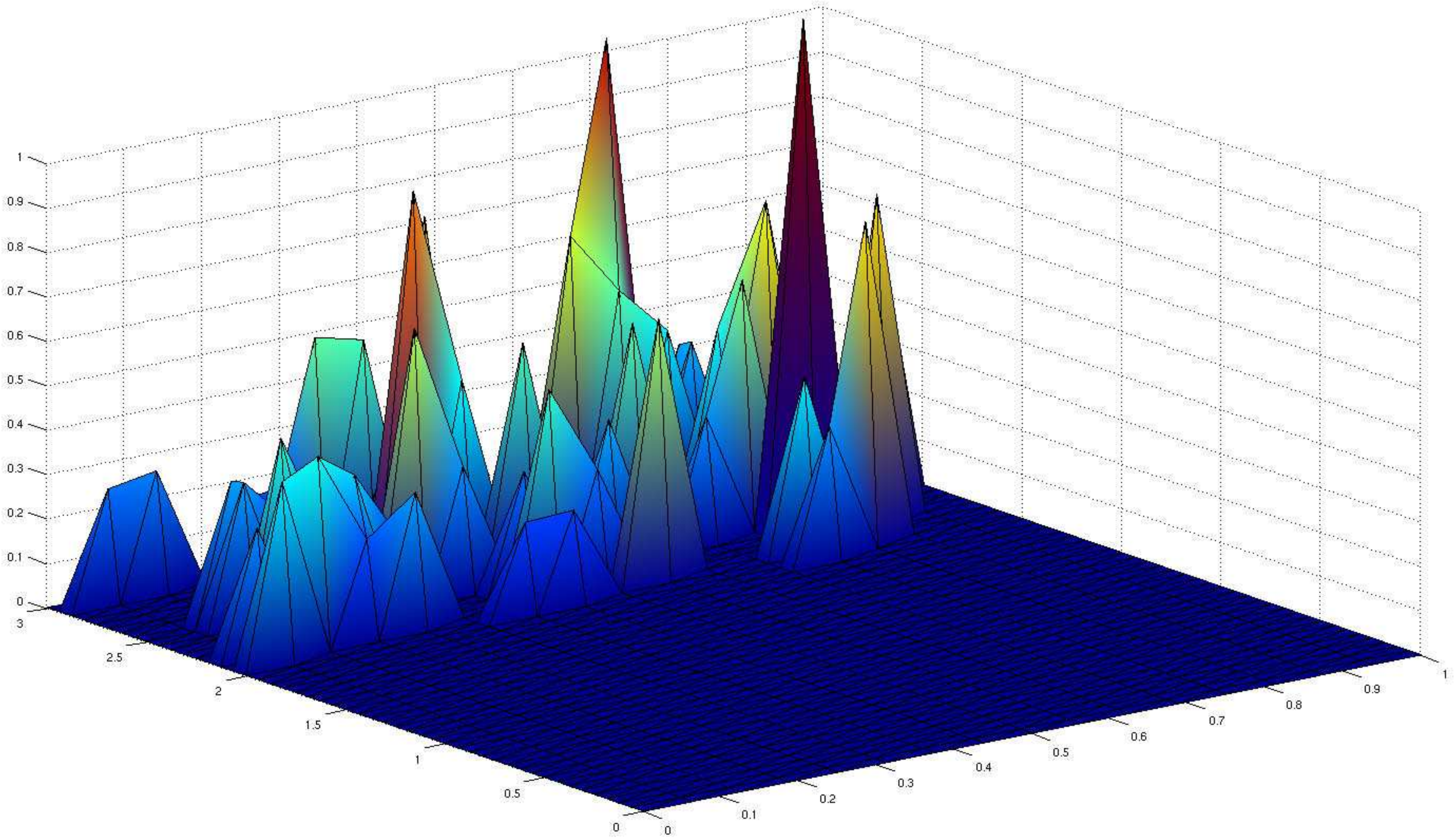
- $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
- $S = \mathcal{O} \times [0, \infty)$,
- $\mathcal{S} = \mathcal{B}(\mathcal{O}) \times \mathcal{B}([0, \infty))$
- $\sigma = \lambda_{d+1}$.

Then, by definition, the space time Lévy white noise is the measure valued process process

$$t \mapsto L(\ \cdot\ \times [0, t));$$

(for more details we refer to Breźniak and Hausenblas (2009) or Peszat and Zabczyk (2007), Albeverio and Wu 1998, St. Lupert Bié, . . .)

Space - Time - White - Noise



Space - Time - White - Noise

Space - Time - White - Noise

Definition 9 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and let* $(S, \mathcal{S}, \sigma)$ *be a measurable space. Then the Poisson white noise on* S *based on* σ *with characteristic jump size measure* $\nu \in \mathcal{L}(\mathbb{R})$ *is a measurable mapping*

$$\eta : (\Omega, \mathcal{F}) \to \left(M\left(M_I(S \times \mathbb{R})\right), \mathcal{M}\left(M_I(S \times \mathbb{R})\right)\right)$$

such that

- *for* $A \times B \in \mathcal{S} \times \mathcal{B}(\mathbb{R})$, $\eta(A \times B)$ *is a Poisson random variable with parameter* $\sigma(A)\,\nu(B)$, *provided* $\sigma(A)\nu(B) < \infty$;
- *if the sets* $A_1 \times B_1 \in \mathcal{S} \times \mathcal{B}(\mathbb{R})$ *and* $A_2 \times B_2 \in \mathcal{S} \times \mathcal{B}(\mathbb{R})$ *are disjoint, then the random variables* $\eta(A_1 \times B_1)$ *and* $\eta(A_2 \times B_2)$ *are independent and* $\eta\left((A_1 \times B_1) \cup (A_2 \times B_2)\right) = \eta(A_1 \times B_1) + \eta(A_2 \times B_2)$.

Space - Time - White - Noise

Again put

- $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.
- $S = \mathcal{O} \times [0, \infty)$,
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- $\sigma = \lambda_{d+1}$.

Then, by definition, the space time Poisson white noise is the measure valued process process

$$t \mapsto \eta(\ \cdot\ \times [0, t));$$

(for more details we refer to Breźniak and Hausenblas (2009) or Peszat and Zabczyk (2007)).

A typical Example

The Equation:

$$(\star)\quad \begin{cases} \frac{du(t,\xi)}{dt} &=& \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \alpha \nabla u(t,\xi) + g(u(t,\xi))\dot{L}(t,\xi) \\ && +f(u(t,\xi)), \qquad \xi \in \mathcal{O},\ t>0; \\ u(0,\xi) &=& u_0(\xi) \quad \xi \in \mathcal{O}; \\ u(t,\xi) &=& u(t,\xi)=0, \quad t\ge 0, \xi \in \partial\mathcal{O}; \end{cases}$$

where $u_0 \in L^p(0,1)$, $p \ge 1$, g a certain mapping and L is a Lévy process taking values in a certain Banach space E.

Let B a Banach space. A mild solution of Equation $(\star)$ on B is a B–valued, adapted, càdlàg process $u = \{u(t) : t \in [0,T]\}$ such that for $t \ge 0$ we have a.s.

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A}\, F^{a}(u(s))\, dt + \int_0^t e^{-(t-s)A} G(u(s))[L(ds)]$$

[a] F and G denote the to f and g associated Nemytskii operators, $\{e^{-tA}\}_{t\ge 0}$ denotes the from operator A in E generated semigroup.

Banach spaces of M type p

Definition 10 [a] *Let $0 < p < \infty$. A Banach space E is of M type p, iff there exists a constant $C = C(E;p)$, such that for each discrete E-valued martingale $M = (M_1, M_2, \ldots)$ one has*

$$\sup_{n\ge 1} \mathbb{E}|M_n|_E^p \le C \sum_{n\ge 1} \mathbb{E}|M_n - M_{n-1}|_E^p.$$

[a] see Pisier (1986), Maurey, Schwartz.

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$$\sup_{n\geq 1} \mathbb{E}|M_n|_E^p \leq C \sum_{n\geq 1} \mathbb{E}|M_n - M_{n-1}|_E^p.$$

- If $(S, \mathcal{S}, \sigma)$ is a probability space and $p > 1$, then the space $L^p(S, \mathcal{S}, \sigma)$ is of M-type $p \wedge 2$. Additionally, $L^\infty(S, \mathcal{S}, \sigma)$, $L^1(S, \mathcal{S}, \sigma)$ and $\mathcal{C}([0,1];\mathbb{R})$ are not of M type p.

- Let $0 < p \leq 2$. Let E be of M-type p and $A : E \to E$ an operator with domain dom(A). If A^{-1} is bounded, then dom(A) is isomorphic to E and therefore of M–type p.

- (Brzeźniak (1990)) Assume E_1 and E_2 are a Banach space of M-type p, where E_2 is continuously and densely embedded in E_1. Then for any $\vartheta \in (0,1)$ the complex interpolation space $[E_1, E_2]_\vartheta$ and the real interpolation space $(E_1, E_2)_{\vartheta,p}$ are of M–type p.

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see Pisier (1986), Maurey, Schwartz.

Burkholder Davis Gundy inequality

Proposition 1 *Let E be a Banach space of M-type p, $1 \leq p \leq 2$. Then there exists a constant $C = C(E;p) < \infty$, such that we have for any discrete E-valued martingale $M = (M_1, M_2, \dots)$ and for all $1 \leq r < \infty$*

$$\mathbb{E} \sup_{n \geq 1} |M_n|_E^r \leq C \mathbb{E} \left[\sum_{n \geq 1} |M_{n-1} - M_n|_E^p \right]^{\frac{r}{p}} .$$

The Itô Stochastic Integral

- E be a separable Banach spaces of M-type p, $1 \leq p \leq 2$;
- $(Z, \mathcal{Z})$ a measurable space and ν a non negative measure on $(Z, \mathcal{Z})$;
- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space and
- η be a time homogeneous Poisson random measure on Z over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with intensity measure ν;

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[a] $\pi_{s,t} \circ f$ is the projection of f onto the time interval (s, t).

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- η be a time homogeneous Poisson random measure on Z over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ with intensity measure ν;

Remark 3 *Here, it is important that* $(\mathcal{F}_t)_{t\ge 0}$ *is non–anticipated to* η*. That is, that for all* $t \ge 0$ *the random variable* $\pi_{t,\infty} \circ \eta$ *is independent of* $\mathcal{F}_t$*.*

$\pi_{s,t} \circ f$ is the projection of f onto the time interval (s, t).

The Itô Stochastic Integral

Let h be a progressively measurable step function with representation

$$h(t) = \sum_{i=1}^{n} H_i 1_{(t_i,t_{i+1}]}(t), \quad t \in \mathbb{R}_+,$$

where $0 = t_0 \leq \cdots t_n = T$ and $H_i : \Omega \to L^p(Z, \nu; E)$ is $\mathcal{F}_{t_i}$–measurable for $i = 1, \ldots, n$.

Definition 11 *The stochastic integral of h with respect to η is defined by*

$$I(h) := \sum_{i=1}^{n} \int_Z H_i(s)\, \tilde{\eta}(ds; (t_i, t_{i+1}]). \quad (\spadesuit)$$

Definition of the Integral

Let E be a Banach space of martingale type p and

$$\mathcal{M}^p([0,T];L^p(Z,\nu;E)) := \Big\{ h:\Omega\times[0,\infty)\to L^p(Z,\nu;E),$$

$$h \text{ is progressively measurable and } \int_{\mathbb{R}^+}\int_Z \mathbb{E}\,|h(s,z)|^p\;\nu(dz)\,ds^{a} < \infty \Big\}$$

Theorem 1 [b] *There exists a linear bounded operator*

$$I:\mathcal{M}^p([0,T];L^p(Z,\nu;E)) \to L^p(\Omega,\mathcal{F}_T,\mathbb{P};E),$$

which is a unique bounded extension of the operator defined in (♠).

If $h\in\mathcal{M}^p([0,T];L^p(Z,\nu;E))$ and $t>0$ then we put

$$\int_0^t\int_Z h(s)\tilde\eta(ds,dz) := I(1_{(0,t]}h)$$

and we call the LHS the Itô integral of the process h up to time t.

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[a] ν is the intensity of η

[b] $p=1,2$ B. Rüdiger (2005), $p\in(1,2]$, EH (2005), EH and Brzeźniak (2008), Filipovic and Tappe (2008)

for Burkholder Davies Gundy type inequalities see Preprint E. Hausenblas (2009) and Röckner, Marinelli, Pivot (2009)

Properties of the Stochastic Integral

- If $h \in \mathcal{M}^p([0,T]; L^p(Z,\nu;E))$, then the process

$$X(t) = \int_0^t \int_Z h(s,z)\, \tilde{\eta}(dz;ds), \quad t \ge 0$$

is an E–valued martingale having a càdlàg modification [a].

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- If $h \in \mathcal{M}^p([0,T]; L^p(Z,\nu;E))$, then the process

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is an E–valued martingale having a càdlàg modification .

- There exists a constant $C = C(p,E) < \infty$, such that for any $h \in \mathcal{M}^p([0,T]; L^p(Z,\nu;E))$ and for any $0 < r \leq p$

$$\mathbb{E} \sup_{0<t\leq T} \left| \int_0^t \int_Z h(s,z)\, \tilde{\eta}(dz;ds) \right|^r \leq$$

$$C \left(\int_0^T \int_Z \mathbb{E}\, |h(s,z)|_E^p\, \nu(dz)\, ds \right)^{\frac{r}{p}} .$$

$p = 1,2$ B. Rüdiger (2005), $p \in (1,2]$ EH and Brzeźniak (2009).

A typical Example

The Equation:

$$(\star)\quad \begin{cases} \frac{du(t,\xi)}{dt} &= \frac{\partial^2}{\partial\xi^2}u(t,\xi)+\alpha\nabla u(t,\xi)+g(u(t,\xi))\dot{L}(t,\xi) \\ & \quad +f(u(t,\xi)), \qquad \xi\in\mathcal{O},\ t>0; \\ u(0,\xi) &= u_0(\xi) \quad \xi\in\mathcal{O}; \\ u(t,\xi) &= u(t,\xi)=0, \quad t\ge 0,\ \xi\in\partial\mathcal{O}; \end{cases}$$

where $u_0\in L^p(0,1)$, $p\ge 1$, g a certain mapping and L is a Lévy process taking values in a certain Banach space Z.

Let B a Banach space. A mild solution of Equation $(\star)$ on B is an adapted B–valued càdlàg process $u=\{u(t):t\in[0,T]\}$ such that for $t\ge 0$ we have a.s.

$$u(t)=e^{-tA}u_0+\int_0^t e^{-(t-s)A}\,F^{a}(u(s))\,dt+\int_0^t e^{-(t-s)A}G(u(s))[L(ds)]$$

[a] F and G denote the to f and g associated Nemytskii operators.

SPDEs - Existence and Uniqueness

Theorem 2 *(EH, 2005 EJP) Assume that there exist some* $\delta_g < \frac{1}{p}$ *and* $\delta_f, \delta_I < 1$ *such that*

- u_0 *satisfies* $\mathbb{E}|(-A)^{-\delta_I} u_0|^p < \infty$*;*
- $(-A)^{-\delta_f} F : E \to E$ *is Lipschitz continuous;*
- $(-A)^{-\delta_g} G : E \to L^p(Z, \nu; E)$ *satisfies*

$$\int_Z \left|(-A)^{-\delta_g} \left[g(x,z) - g(y,z)\right]\right|_E^p \nu(dz) \le C\, |x-y|_E^p, \quad x, y \in E.$$

Then, there exists a unique mild solution to Problem (1), such that for any $T > 0$

$$\int_0^T \mathbb{E}|u(s)|_E^p \, ds < \infty,$$

and $(-A)^{-\gamma} u \in L^0(\Omega; \mathbb{D}([0,T]; E))$*, where* $\gamma > \frac{1}{p} + 1$*.*

Space Time Lévy Noise

By means of Besov-spaces it is possible to show existence of an integral solution if the driving proces is a space time Levy white noise.

Corollary 1 *Let* A *be the Laplace operator. If there exists a* $p \in (1, 2]$ *with* $1 < p < \frac{2+d}{d}$*, then there exists a solution to the SPDE above with space time Lévy noise.*

SPDEs of Reaction Diffusion Type

We are interested in SPDEs of the following type:

$$(\Diamond)\quad \begin{cases} du(t) &= \big(\Delta u(t) - u^3(t) + u(t)\big)\,dt + dL(t), \quad t \geq 0, \\ u(0,\xi) &= u_0(\xi) \quad 0 \leq \xi \leq 1, \\ u(t,0) &= u(t,1) = 0, \quad t \geq 0, \end{cases}$$

where $u_0 \in L^p(0,1)$, $p \geq 1$, and $L(t)$ is a Lévy process.

Or an SPDE given by

$$(\clubsuit)\quad \begin{cases} du(t) &= Au(t)\,dt + F(t,u(t))\,dt \\ & \qquad + \int_Z G(t,u(t);z)\tilde{\eta}(dz;dt), \\ u(0) &= u_0 \in E, \end{cases}$$

where F and G are not global Lipschitz, but continuous and bounded, E is a Banach space.

Solution of Martingale Type

Definition 12 *A martingale solution to equation (♣) is a system*

$$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \{\eta(t)\}_{t\geq 0}, \{u(t)\}_{t\geq 0})$$

such that $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete probability space,* $\{\mathcal{F}_t\}_{t\geq 0}$ *a filtration on it,* $\{\eta(t)\}_{t\geq 0}$ *is a time homogeneous Poisson Random measure on* Z *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *with intensity* ν *and* $u(t)$ *is a* B*–valued adapted process such that for any* $t \in [0, T]$

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} F(s, u(s))\, ds + \int_0^t \int_Z e^{-(t-s)A} G(s, u(s); z)\, d\tilde{\eta}(dz, ds),\ a.s..$$

Assumptions

- there exists some $0 \le \delta_F < 1$ such that the map
$$A^{-\delta_F} F : [0,\infty) \times E \to E$$
is *bounded* and continuous with respect to the second variable.
- there exists some δ_G, $0 \le \delta_G < \frac{1}{p}$ such that

(i) there exists some $M < \infty$ with
$$\int_Z |A^{-\delta_G} G(t,u;z)|_E^p \, \nu(dz) \le M \quad \text{(boundedness);}$$

(ii) for all $u_0 \in E$ and $t \in \mathbb{R}^+$ and for all $u_0 \in E$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in \mathbb{R}^+$ we have
$$\int_Z |A^{-\delta_G} \left(G(t,u;z) - G(t,u_0;z) \right) |_E^p \, \nu(dz) \le \varepsilon \quad \text{(continuity)}$$
provided $u \in E$ satisfies $|u - u_0|_E \le \delta$.

The Result

Given

$$(\clubsuit)\begin{cases} du(t) &= Au(t)\,dt + F(t,u(t))\,dt + \int_Z G(t,u(t);z)\tilde\eta(dz;dt), \\ u(0) &= u_0 \in E. \end{cases}$$

Theorem 3 *(Brzeźniak, E.H.) Assume there exists some $\epsilon < 1$ such that $A^{-\epsilon}x \in E$. Then, under the assumption before, there exists a martingale solution $u = \{u(t), t \geq 0\}$ of (♣), such that*

$$\int_0^\infty e^{-\lambda t}\mathbb{E}|u(t)|_E^p\,dt < \infty$$

and for any $\delta > \max(0, \delta_F - 1 + \frac{1}{p}, \delta_G + \frac{1}{p}, \epsilon)$ and we have a.s. $u \in \mathbb{D}(\mathbb{R}^+; B)$, where $B = V_{-\delta}$.

Proof - The Approximating Sequence

Let $s_n = \frac{k}{2^n}T$ if $\frac{k}{2^n}T \leq s < \frac{k+1}{2^n}T$. Define a sequence of adapted E–valued processes by

$$\begin{aligned} \bar{u}_n(t) &= e^{-tA}x_n + \int_0^t e^{-(t-s)A}F(s, \hat{u}_n(s_n))\, ds \\ &\quad + \int_0^t \int_Z e^{-(t-s)A}G(s, \hat{u}_n(s_n); z)\, \tilde{\eta}(dz; ds), \end{aligned}$$

where $u(s_n)$ is defined by $\hat{u}(s_n) = u_0$ for $0 \leq s < 2^{-n}$, and

$$\hat{u}(s_n) := 2^n \int_{s_n - 2^{-n}}^{s_n} \bar{u}_n(r)\, dr.$$

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Moreover, let $\bar{\eta}_n := \eta$, $n \in \mathbb{N}$.

In this way we constructed a sequence

$$\{(\bar{u}_n, \bar{\eta}_n)\,, n \in \mathbb{N}\}\,.$$

Proof - The Deterministic Convolution

For fixed $\alpha \in (0,1]$ and $f \in \mathbb{L}^p_\lambda(\mathbb{R}^+; E)$ let $\Lambda^{-\alpha}$ be defined by

$$\left(\Lambda^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)A} f(s)\, ds, \quad t \in \mathbb{R}^+ .$$

Then, for any $\alpha \in (0,1]$ the operator satisfies following properties:

- $\Lambda^{-\alpha} : \mathbb{L}^p([0,T]; E) \to \mathbb{L}^p([0,T]; E)$ is bounded and compact;
- for $0 < \beta < \alpha - \frac{1}{q} + \gamma - \delta$, $\Lambda^{-\alpha} : \mathbb{L}^q([0,T]; V_\gamma) \to \mathcal{C}^{(\beta)}([0,T]; V_\delta)$ is bounded and compact;

[Here we use results of Brzeźniak (1997)]

Proof - the Stochastic Convolution Term

In contrary to the stochastic convolution driven by Wiener noise, the convolution driven by Lévy noise cannot be decomposed !

$$(\mathfrak{G}u)\,(t) = \int_0^t e^{-(t-s)A}\, G(s, u(s); z)\tilde{\eta}(dz, ds), \quad t \in \mathbb{R}^+ \,.$$

Under the assumption of the Theorem the set

$$\left\{ \mathfrak{G}x \;\middle|\; x \in \mathbf{L}^0 \left(\Omega; \mathbb{L}^p(\mathbb{R}^+; E) \right) \right\}$$

is tight on $\mathbb{L}^p(\mathbb{R}^+; E)$ and on $\mathbb{D}(\mathbb{R}^+; B)$.

Proof - The Approximating Sequence

(1) The laws of the set $\{(\bar{u}_n, \bar{\eta}_n), n \in \mathbb{N}\}$ are tight on $\mathbb{L}^p_\lambda(\mathbb{R}^+; E) \times M(\mathbb{R}^+ \times Z)$.

(2) The laws of the set $\{(\bar{u}_n, \bar{\eta}_n), n \in \mathbb{N}\}$ are tight on $\mathbb{D}(\mathbb{R}^+; B) \times M(\mathbb{R}^+ \times Z)$.

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(1) and (2) $\Rightarrow$ there exist a subsequence of $\{(\bar{u}_n, \bar{\eta}_n), n \in \mathbb{N}\}$ and

$$(\chi^*, \mu^*) \in \mathcal{M}\left(\mathbb{L}^p_\lambda(\mathbb{R}^+; E) \cap \mathbb{D}(\mathbb{R}^+; B) \times M(\mathbb{R}^+ \times Z)\right)$$

such that the laws of $(\bar{u}_n, \eta_n)$ converge to (χ^*, μ^*).

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such that the laws of $(\bar{u}_n, \eta_n)$ converge to (χ^*, μ^*).

By the Skorohod embedding Theorem we know that there exists a probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$, random variables $(\check{u}_n, \check{\eta}_n)$, $n \in \mathbb{N}$, and $(\check{u}^*, \check{\eta}^*)$ over $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ such that

$$\begin{aligned} (\check{u}_n, \check{\eta}_n) &\rightarrow (\check{u}^*, \check{\eta}^*) \quad \check{\mathbb{P}}\text{-a.s.}; \\ \mathcal{L}((\check{u}_n, \check{\eta}_n)) &= \mathcal{L}((\bar{u}_n, \bar{\eta}_n)). \end{aligned}$$

Proof - The Approximating Sequence

Let
$\check{\mathcal{F}}_t := \sigma\left(\check{u}^*(s), \check{\eta}^*(A \times [0,s)), \check{u}_n(s), \check{\eta}_n(A \times [0,s)), n \in \mathbb{N}, 0 \le s \le t, A \in \mathcal{B}(Z)\right)$.

It remains to show that the system

$$(\check{\Omega}, \check{\mathcal{F}}, (\check{\mathcal{F}})_{t\ge 0}, \check{\mathbb{P}}, \check{\eta}^*, \check{u}^*)$$

is a martingale solution to (•). In particular, that $\check{\mathbb{P}}$ a.s.

$$\begin{aligned} \check{u}^*(t) &= e^{-tA}x + \int_0^t e^{-(t-s)A}\, F(\check{u}^*(s))\, ds \\ &+ \int_0^t \int_Z e^{-(t-s)A}\, G(\check{u}^*(s); z)\, \tilde{\check{\eta}}^*(dz; ds), \quad t > 0. \end{aligned}$$

Proof - The Approximating Sequence

Here we have the following setting:

$$\begin{aligned}(\check{u}_n, \check{\eta}_n) &\to (\check{u}^*, \check{\eta}^*) \quad \check{\mathbb{P}}\text{-a.s.};\\ \mathcal{L}\left((\check{u}_n, \check{\eta}_n)\right) &= \mathcal{L}\left((\bar{u}_n, \bar{\eta}_n)\right);\end{aligned}$$

Moreover, we know how $\bar{u}_n$ is constructed. In particular,

$$\begin{aligned}\bar{u}_n(t) &= e^{-tA}x_n + \int_0^t e^{-(t-s)A} F(s, \hat{u}_n(s_n))\, ds \\ &\quad + \int_0^t \int_Z e^{-(t-s)A} G(s, \hat{u}_n(s_n); z)\, d\tilde{\eta}(dz; ds),\end{aligned}$$

where $\hat{u}(s_n)$ is defined by

$$\hat{u}(s_n) := 2^n \int_{s_n - 2^{-n}}^{s_n} \bar{u}_n(r)\, dr.$$

Observe, the latter implies $\mathbb{E}\|\hat{u}_n - \bar{u}_n\|^p_{L^p_\lambda(\mathbb{R}_+, E)} \to 0.$

Definition of the Integral

Theorem 4 [a] *Assume the following holds:*

- $(\Omega_1, \mathcal{F}_1, (\mathcal{F}_t^1)_{t\ge 0}, \mathbb{P}_1)$ *and* $(\Omega_1, \mathcal{F}_1, (\mathcal{F}_t^1)_{t\ge 0}, \mathbb{P}_1)$ *are two probability spaces;*
- (ξ_1, η_1) *and* (ξ_2, η_2) *belong a.s. to* $L^p(\mathbb{R}_+; L^p(Z, \nu, E)) \times \mathcal{M}_I(S \times \mathbb{R}_+)$*;*
- η_1 *is a time homog. Prm on* Z *over* $(\Omega_1, \mathcal{F}_1, (\mathcal{F}_t^1)_{t\ge 0}, \mathbb{P}_1)$ *with intensity* ν*;*
- $\xi_1 \in \mathcal{M}^p(\Omega_1 \times \mathbb{R}_+; L^p(Z, \nu, E))$ *with respect to* $(\mathcal{F}_t^1)_{t\ge 0}$*;*

Then, if

$$\mathcal{L}\text{aw}((\xi_1, \eta_1)) = \mathcal{L}\text{aw}((\xi_2, \eta_2))$$

on $L^p(\mathbb{R}_+; L^p(Z, \nu, E)) \times \mathcal{M}_I(S \times \mathbb{R}_+)$*, then*

$$\mathcal{L}\text{aw}\left(\left(\int_0^\cdot \int_Z \xi_1(s,z)\,\tilde{\eta}_1(dz;ds), \xi_1, \eta_1\right)\right) = \mathcal{L}\text{aw}\left(\left(\int_0^\cdot \int_Z \xi_2(s,z)\,\tilde{\eta}_2(dz;ds), \xi_2, \eta_2\right)\right)$$

on $\mathbb{D}(\mathbb{R}_+, E) \cap L^p(\mathbb{R}_+, E) \times L^p(\mathbb{R}_+; L^p(S, \nu, E)) \times \mathcal{M}_I(S \times \mathbb{R}_+)$*.*

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[a] EH and Brzeźniak (2009), Proceeding of the Ascona workshop 2008

The Stochastic Convolution Process

Content of the work: Brzeźniak and E.H. (2008):

Let us assume that $1 < p \le 2$, $1 \le q \le p$, E is a M-type p Banach space, and $-A$ is an infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{0 \le t < \infty}$in E. We consider the following SPDE written in the Itô-form

$$\begin{cases} du(t) &= Au(t)\, dt + \int_Z \xi(t;x)\tilde{\eta}(dx;dt), \\ u(0) &= 0, \end{cases}$$

where $\xi : [0,T] \times \Omega \to L^p(Z,\nu;E)$ is a progressively measurable process.

Then, we have

$$\boxed{\mathbb{E} \int_0^T |u(t)|^p_{D_A(\theta+\frac{1}{p},q)}\, dt \le C\mathbb{E} \int_0^T \int_Z |\xi(t,z)|^p_{D_A(\theta,q)}\, dt.} \qquad \text{(-8)}$$

The Stochastic Convolution Process

Theorem 5 *(EH and Brzeźniak (2009)) Assume the setting of Theorem (4). In addition, assume that* $-A$ *is an infinitesimal generator of an analytic semigroup* $\{e^{-t}\}_{0\le t<\infty}$ *in* E. *Let us consider the following SPDE written in the Itô-form*

$$\begin{cases} du(t) &= Au(t)\,dt + \int_Z \xi(t;x)\tilde\eta(dx;dt), \\ u(0) &= 0, \end{cases}$$

where $\xi : [0,T]\times\Omega \to L^p(Z,\nu;E)$ *is a progressively measurable process. Then, if*

$$\mathcal{L}\text{aw}((\xi_1,\eta_1)) = \mathcal{L}\text{aw}((\xi_2,\eta_2))$$

on $L^p(\mathbb{R}_+;L^p(Z,\nu,E))\times\mathcal{M}_I(S\times\mathbb{R}_+)$*, then*

$$\mathcal{L}\text{aw}\Big(\big(\int_0^\cdot\int_Z e^{-(\cdot-s)}\xi_1(s,z)\,\tilde\eta_1(dz;ds),\xi_1,\eta_1\big)\Big) = \mathcal{L}\text{aw}\Big(\big(\int_0^\cdot\int_Z e^{-(\cdot-s)}\xi_2(s,z)\,\tilde\eta_2(dz;ds),\xi_2,\eta_2\big)\Big)$$

on $\mathbb{D}(\mathbb{R}_+,B)\cap L^p(\mathbb{R}_+,X)\times L^p(\mathbb{R}_+;L^p(Z,\nu,E))\times\mathcal{M}_I(S\times\mathbb{R}_+)$.

SPDE of Reaction Diffusion Type

We are interested in SPDEs of the following type

$$(\Diamond)\begin{cases} \frac{\partial}{\partial t}u(t,\xi) &= -(-\Delta)^k u(t,\xi) - |u(t,\xi)|^q \,\mathrm{sgn}(u(t,\xi)) + b\,u(t,\xi) \\ & \quad + g(u(t-,\xi),\xi;\zeta)\,\dot{L}(\xi,t),\ \xi\in\mathcal{O},\ t>0, \\ u(0,\xi) &= u_0(\xi), \quad \xi\in\mathcal{O}, \\ u(t,\xi) &= 0,\ \text{for } \xi\in\partial\mathcal{O},\ t>0\,. \end{cases}$$

where $b\in\mathbb{R}$ and $\dot{L}$ denotes roughly spoken the Radon Nikodym derivative of the space time Poissonian noise.

We are asking for the conditions on q and k under which a solution to the equation $(\Diamond)$ exists.

SPDE of Reaction Diffusion Type

Theorem 6 *(Brzeźniak, E.H. (2009)) If*

$$d < \frac{2k}{q} + 4\left(\frac{1}{q} - \frac{1}{p}\right)$$

and $x_0 \in W_p^{-(d-\frac{d}{p})}(\mathcal{O})$*, then, there exists a martingale solution to (*$\star$*), such that*

$$\mathbb{P}\left(u \in \mathbb{D}(\mathbb{R}_+; B)\right) = 1,$$

for $B = W_p^{-\gamma}(\mathcal{O})$ *for any* $\gamma \in \mathbb{R}$ *with* $\gamma > d - \frac{d}{p}$*.*

SPDE of Reaction Diffusion Type

Let F be defined on $X = H_p^{\gamma_1}(\mathcal{O})$[a] for $\gamma_1 > \frac{d}{p}$ by

$$F(x)(\xi) := -|x(\xi)|^q \operatorname{sgn}(x(\xi)) + x(\xi) \ , \xi \in \mathcal{O}.$$

Let $F_n : X \to X$ be defined by

$$F_n(x) = \begin{cases} F(x), & \text{if } |x|_X \le n, \\ F\left(\frac{n}{|x|_X} x\right), & \text{otherwise.} \end{cases}$$

Since we have $|F_n(y)|_X \le a(n)$ for all $y \in X$, it follows in view of the Theorem before, that there exists a martingale solution.

Let us denoted the family of martingales solutions by $\{u_n, n \in \mathbb{N}\}$. The next step is to show, that the laws of the family of martingale solutions $\{u_n, n \in \mathbb{N}\}$ are tight in $\mathcal{M}(\mathbb{D}(\mathbb{R}^+; B))$.

[a] Note, that $H_p^{\gamma_1}(\mathcal{O}) \hookrightarrow C_b^0(\mathcal{O})$.

SPDE of Reaction Diffusion Type

Lemma 1 *(Da Prato) Assume that X is a Banach space, $-A$ a generator of a strongly continuous semigroup of bounded linear operators on X and a mapping $F : X \to X$ such that*

$$< -Ax + F(t, x + y), z >\leq (1 + |y|_X^q) - k|x|_X, \tag{-7}$$

for any $z \in x^* = \partial|x|$. Assume that for some $\tau > 0$ two continuous functions $z, v : [0, \infty) \to X$ satisfy

$$z(t) \quad = \quad \int_0^t e^{-(t-s)A} F(z(s) + v(s))\, ds,\ t \leq \tau.$$

Then

$$|z(t)|_X \quad \leq \quad \int_0^t e^{-k(t-s)} \, (1 + |v(s)|_X^q)\ ds, \ \ 0 \leq t \leq \tau.$$

SPDE of Reaction Diffusion Type

Let

$$v_n(t) := \left(\Lambda^{-1}u\right)(t) = \int_0^t e^{-(t-s)A} G(z, u_n(t))^{a}\, \tilde{\eta}(dz, ds), \quad t \in \mathbb{R}^+ ,$$

and

$$z_n(t) = \int_0^t e^{-(t-s)A} F_n(z_n(s) + v_n(s))\, ds, \ t \le \tau.$$

Then $z_n + v_n = u_n$ is the solution to

$$\begin{aligned} \frac{\partial}{\partial t} u_n(t,\xi) &= -(-\Delta)^k u_n(t,\xi) + F_n\left(u_n(t,\xi)\right) + g(u_n(t,\xi), \xi; \zeta)\, \dot{L}(\xi, t), \\ u_n(0,\xi) &= u_0(\xi), \quad \xi \in \mathcal{O}, \\ u_n(t,\xi) &= 0, \text{ for } \xi \in \partial\mathcal{O},\, t > 0\,. \end{aligned}$$

---

[a] G is the E valued operater associated to g

SPDE of Reaction Diffusion Type

Let $\gamma_0 \geq \gamma_1 \geq \gamma_2$ and

$$\begin{aligned} E &:= H_p^{\gamma_0}(\mathcal{O}), \\ X &:= H_p^{\gamma_1}(\mathcal{O}) = (E, B)_{[1-\frac{p}{q}]}, \qquad \gamma_1 := \gamma_2 + \frac{p}{q}(\gamma_0 - \gamma_2). \\ B &:= H_p^{\gamma_2}(\mathcal{O}) \quad \text{and} \end{aligned}$$

Theorem (Bergh and Löfström)

$$\left(\mathbb{L}^p(\mathbb{R}^+; E), \mathbb{L}^\infty(\mathbb{R}^+; B)\right)_{[1-\frac{p}{q}]} = \mathbb{L}^q\left(\mathbb{R}^+; (E, B)_{[1-\frac{p}{q}]}\right)$$

SPDE of Reaction Diffusion Type

- if $\gamma_2 < \frac{d}{p} - d$ then the space time Poissonian noise can be identified in $H_p^{\gamma_2}(\mathcal{O})$.
- if $\gamma_0 - \gamma_2 < \frac{2}{p}$, that the set

$$\left\{ \Lambda^{-1} x \middle| x \in \mathbf{L}^0 \left(\Omega; \mathbb{L}^p(\mathbb{R}^+; E) \right) \cap \mathbf{L}^0 \left(\Omega; \mathbb{D}(\mathbb{R}^+; B) \right) \right\}$$

  is tight in $\mathcal{M}\left(\mathbb{L}(\mathbb{R}^+; E)\right)$ and $\mathcal{M}\left(\mathbb{D}(\mathbb{R}^+; B)\right)$.
- if $\gamma_1 > \frac{d}{p}$ then $C_b^0(\mathcal{O}) \hookrightarrow H_p^{\gamma_1}(\mathcal{O})$ and the mappings F_n are satisfying the assumption of the Lemma before;

SPDE of Reaction Diffusion Type

- $\Longrightarrow$ the set $\{v_n, n \in \mathbb{N}\}$ is tight in $\mathcal{M}(\mathbb{L}^q(\mathbb{R}^+; X))$;
- $\Longrightarrow$ By the lemma before, one knows, that the set $\{z_n, n \in \mathbb{N}\}$ given by

$$z_n(t) \quad = \quad \int_0^t e^{-(t-s)A} F_n(z_n(s) + v_n(s))\, ds, \qquad t \le \tau.$$

  is tight in $\mathcal{M}\left(\mathcal{C}(\mathbb{R}^+; X)\right)$;
- $\Longrightarrow$ the set $\{u_n = v_n + z_n, n \in \mathbb{N}\}$ is tight in $\mathcal{M}(\mathbb{D}(\mathbb{R}^+; B))$.

The End

Thank you for your attention