Some computational aspects of stochastic phase-field models

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based on joint work with
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17 November 2009

3rd
Workshop
On
Random
Dynamical
Systems

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Outline

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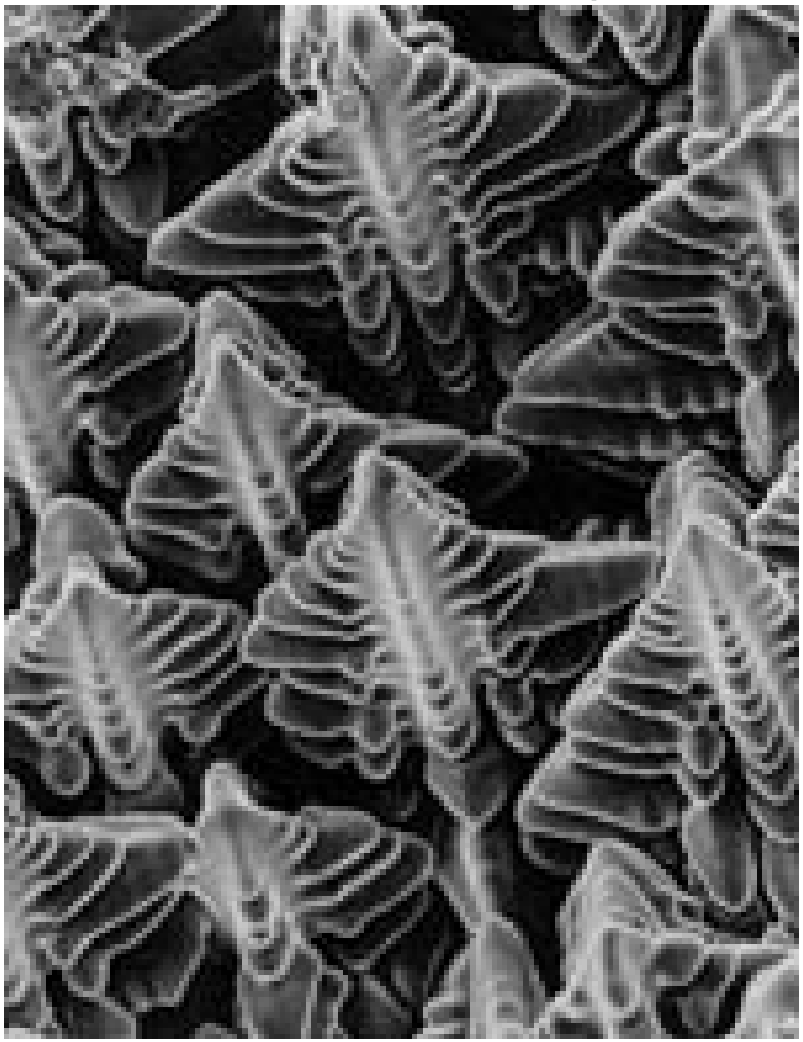
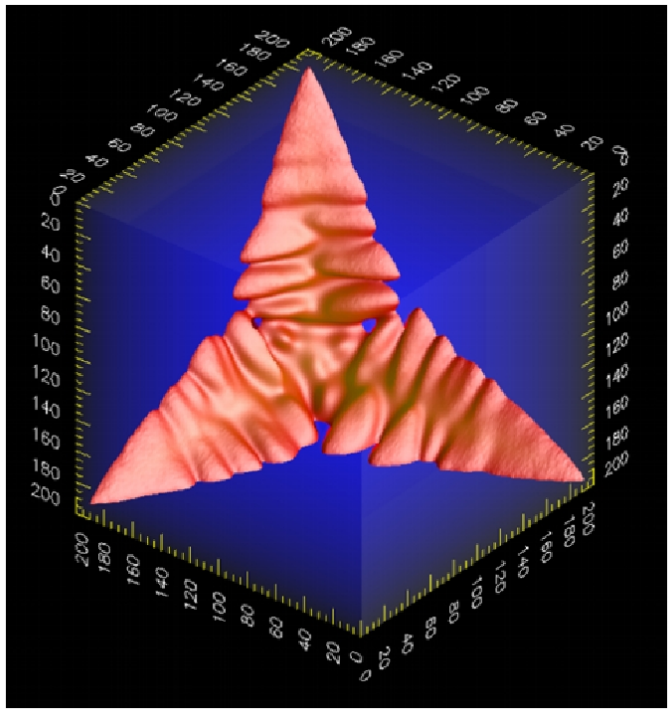
Motivation

Dendrites

A real-life dendrite lab picture:

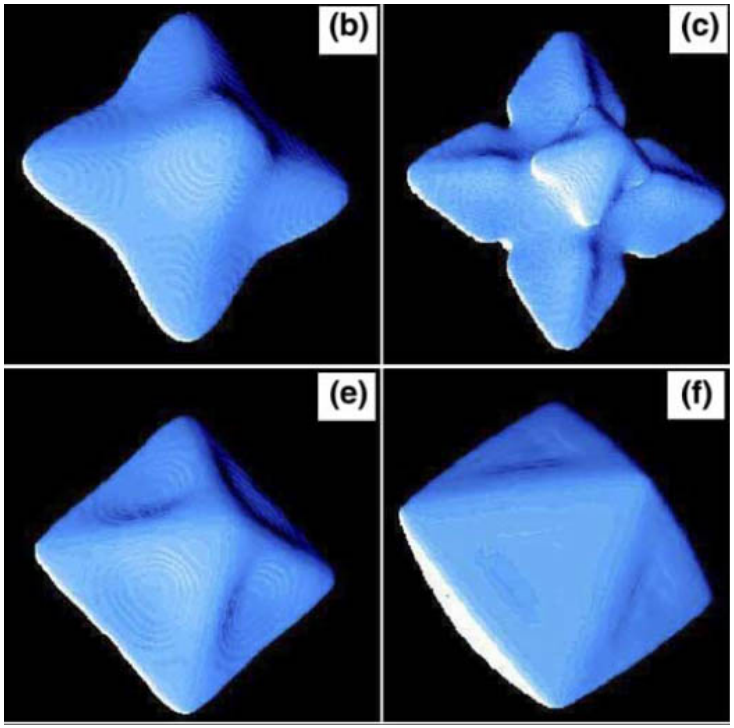


A phase-field simulation:

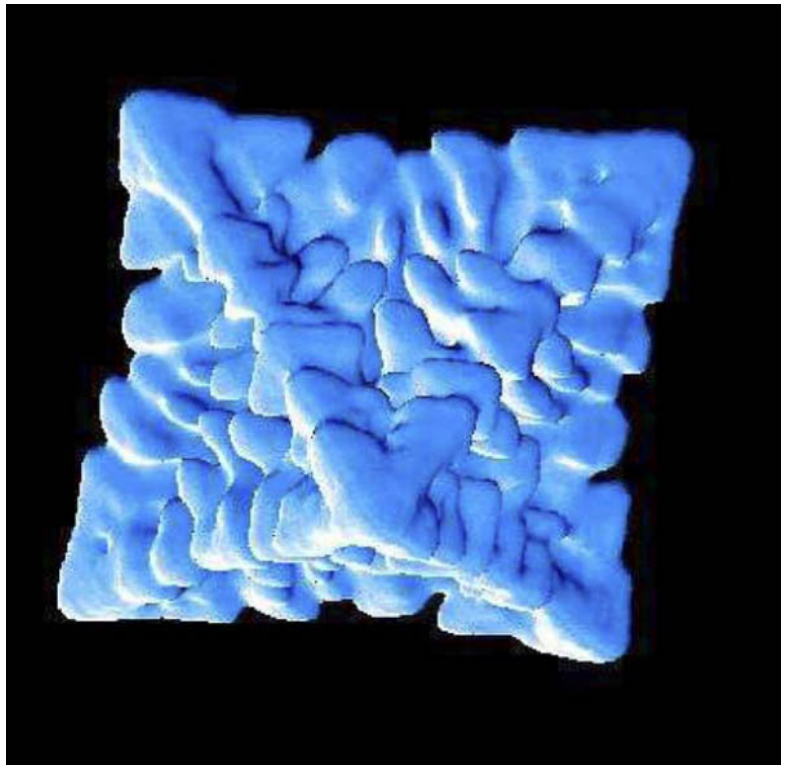


Dendrites

Computed dendrites without thermal noise [Nestler et al., 2005]



A computed dendrite with thermal noise [Nestler et al., 2005]



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Deterministic Phase Field

following [Chen, 1994]

Phase transition in solidification process

$$\alpha\epsilon\partial_t u - \epsilon\Delta u + (u^3 - u)/\epsilon = \sigma w \qquad \text{("interface" motion)}$$
$$c\partial_t w - \Delta w = -\partial_t u \qquad \text{(diffusion in bulk)},$$

where

u : order parameter/phase field $\begin{cases} \approx 1 & \text{solid phase,} \\ \in (-1+\delta_\epsilon, 1-\delta_\epsilon), & \text{interface (region)} \\ \approx -1 & \text{liquid phase.} \end{cases}$

w : temperature in liquid/solid bulk

The Stochastic (or Noisy) Allen-Cahn Problem

is the following semilinear parabolic stochastic PDE with additive white noise

$$\begin{aligned}
&\partial_t u(x,t) - \Delta u(x,t) + f_\epsilon(u(x,t)) = \epsilon^\gamma \partial_{xt} W(x,t), && x \in (0,1), t \in \mathbb{R}^+.\\
&u(x,0) = u_0(x), && x \in (0,1)\\
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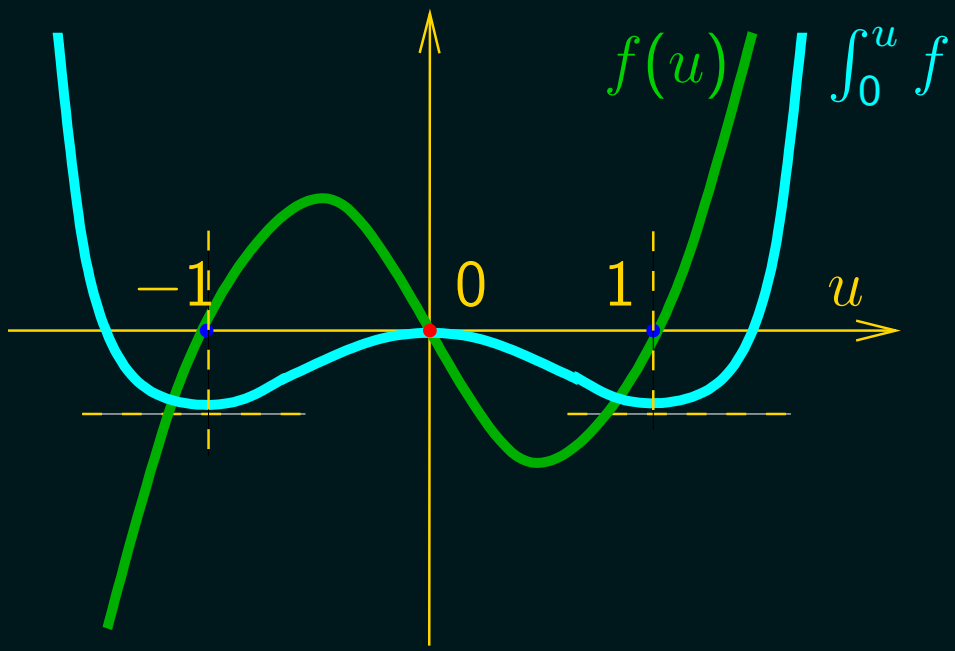
\end{aligned}$$



$f_\epsilon(\xi) = \frac{1}{\epsilon^2}\left(\xi^3 - \xi\right)$: first derivative of double well potential

$\epsilon \in \mathbb{R}^+$: "diffuse interface" parameter

$\gamma \in \mathbb{R}$: noise intensity parameter

$\partial_{xt} W$: space-time white noise

Why do we care?
about the (deterministic) Allen–Cahn

$$\partial_t u - \Delta u + \frac{1}{\epsilon^2}(u^3 - u) = 0$$

- Phase-separation models in metallurgy [Allen and Cahn, 1979].
- Simplest model of more complicated class [Cahn and Hilliard, 1958].
- Cubic nonlinearity approximation of "harder" (logarithmic) potential.
- Double obstacle can replace cubic by other.
- Phase-field models of phase separation and geometric motions [Rubinstein et al., 1989], [Evans et al., 1992], [Chen, 1994], [de Mottoni and Schatzman, 1995], . . . ;
- Metastability, exponentially slow motion in $d = 1$, [Carr and Pego, 1989], [Fusco and Hale, 1989], [Bronsard and Kohn, 1990];

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Why do we care?
about the stochastic Allen–Cahn

$$\partial_t u - \Delta u + \frac{1}{\epsilon^2}(u^3 - u) = \epsilon^\gamma \partial_{xt} W$$

- Noise = stabilizing/destabilizing mechanism [Brassesco et al., 1995], [Funaki, 1995].
- Stochastic 1d: Basic existence theory [Faris and Jona-Lasinio, 1982].
- Stochastic MCF of interfaces colored space-time/time-only noises possible [Souganidis and Yip, 2004], [Funaki, 1999], [Dirr et al., 2001].
- Rigorous mathematical setting to noise-induced dendrites.

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Why do we care?
Modeling and computation with stochastic Allen–Cahn

Stochastic Allen–Cahn (aka Ginzburg–Landau) with noise in materials science:

- Modeling: in phenomenological/lattice approximation, noise = unknown meso/micro-scopic fluctuation with known statistics effect in macroscopic scale [Halperin and Hoffman, 1977], [Katsoulakis and Szepessy, 2006, cf.];
- Simulation: noise as ad-hoc nucleation/instability inducing mechanism [Warren and Boettinger, 1995, Nestler et al., 2005, e.g.,];

Numerics for $d = 1$ SDE approach [Shardlow, 2000], spectral methods [Liu, 2003], interface nucleation/annihilation [Lythe, 1998, Fatkullin and Vanden-Eijnden, 2004].

Numerics for $d \geq 2$?

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Deterministic Allen–Cahn in 1 dimension

metastability and profiles

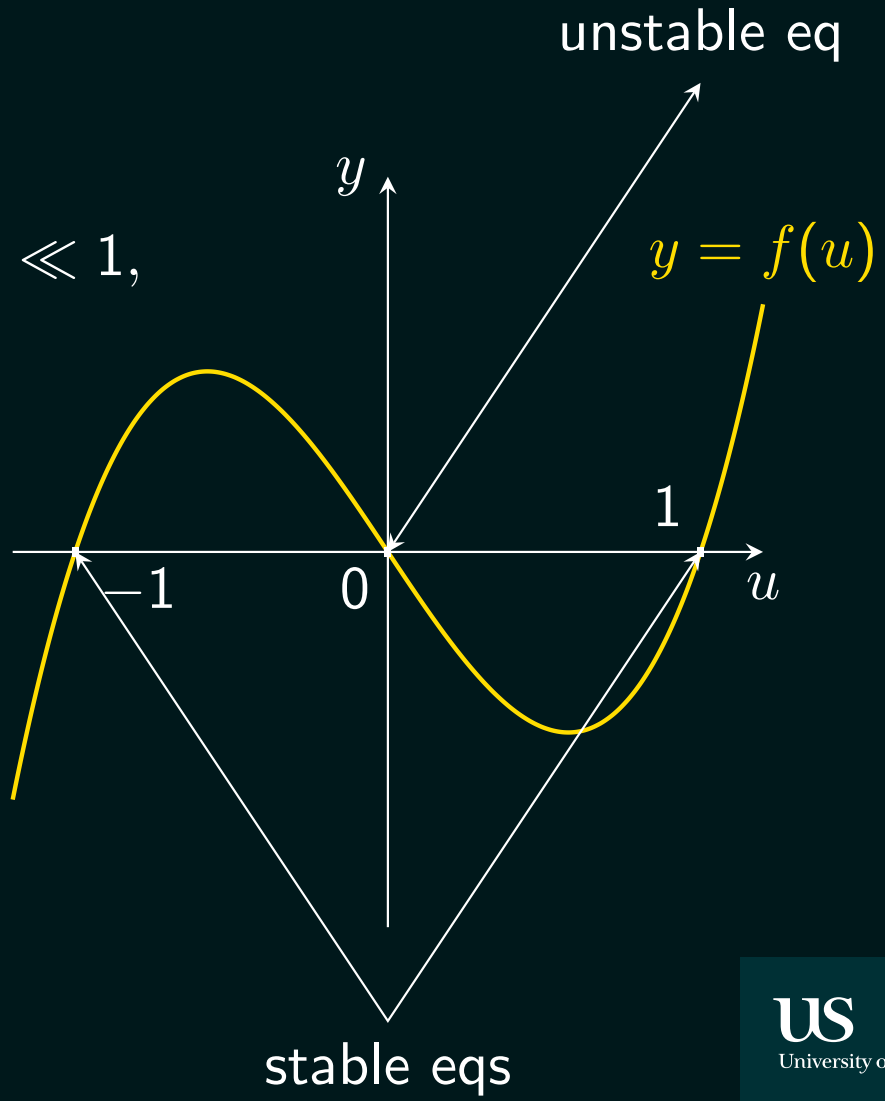
PDE: $\partial_t u - \Delta u + \frac{1}{\epsilon^2} f(u) = 0, \quad 0 < \epsilon \ll 1,$

potential: $\frac{1}{\epsilon^2} f(\xi) = \frac{1}{\epsilon^2} (\xi^3 - \xi)\,,$

ODE (no diffusion): $\dot{u} = -\frac{1}{\epsilon^2} f(u)$

equilibriums: $f(\pm 1) = 0, f(0) = 0,$

linearize: $\begin{array}{l} \pm 1 \text{ stable } - f'(\pm 1) < 0, \\ 0 \text{ unstable } - f'(0) > 0. \end{array}$



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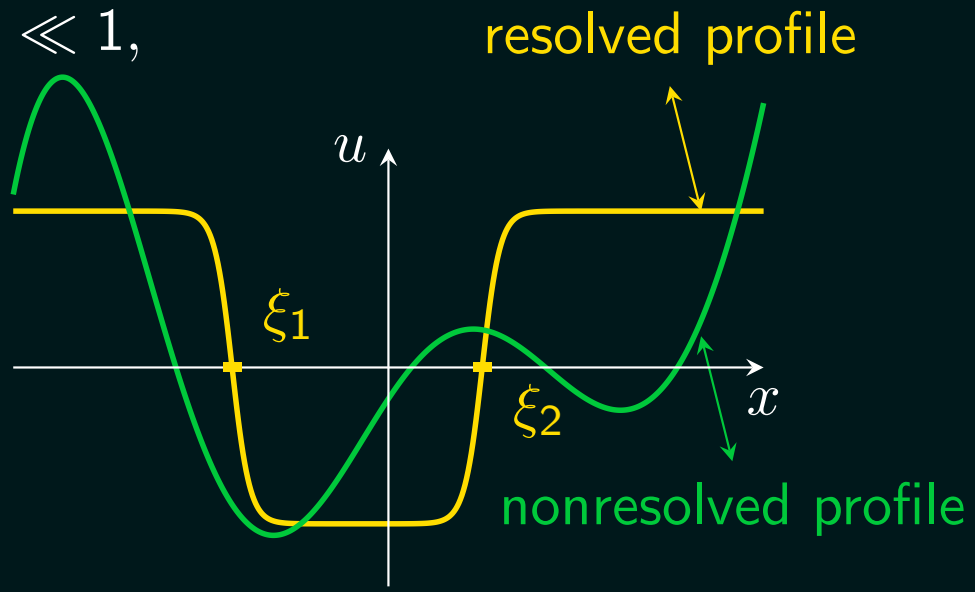

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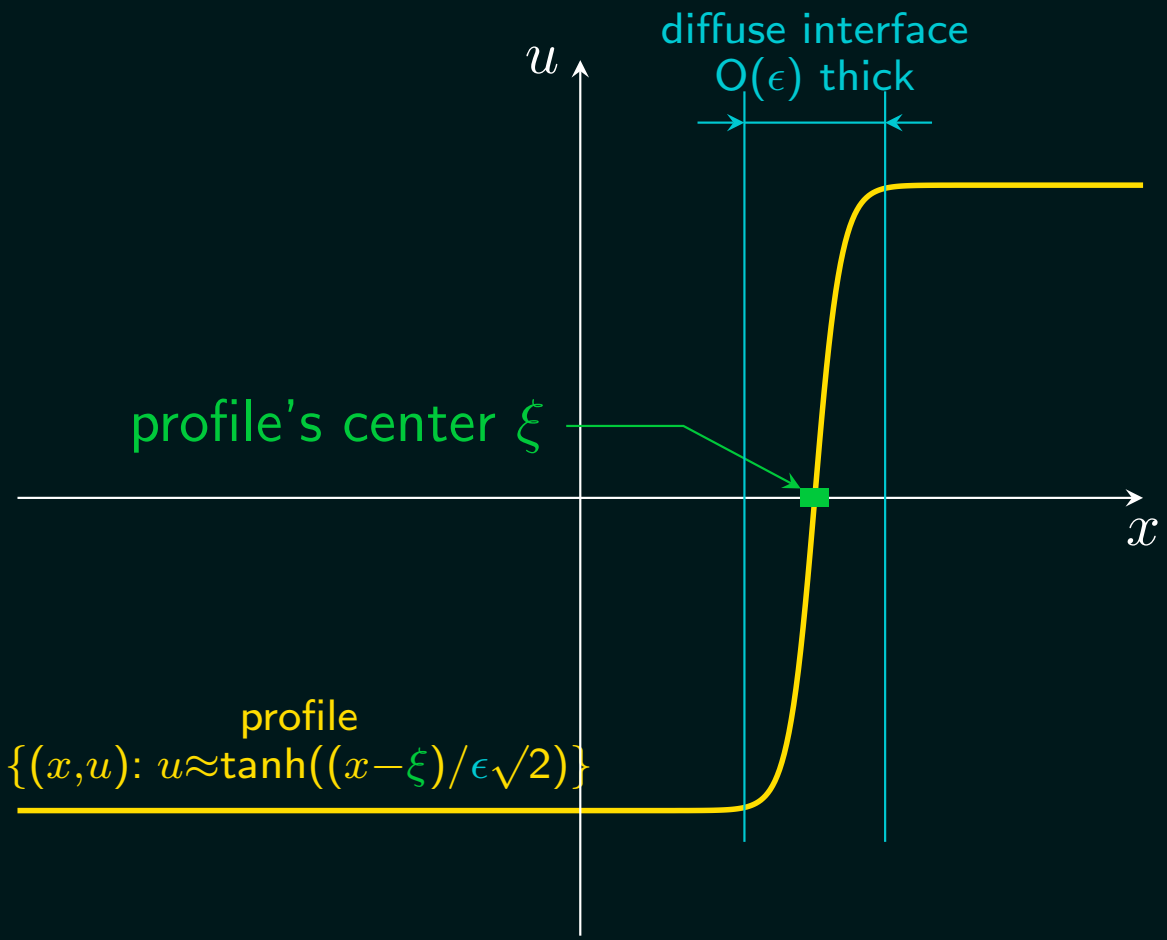
Deterministic Allen–Cahn in 1 dimension

resolved profiles terminology



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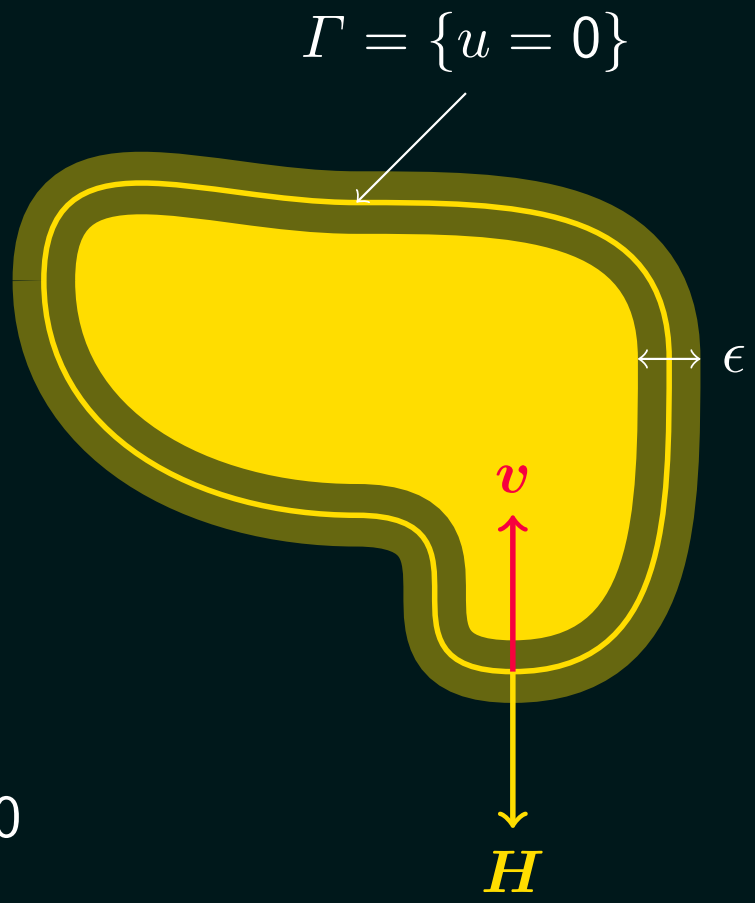
Deterministic Allen–Cahn in dimension $d > 1$

close relation to Mean Curvature Flow (MCF)



Level set $\Gamma = \{u = 0\}$, moves, as $\epsilon \to 0$
a mean curvature flow:

$$\underset{\downarrow}{\epsilon\, \partial_t u|_\Gamma} \quad \underset{\downarrow}{-\epsilon \Delta\, u|_\Gamma} \quad \underset{\downarrow}{+\tfrac{1}{\epsilon}\,(u^3 - u)\big|_\Gamma} = 0$$

$$\underset{\text{normal velocity}}{-v} \quad \underset{\text{curvature}}{-H} \quad +0 = 0$$

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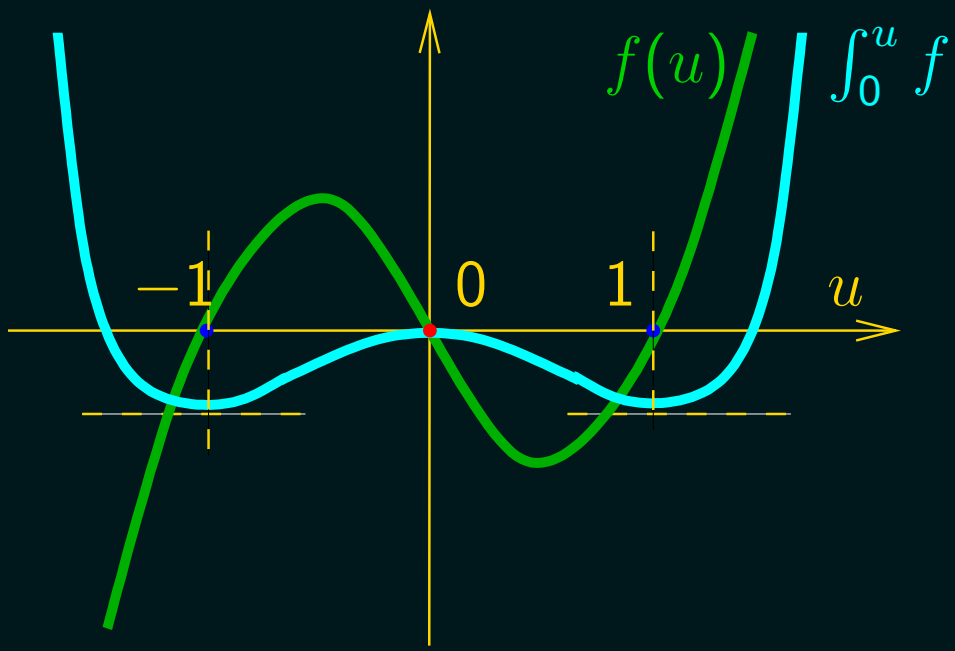

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$\partial_{xt} W$: space-time white noise

White noise $\partial_{xt}W$: an informal definition

- Informally: white noise is mixed derivative of (2 dimensional) Brownian sheet $W = W_{x,t}$.
- Brownian sheet W: extension of 1-dim Brownian motion to multi-dim, built on Ω.
- Solutions of PDE understood as mild solution
- Notation for $\partial_{xt}W$

$$\int_0^\infty \int_0^1 f(x,t)\,\mathrm{d}W(x,t) := \int_0^\infty \int_0^1 f(x,t)\partial_{xt}W(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

stochastic integral [Walsh, 1986].

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White noise 101

- $\forall A \in \text{Borel}(\mathbb{R}^2) \int_A \mathrm{d}W(x,t) =: W(A) \in \mathsf{N}\left(0, |A|\right)$, I.e, $W(A)$ is a 0-mean Gaussian random variable with variance $= |A|$.
- $A \cap B = \emptyset \Rightarrow$ $W(A)$,$W(B)$ independent and $W(A \cup B) = W(A) + W(B)$.
- Brownian sheet: $W_{x,t} = W([0,t] \times [0,x]) = \int_0^t \int_0^x \mathrm{d}W(x,t)$.
- Basic yet crucial property:
  $\mathsf{E}\left[\left(\int_I \int_D f(x,t)\, \mathrm{d}W(x,t)\right)^2\right] = \mathsf{E}\left[\int_I \int_D f(x,t)^2\, \mathrm{d}x\, \mathrm{d}t\right]$, $\forall f \in \mathsf{L}_2$.
  (Aka $\mathrm{d}W = \sqrt{\mathrm{d}x\, \mathrm{d}t}$.)

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Solutions of the stochastic linear heat equation

$$\partial_t w - \Delta w = \partial_{xt} W, \text{ in } D \times \mathbb{R}_0^+$$
$$w(0) = 0, \text{ on } D$$
$$\partial_x w(1,t) = \partial_x w(0,t) = 0, \ \forall t \in [0,\infty).$$

Solution is defined as Gaussian process produced by the stochastic integral

$$Z_t(x) = Z(x,t) := \int_0^t \int_D G_{t-s}(x,y)\, \mathrm{d}W(x,y).$$

where G is the heat kernel:

$$G_t(x,y) = 2\sum_{k=0}^{\infty} \cos(\pi k x)\cos(\pi k y)\exp(-\pi^2 k^2 t).$$

(An "explicit semigroup" approach.)

Solutions of the stochastic Allen–Cahn problem

$$\partial_t u(x,t) - \Delta u(x,t) + f_\epsilon(u(x,t)) = \epsilon^\gamma \partial_{xt} W(x,t), \text{ for } x \in D = [0,1], t \in [0,$$
$$u(x,0) = u_0(x), \ \forall x \in D$$
$$\partial_x u(0,t) = \partial_x u(1,t) = 0, \ \forall t \in \mathbb{R}^+.$$

Defined as the continuous solution of integral equation

$$u(x,t) = -\int_0^t \int_D G_{t-s}(x,y) f_\epsilon(u(y,s)) \, \mathrm{d}y \, \mathrm{d}s + \int_D G_t(x,y) u_0(y) \, \mathrm{d}y + \epsilon^\gamma Z_t(x).$$

Unique continuous integral solution exists in 1d provided the initial condition u_0 fulfills boundary conditions.

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Remarks on solution process

- The solution u of Stochastic Allen-Cahn, is adapted continuous Gaussian process (it is Hölder of exponents 1/2,1/4).
- The process u has continuous, but nowhere differentiable sample paths.
- This approach works only in 1d+time, for higher dimensions; when defined solutions are extremely singular distributions (let alone continuous).
- If white noise (uncorrelated) is replaced by colored noise (correlated), then "regular" u can be sought in higher dimensions.
- Direct numerical discretization of this problem not obvious.

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Discretization strategy

In two main steps:

1. Replace white noise $\partial_{xt} W$ by a smoother object: the approximate white noise $\partial_{xt} \bar{W}$.
2. Discretize the approximate problem with $\partial_{xt} \bar{W}$ via a finite element scheme for the Allen-Cahn equation.

Inspired by [Allen et al., 1998], [Yan, 2005].

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Approximate White Noise (AWN)

Fix a final time $T > 0$ and consider uniform space and time partitions

in space: $D = [0,1]$, $D_m := (x_{m-1}, x_m)$, $x_m - x_{m-1} = \sigma$, $m \in [1:M]$,

in time: $I = [0,T]$, $I_n := [t_{n-1}, t_n)$, $t_n - t_{n-1} = \rho$, $n \in [1:N]$.

Regularization of the white noise is projection on piecewise constants:

$$\bar{\eta}_{m,n} := \frac{1}{\sigma\rho} \int_I \int_D \chi_m(x)\psi_n(t)\, \mathrm{d}W(x,t).$$

$$\partial_{xt}\bar{W}(x,t) := \sum_{n=1}^{N}\sum_{m=1}^{N} \bar{\eta}_{m,n}\chi_m(x)\varphi_n(t).$$

$\chi_m = \mathbb{1}_{D_m}$, $\varphi_n = \mathbb{1}_{I_n}$ (characteristic functions).

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Regularity of AWN

$$\partial_{xt}\bar{W}(x,t) := \sum_{n=1}^{N}\sum_{m=1}^{N} \bar{\eta}_{m,n}\chi_m(x)\varphi_n(t).$$

$\bar{\eta}_{m,n}$ are independent $\mathsf{N}(0, 1/(\sigma\rho))$ variables and

$$\mathsf{E}\left[\left(\int_I\int_D f(x,t)\,\mathrm{d}\bar{W}(x,t)\right)^2\right] \leq \mathsf{E}\left[\left(\int_I\int_D f(x,t)\,\mathrm{d}W(x,t)\right)^2\right].$$

$$\mathsf{E}\left[\left(\int_I\int_D \mathrm{d}\bar{W}(x,t)\right)^2\right] = T$$

$$\mathsf{E}\left[\int_I\int_D \left|\partial_{xt}\bar{W}(x,t)\right|^2 \,\mathrm{d}x\,\mathrm{d}t\right] \leq \frac{1}{\sigma\rho}$$

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The regularized problem

$$\partial_t \bar{u} - \Delta \bar{u} + f_\epsilon(\bar{u}) = \epsilon^\gamma \partial_{xt} \bar{W},$$
$$\partial_x \bar{u}(t,0) = \partial_x \bar{u}(t,1) = 0,$$
$$\bar{u}(0) = u_0,$$

Admits a "classical" solution. $\forall \omega \in \Omega \Rightarrow$ corresponding realization of the AWN $\partial_{xt}\bar{W}(\omega) \in \mathsf{L}_\infty([0,T] \times D)$ (parabolic regularity) $\Rightarrow$
$\partial_t \bar{u}(\omega) \in \mathsf{L}_2([0,T]; \mathsf{L}_2(D))$.
$\Rightarrow$ variational formulation and thus FEM (or other standard methods) now possible.
Error estimate schedule:

1. Compare $\bar{u}$ with u.
2. Approximate $\bar{u}$ by U in a finite element space and compare.

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The Finite Element Method

- Parameters (numerical mesh) $\tau, h > 0$ (approximation mesh) ρ, σ.
- Not necessary, but "natural" coupling: $\tau = \rho = h^2 = \sigma^2$.
- Linearized Semi-Implicit Euler

$$\left\langle \frac{U^n - U^{n-1}}{\tau}, \Phi \right\rangle + \langle \partial_x U^n, \partial_x \Phi \rangle + \left\langle f'_\epsilon(U^{n-1}) U^n, \Phi \right\rangle$$
$$= \left\langle f'_\epsilon(U^{n-1}) U^{n-1} + f_\epsilon(U^{n-1}), \Phi \right\rangle + \left\langle \epsilon^\gamma \partial_{xt} \bar{W}, \Phi \right\rangle, \quad \forall\, \Phi \in \mathbb{V}^h.$$

$\mathbb{V}^h$ is a finite element space of piecewise linear [Kessler et al., 2004, Feng and Wu, 2005].

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Implementation

$$\left[\frac{1}{\rho}\boldsymbol{M}+\boldsymbol{A}+\frac{1}{\epsilon^2}\boldsymbol{N}(\boldsymbol{u}^{n-1})\right]\boldsymbol{u}^n=\frac{1}{\epsilon^2}\boldsymbol{g}(\boldsymbol{u}^{n-1})+\frac{1}{\rho}\boldsymbol{M}\boldsymbol{u}^{n-1}+\epsilon^{\gamma}\boldsymbol{w}.$$

- $\boldsymbol{M}$, $\boldsymbol{A}$ usual mass and stiffness matrices,
- $\boldsymbol{N}$ and $\boldsymbol{g}$ nonlinear mass matrix and load vector
- $\boldsymbol{w}=(w_i)$ a random load vector generated at each time-step: for each node i we have

$$w_i=\frac{\epsilon^{\gamma}}{2}\sqrt{\frac{h}{\rho}}(\eta_{i-1}+\eta_i)\quad \eta_i\in \mathrm{N}(0,1)\,\text{pseudo-random.}$$

- Simple Monte Carlo method on $\boldsymbol{w}$.

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Convergence of regularized to stochastic solution

Theorem (quasi-strong (least squares) convergence)

Let $\gamma > -1/2, T > 0$. For some c_1, c_2 ($\not\mapsto \epsilon$), and for each $\epsilon \in (0,1)$ there correspond (i) an event Ω^∞, (ii) constants C_ϵ, C_1, C_2 s.t.

$$P(\Omega^\infty) \geq 1 - 2c_1 \exp(-c_2/\epsilon^{1+2\gamma})$$

$$\int_{\Omega^\infty} \int_0^r \int_D |\bar{u} - u|^2 \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}P \leq C_\epsilon \left(C_1 \rho^{1/2} + C_2 \frac{\sigma^2}{\rho^{1/2}} \right), \ \forall \sigma, \rho > 0.$$

Remark

1. Event $\Omega^\infty = \{|u, \bar{u}| < 3\}$ (measured by maximum principle & exponential decay of "chi-squared" distribution).
2. C_ϵ grows exponentially with $1/\epsilon$.
3. Constant improves for $\gamma \gg 1$ (exploiting spectral gap).

Maximum principle in probability sense

Lemma

Suppose $\gamma > -1/2$. *Given* T, *there exist* c_1, $c_2, \delta_0 > 0$ *such that if* $\|u_0\|_{\mathsf{L}_\infty(D)} \leq 1 + \delta_0$ *then*

$$P\left\{ \sup_{t\in[0,T]} \|(u,\bar{u})(t)\|_{\mathsf{L}_\infty(D)} > 3 \right\} \leq c_1 \exp(-c_2/\epsilon^{1+2\gamma}).$$

Remark

1. The constant 3 is for convenience, can be replaced by $1 + \delta_1$, if needed, by adjusting c_1, c_2 and δ_0.
2. This result is used to determine the event Ω^∞ for convergence to hold.
3. Because, nonlinearity is *not* globally Lipschitz.

Small noise resolution (for γ big: weak intensity)

Theorem (small noise)

Let q solution of deterministic the Allen–Cahn problem

$$\partial_t q - \Delta q + f_\epsilon(q) = 0, \quad q(0) = u_0, \text{ on } D.$$

Then $\forall\, T > 0 : \exists\, K_1(T) > 0, \epsilon_0(T) > 0$

$$P\left(\sup_{[0,T]} \|\bar{u} - q\|_{\mathsf{L}_2(D)} \leq \epsilon^3\right)$$

$$\geq 1 - \left(1 + \frac{K_1(T)}{2}\epsilon^{6-2\gamma}\right)^{T/(\sigma\rho)-1} \exp\left(-\frac{K_1(T)}{2}\epsilon^{6-2\gamma}\right)$$

for all $\epsilon \in (0, \epsilon_0)$, $\gamma > 3$ *and* $\rho, \sigma > 0$.

Spectrum estimate

Theorem ([Chen, 1994], [de Mottoni and Schatzman, 1995])

Let q be the (classical) solution of the problem

$$\partial_t q - \Delta q + f_\epsilon(q) = 0, \quad q(0) = u_0, \text{ on } D$$

Key to argument in convergence for weak noise is the use of spectral estimate for q: There exists a constant $\lambda_0 > 0$ independent of ϵ such that for any $\epsilon \in (0, 1]$ we have

$$\|\partial_x \phi\|^2_{\mathsf{L}_2(D)} + \left\langle f'_\epsilon(q)\phi, \phi \right\rangle \geq -\lambda_0 \|\phi\|^2_{\mathsf{L}_2(D)}, \ \forall \phi \, \mathsf{H}^1(D).$$

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$X(0,T;\mathsf{L}_2(D))$ bounds for AWN

Lemma

$\forall K > 0$

$$P\left\{\sup_{t\in[0,T]}\left\|\partial_{xt}\bar{W}(t)\right\|_{\mathsf{L}_2(D)} \leq K\right\}$$

$$\geq \left[1-\frac{T}{\rho}\left(1+\frac{K^2}{2}\rho\right)^{1/(2\sigma)-1}\exp\left(-\frac{K^2}{2}\rho\right)\right]^+$$

and $P\left\{\int_0^T\left\|\partial_{xt}\bar{W}(t)\right\|^2_{\mathsf{L}_2(D)} \leq K^2\right\}$

$$\geq 1-\left(1+\frac{K^2}{2}\right)^{1/(2\rho\sigma)-1}\exp\left(-\frac{K^2}{2}\right)$$

Convergence of FEM

Taking $\tau = \rho = h^2 = \sigma^2$ to simplify, we have

$$\int_{\Omega^\infty} \int_D |\bar{u}(t_n) - U^n|^2 \, \mathrm{d}x \, \mathrm{d}P \leq C(T, \epsilon)h.$$

Remark

1. compare w.r.t. deterministic case,
2. impossible to take E, but only $\int_{\Omega^\infty}$, due to rare events.

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Regularity of $\bar{u}$

Theorem (Regularity of $\bar{u}$)

$$\int_{\Omega^\infty}\int_{I_m}\int_D |\partial_t \bar{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq c_1\rho^{-1/2} + c_2\sigma^{-1}$$

$$\int_{\Omega^\infty}\int_D |\Delta \bar{u}|^2 \,\mathrm{d}x \leq c_3\rho^{-3/2} + c_4\sigma^{-1}\rho^{-1}.$$

For Ω^∞ an event of high probability.

Remark

Expected loss of regularity, as $\sigma, \rho \to 0$.

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Numerical experiments

1. Computations for $\epsilon = 0.04, 0.02, 0.01$ and $\gamma = 1.0, 0.5, 0.2, 0.0, -0.2$ ($\gamma < 0$ though not discussed in theory can be interesting in practice).
2. Run a Monte-Carlo simulation for each set of parameters (2500 samples).
3. Meshsize $h < \epsilon$.
4. Timestep $\tau = c_0 h^2 \approx \epsilon^2$.

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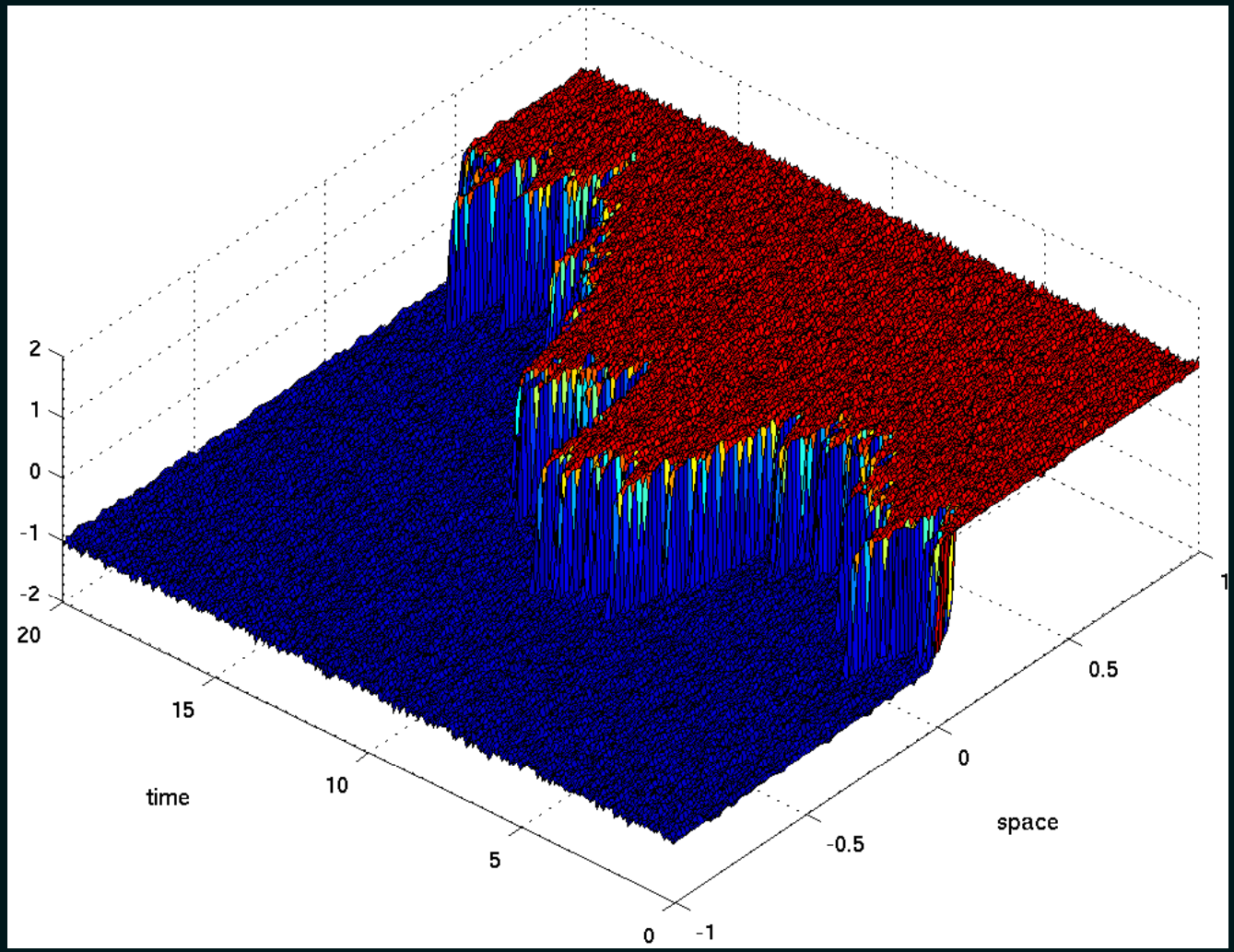
Sample path with $\epsilon = 0.01$, $\gamma = 0$



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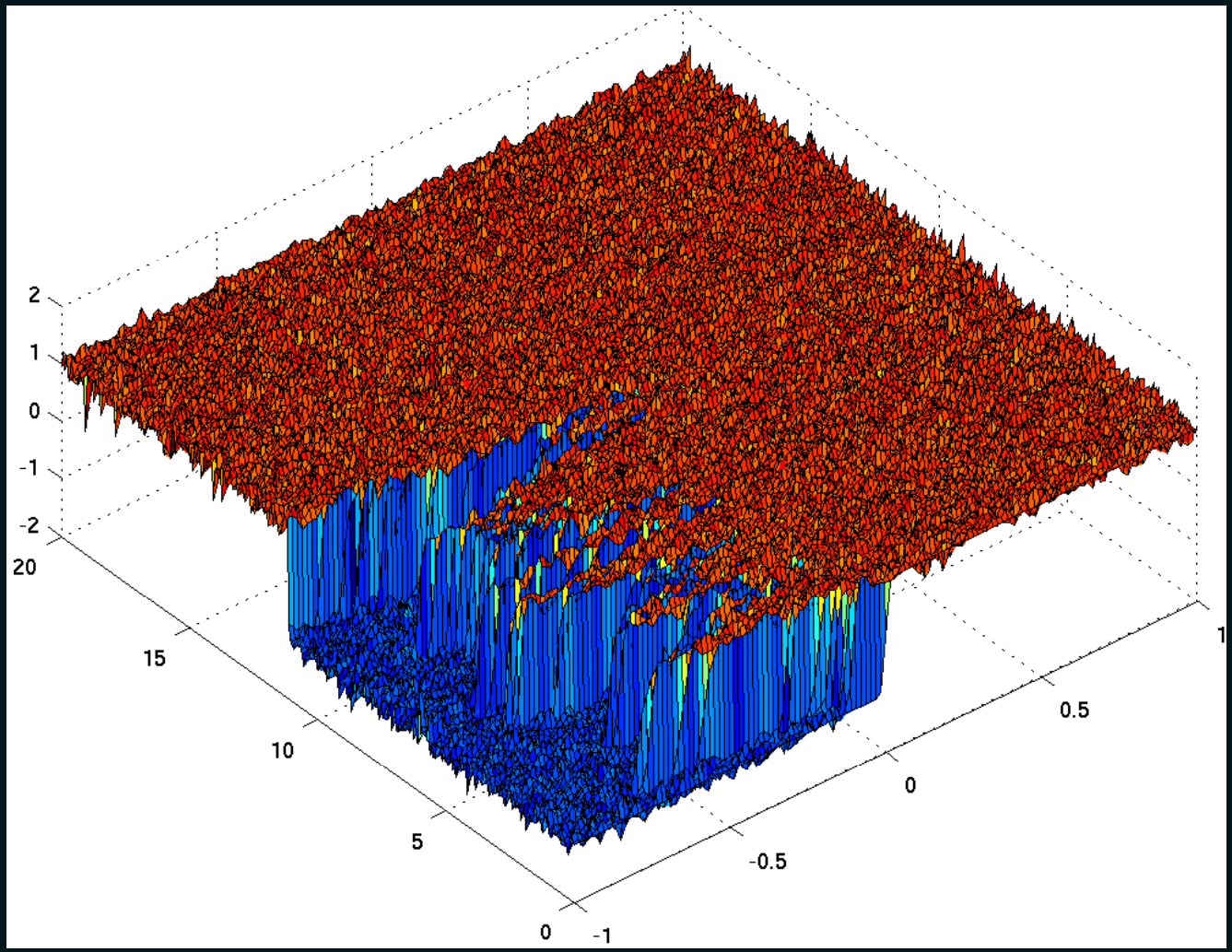
Sample path with $\epsilon = 0.01$, $\gamma = -0.2$



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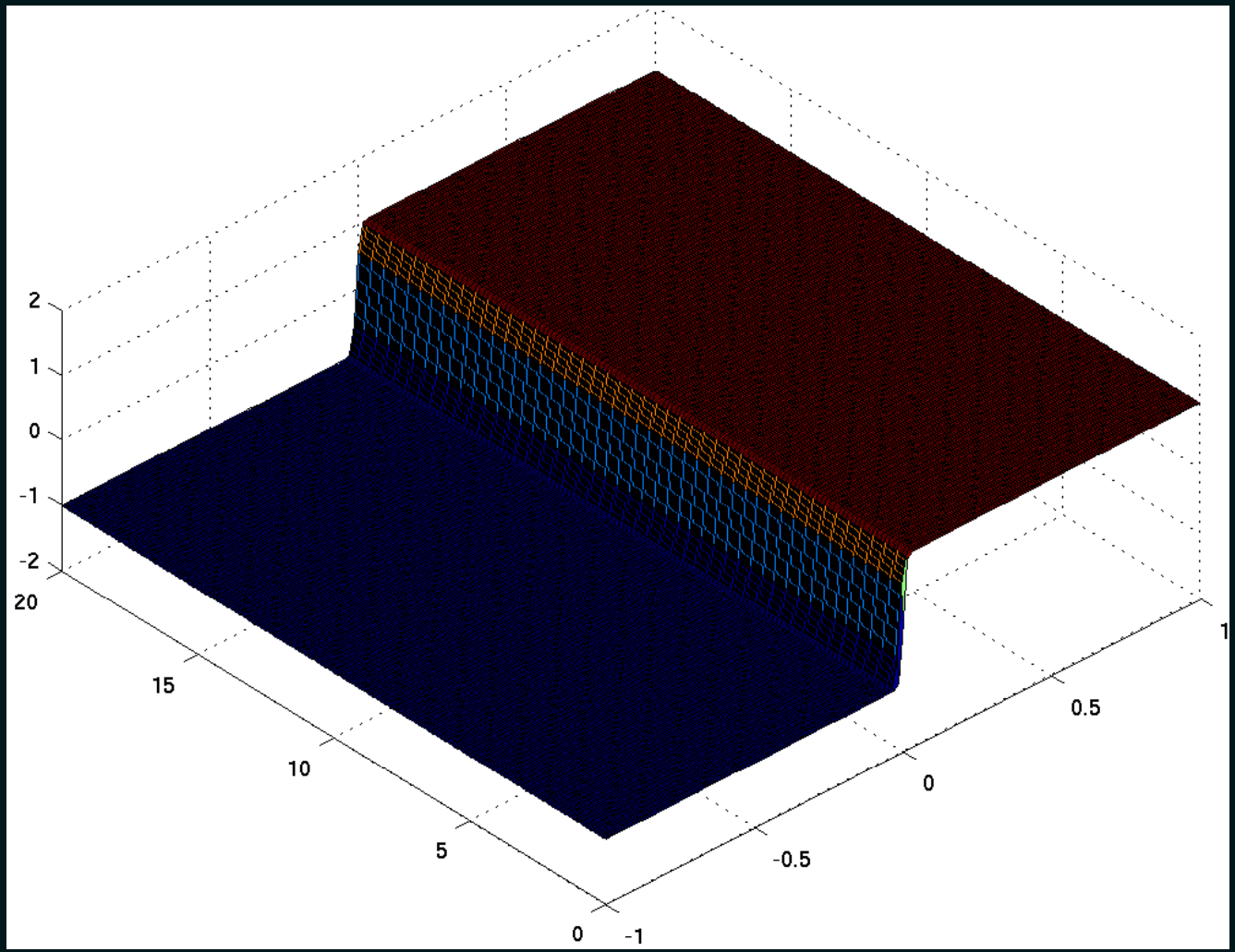
Sample path with $\epsilon = 0.01$, $\gamma = 1.0$



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How to the benchmark our computations?

Benchmarking = "comparing with known solutions" Very few known "exact result".

Theorem ([Funaki, 1995, Brassesco et al., 1995])

The interface motion is asymptotically, as $\epsilon \longrightarrow 0$, a (1d) Brownian motion,

$$\lim_{\epsilon \longrightarrow 0} P \left\{ \sup_{t \leq \epsilon^{-1-2\gamma} T} \left\| \tilde{u}_t - q(x - \tilde{\xi}_t^\epsilon) \right\|_{\mathsf{L}_2(D)} > \delta \right\} = 0,$$

for $\delta > 0$.

Remark

- q is an instanton for the deterministic Allen-Cahn equation
- $\tilde{\xi}_t^\epsilon$ solves a SODE representing the motion of the interface
- $\tilde{u}_t(x)$ is an appropriate rescaling of $u(x,t)$.

- In our scaling, the interface position ξ_t performs a process that looks asymptotically (as $\epsilon \longrightarrow 0$) like

$$\xi_t^\epsilon \approx \frac{\sqrt{3}}{2\sqrt{2}} \epsilon^{\gamma+1/2} B_t,$$

where B_t is the 1d Brownian Motion.
We use statistics on the interface (discarding all cases of nucleation/coalescence).

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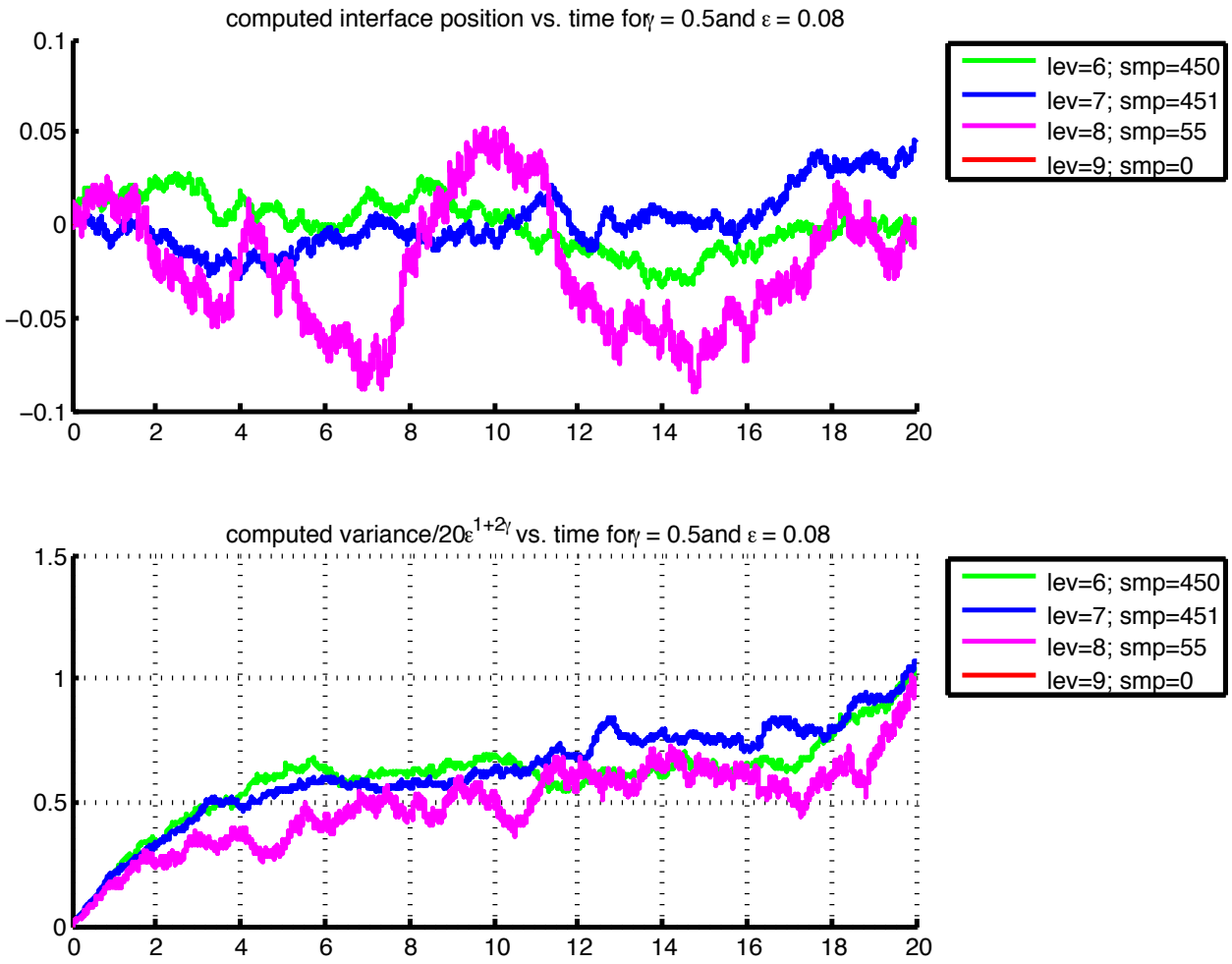
Numerically computed average and variance

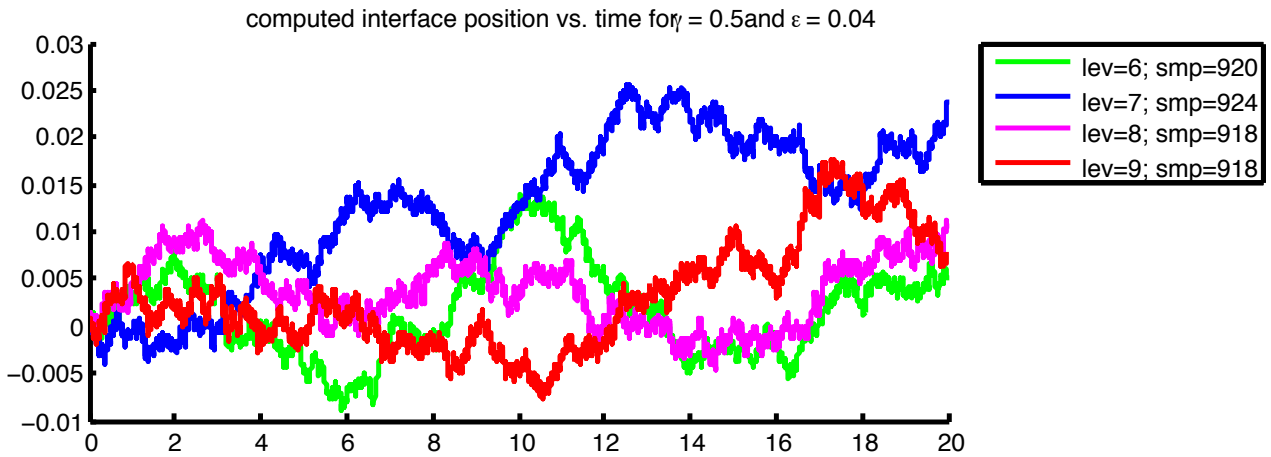
$\gamma = 0.5$ and $\epsilon = 0.08$

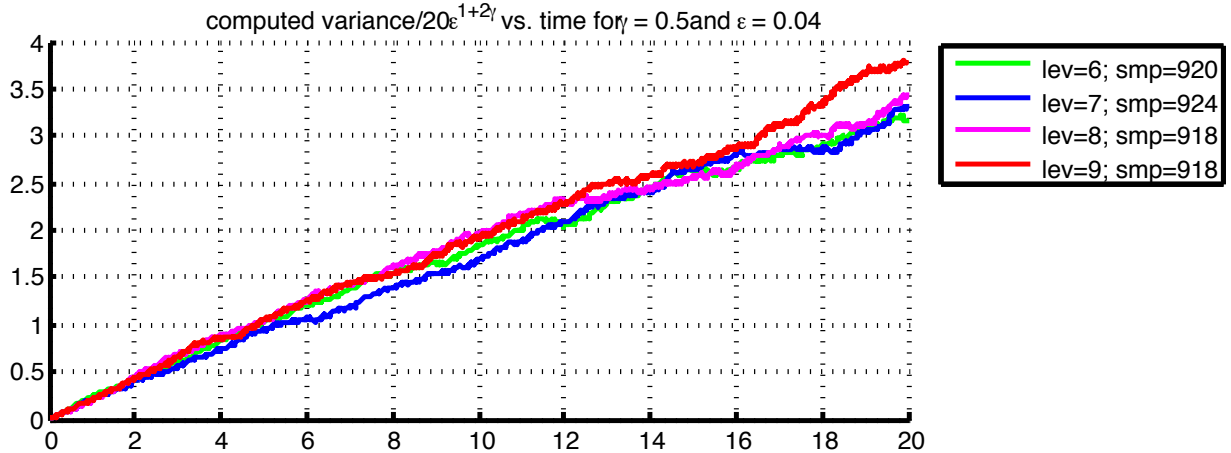


Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$. **Sample number** smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs. **Variance** against time for the various levels. As time grows, variance accuracy deteriorates.

Numerically computed average and variance

$\gamma = 0.5$ and $\epsilon = 0.04$

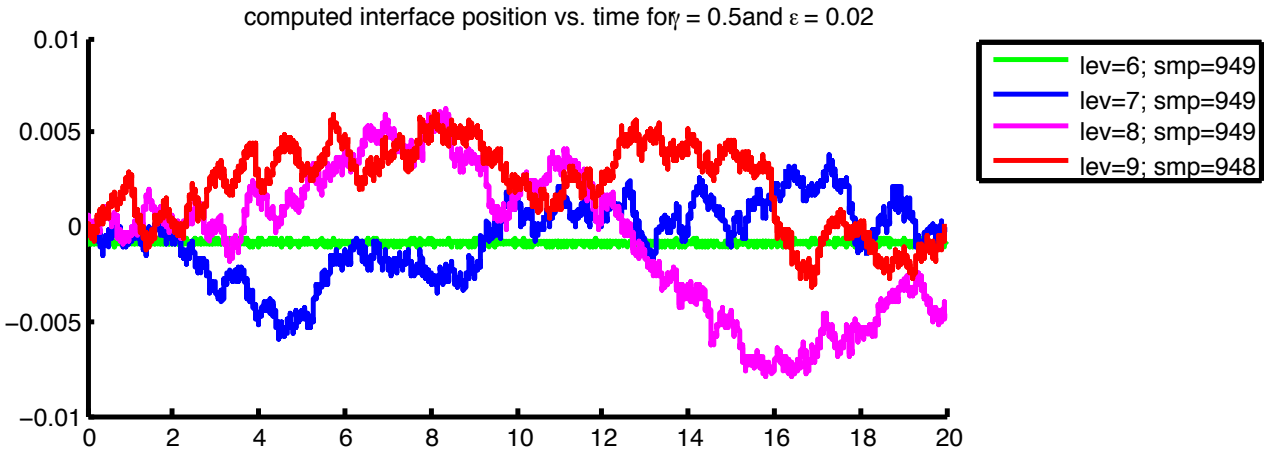


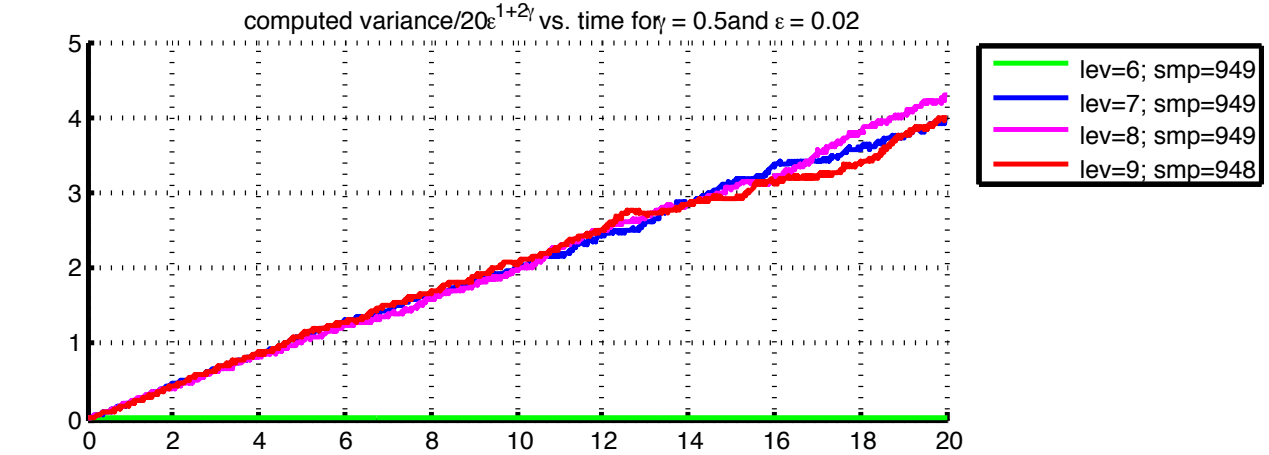


Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$. **Sample number** smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs. **Variance** against time for the various levels. As time grows, variance accuracy deteriorates.

Numerically computed average and variance

$\gamma = 0.5$ and $\epsilon = 0.02$

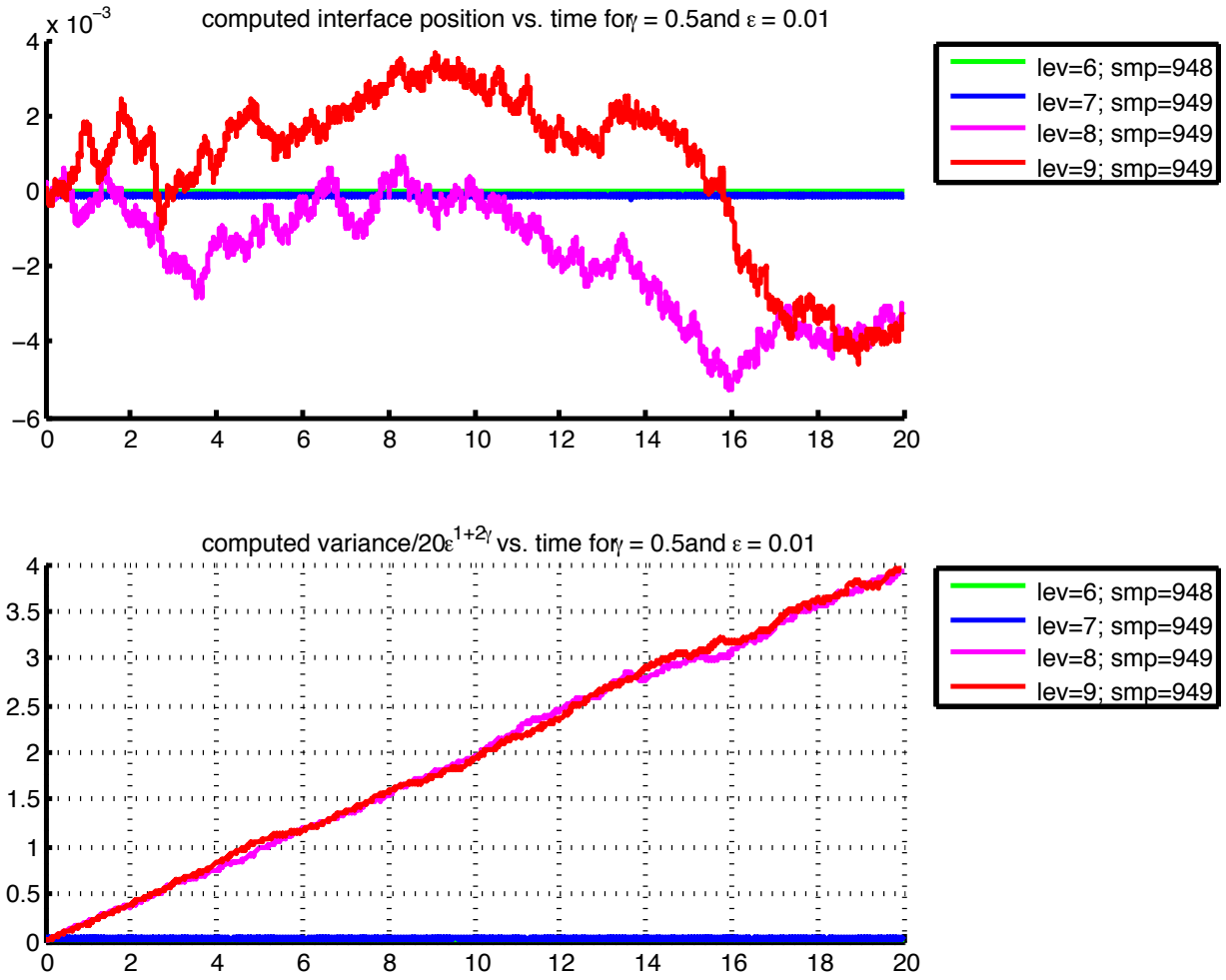




Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$. **Sample number** smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs. **Variance** against time for the various levels. As time grows, variance accuracy deteriorates.

Numerically computed average and variance

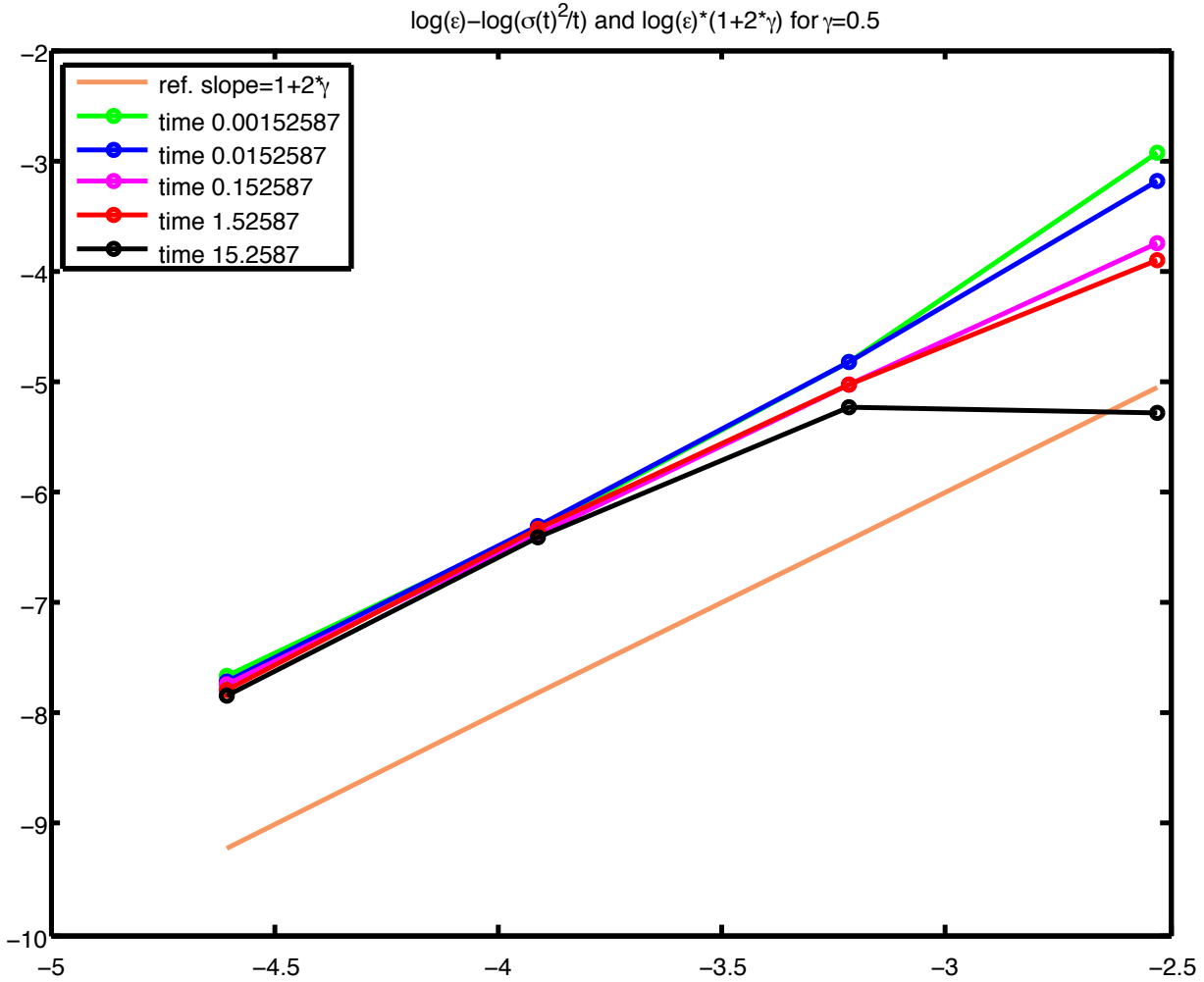
$\gamma = 0.5$ and $\epsilon = 0.01$



Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$. **Sample number** smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs. **Variance** against time for the various levels. As time grows, variance accuracy deteriorates.

Simulations vs. theory

$\log(\mathrm{var}[U_h^i]\,/t_i)$ vs. $\log\epsilon$ plot for $\gamma = 0.5$ at various times $t_i = t_0\,10^i$



Theoretical slope (predicted by [Funaki, 1995] and [Brassesco et al., 1995]) is $1 + 2\gamma$.

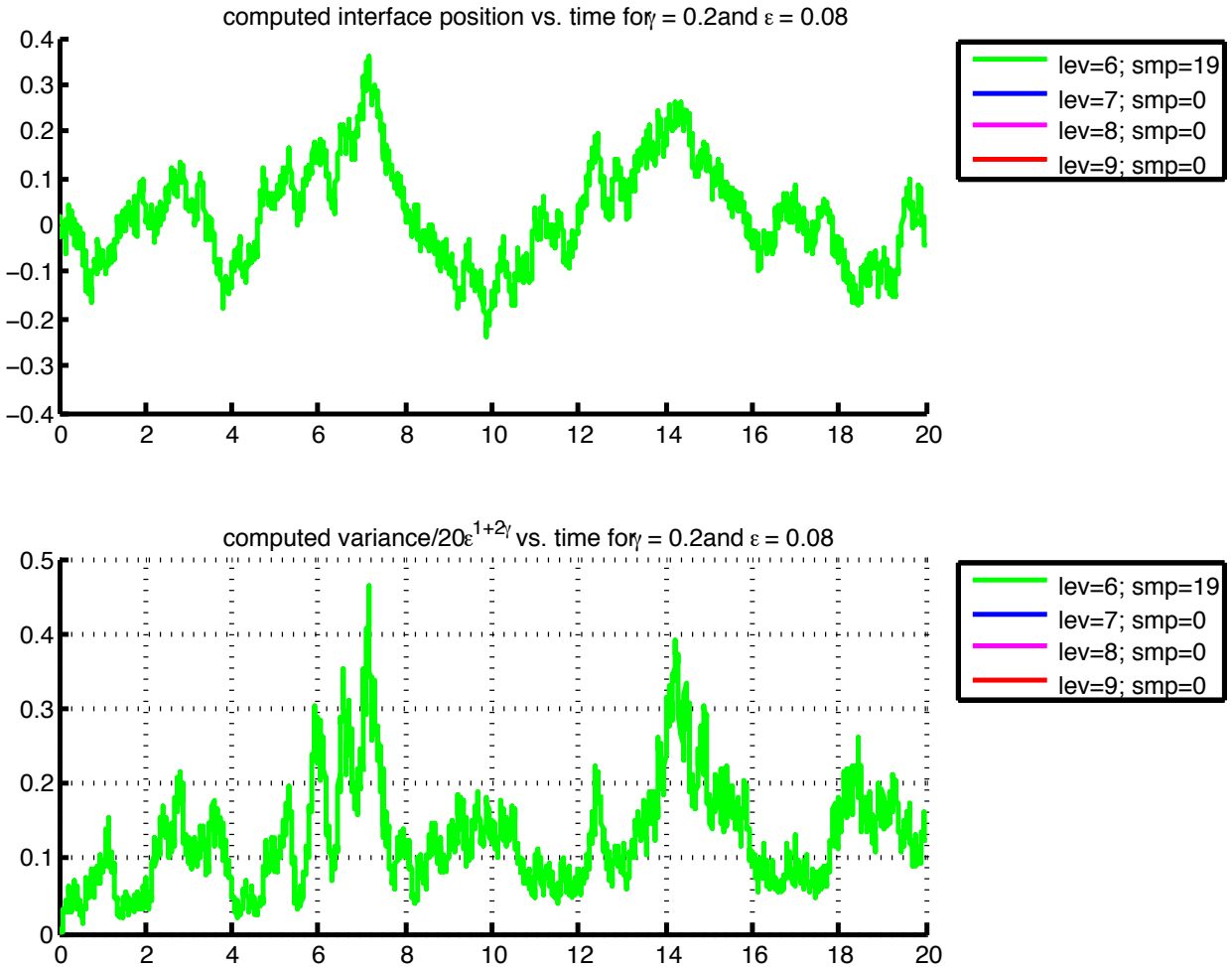
Plotted in **orange** reference line

As $\epsilon \to 0$ slope improves.

Meshsize $h = 1/512$.

Numerically computed average and variance

$\gamma = 0.2$ and $\epsilon = 0.08$



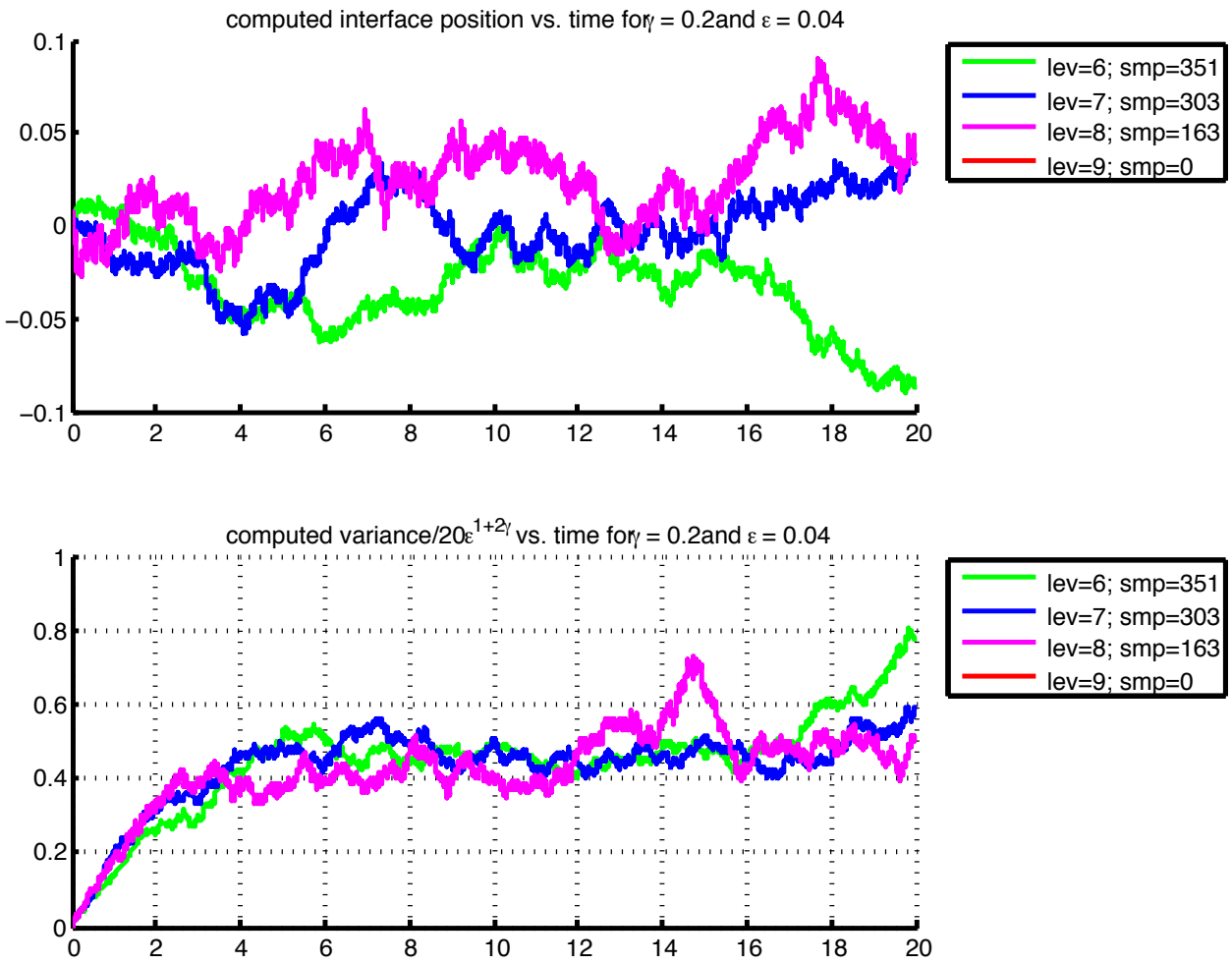
Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$.

Sample number smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs.

Variance against time for the various levels. As time grows, variance accuracy deteriorates.

Numerically computed average and variance

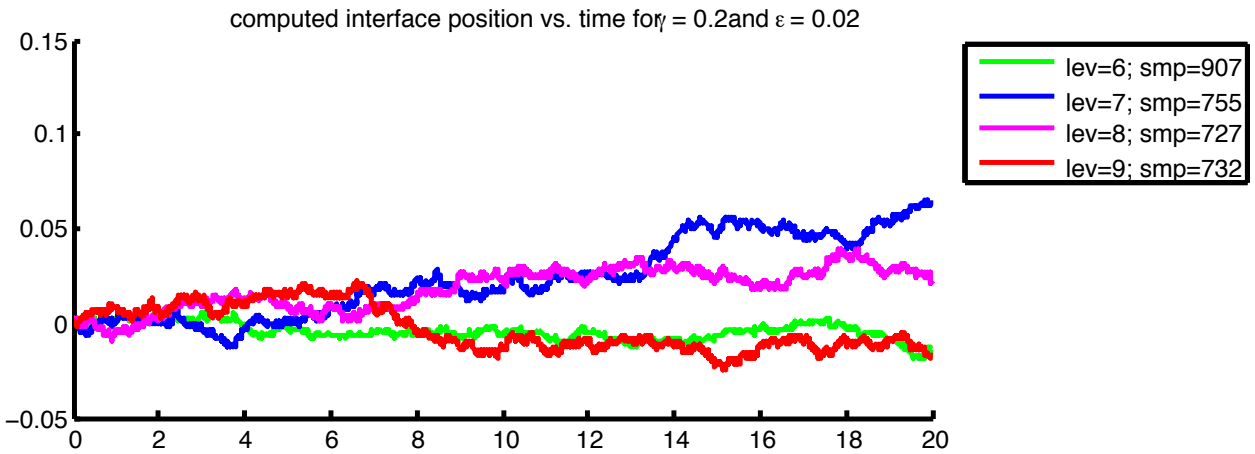
$\gamma = 0.2$ and $\epsilon = 0.04$

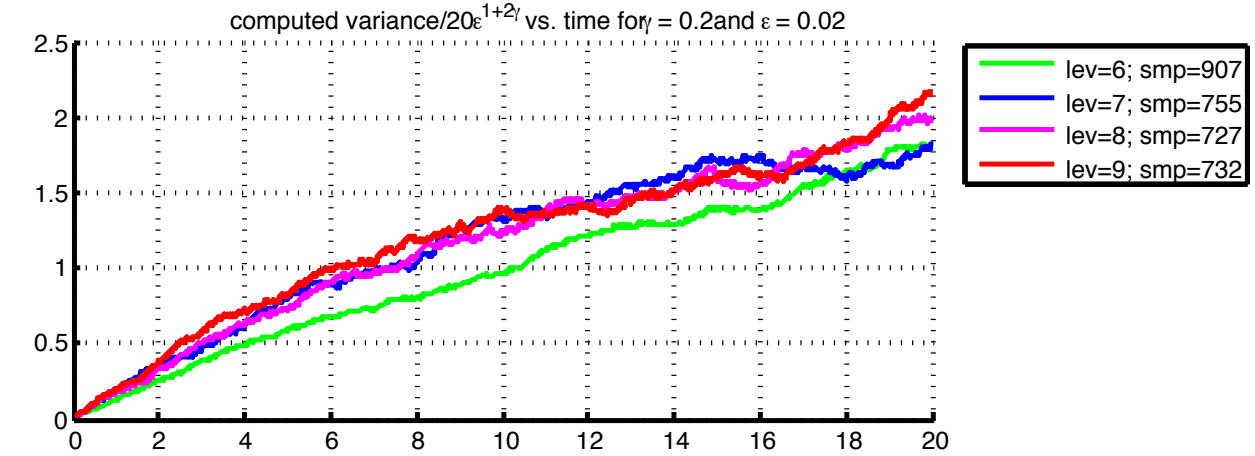


Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$. **Sample number** smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs. **Variance** against time for the various levels. As time grows, variance accuracy deteriorates.

Numerically computed average and variance

$\gamma = 0.2$ and $\epsilon = 0.02$





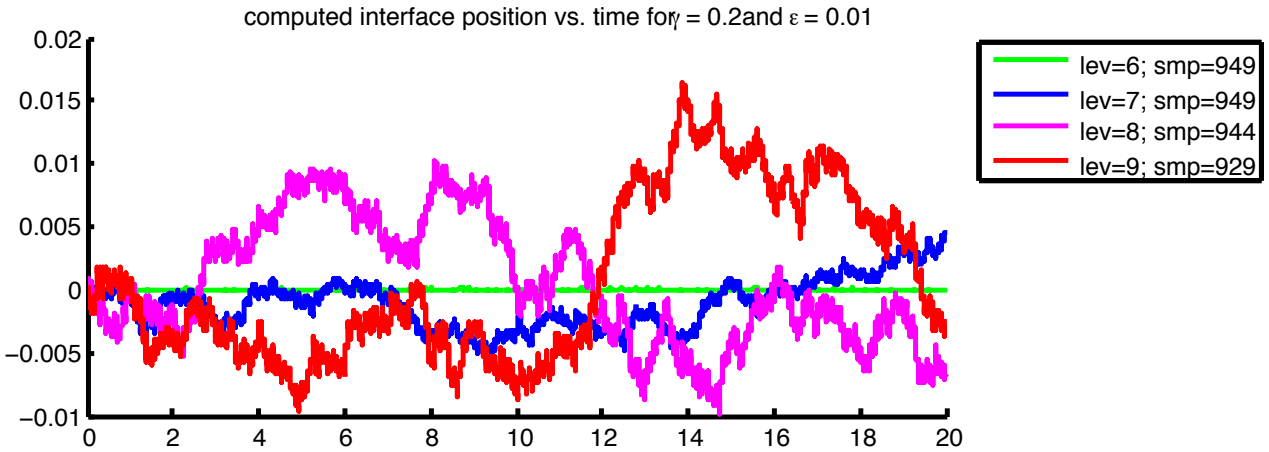
Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$.

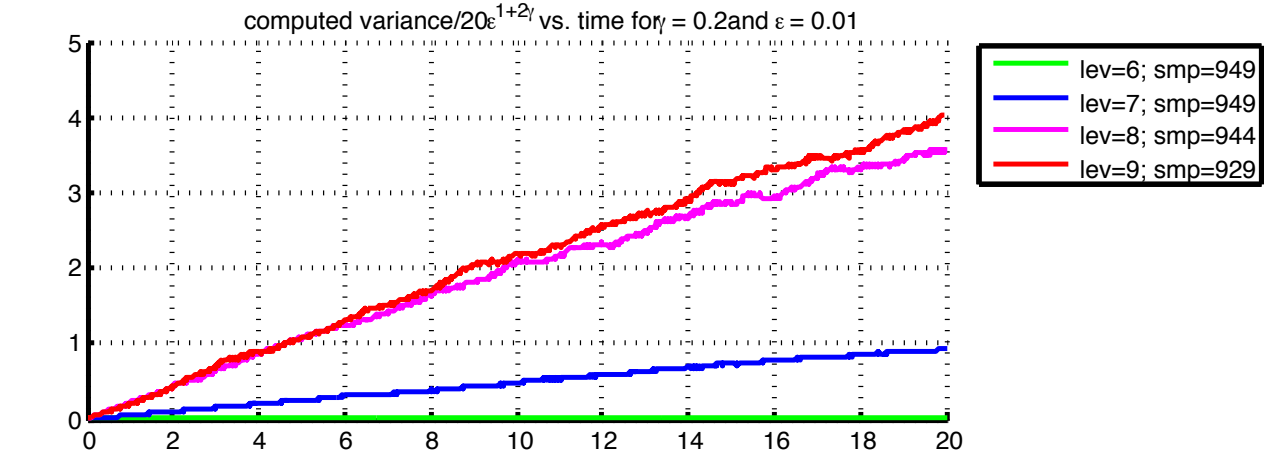
Sample number smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs.

Variance against time for the various levels. As time grows, variance accuracy deteriorates.

Numerically computed average and variance

$\gamma = 0.2$ and $\epsilon = 0.01$





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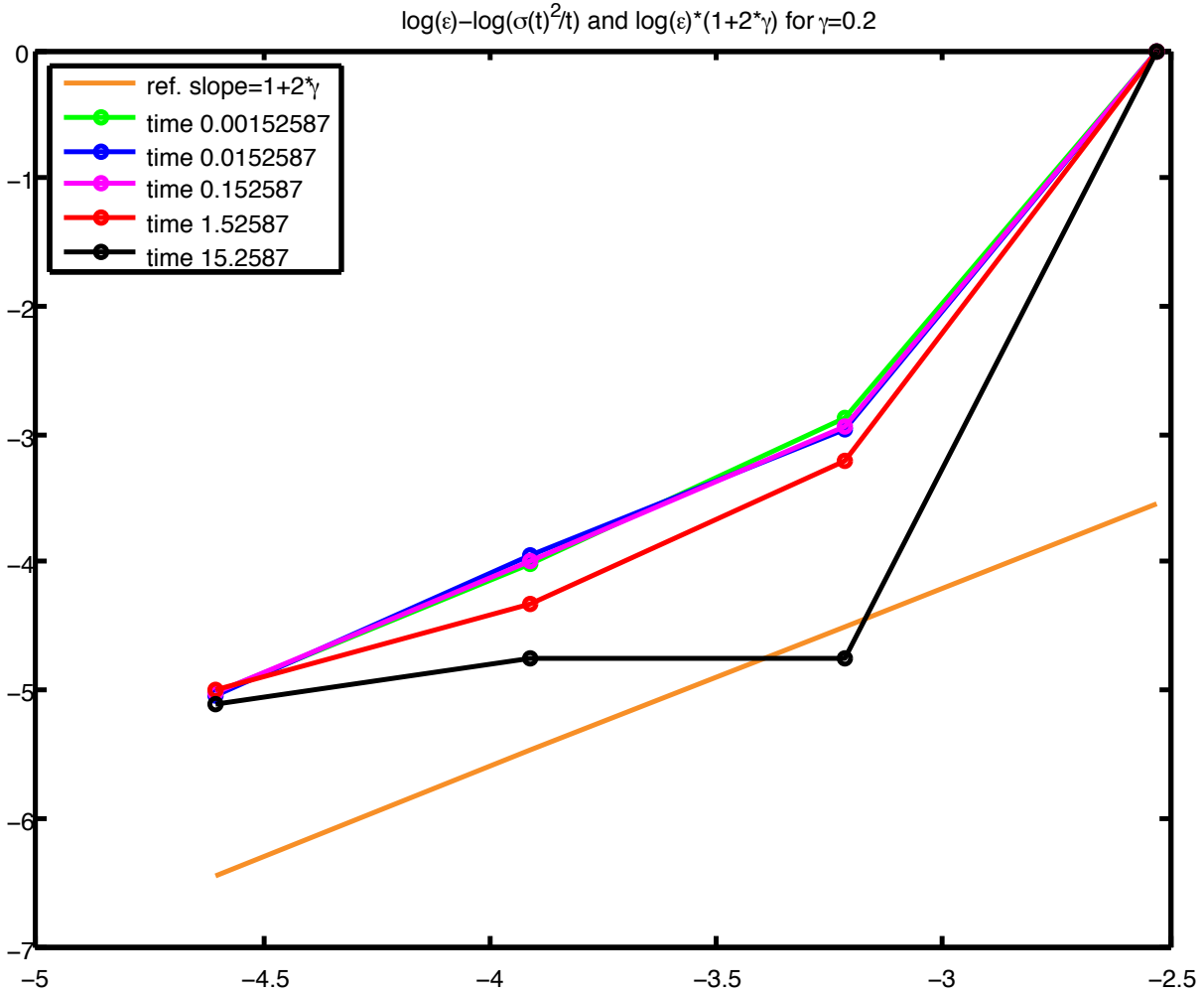
Simulations vs. theory

$\log(\mathrm{var}[U_h^i]\,/t_i)$ vs. $\log\epsilon$ plot for $\gamma = 0.2$ at various times $t_i = t_0\,10^i$



Theoretical slope (predicted by [Funaki, 1995] and [Brassesco et al., 1995]) is $1 + 2\gamma$.

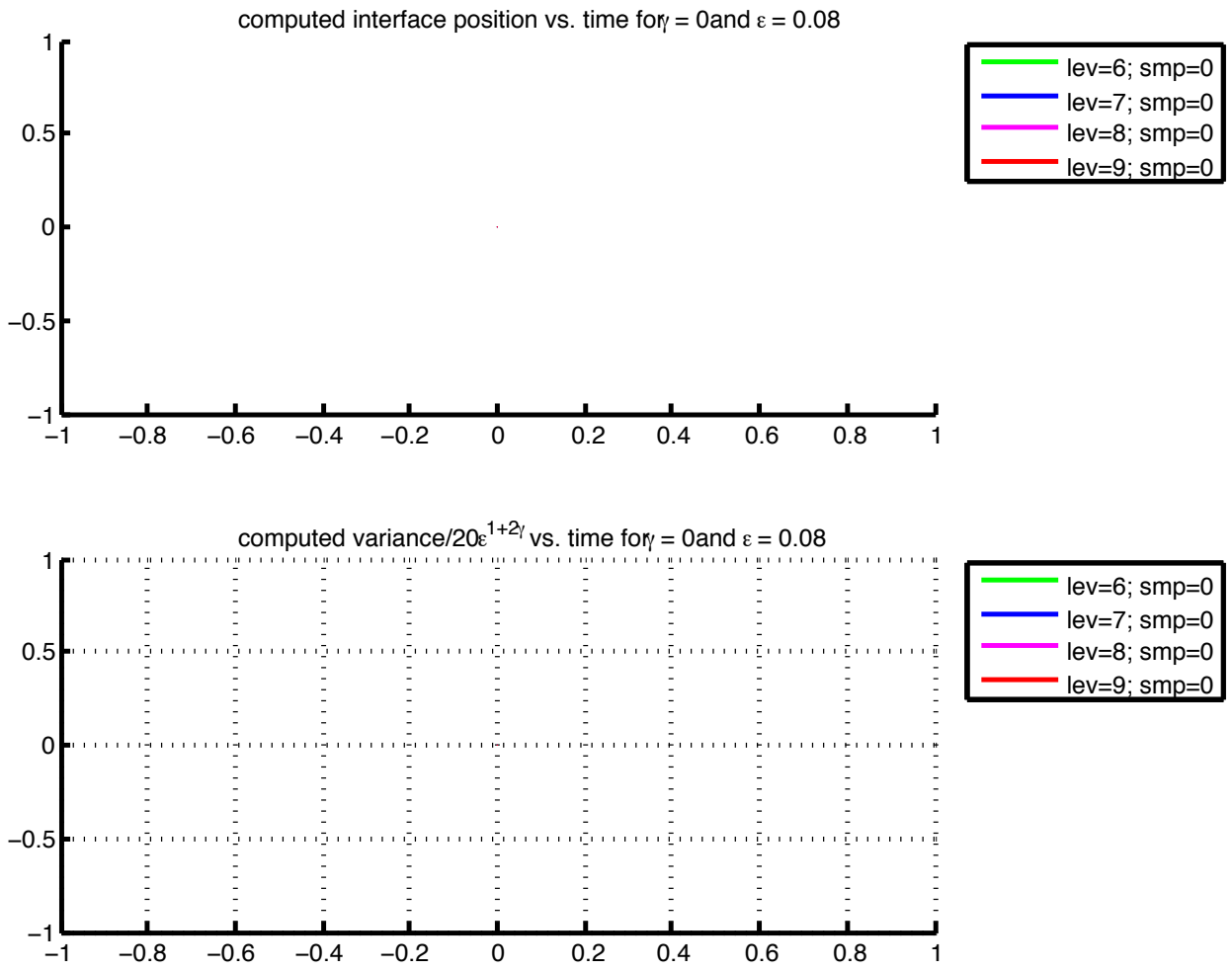
Plotted in **orange** reference line

As $\epsilon \to 0$ slope improves.

Meshsize $h = 1/512$.

Numerically computed average and variance

$\gamma = 0.0$ and $\epsilon = 0.08$



Average of Monte Carlo samples against time for various levels of refinement lev $=$ **6**, **7**, **8**, **9**, and meshsize $h = 1/2^{\text{lev}}$. **Sample number** smp differs with lev. Samples are **rejected** if annihilation/nucleation occurs. **Variance** against time for the various levels. As time grows, variance accuracy deteriorates.

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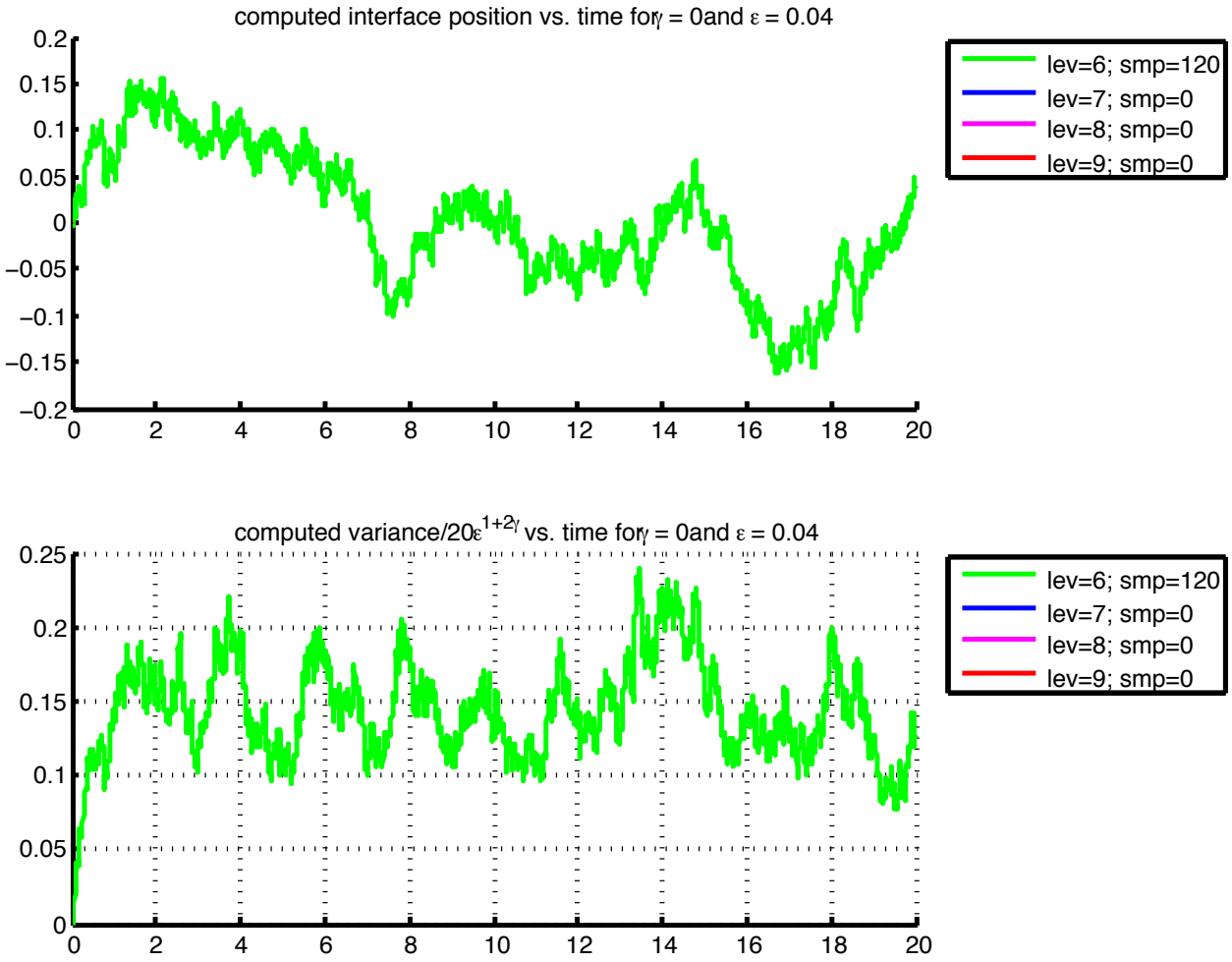

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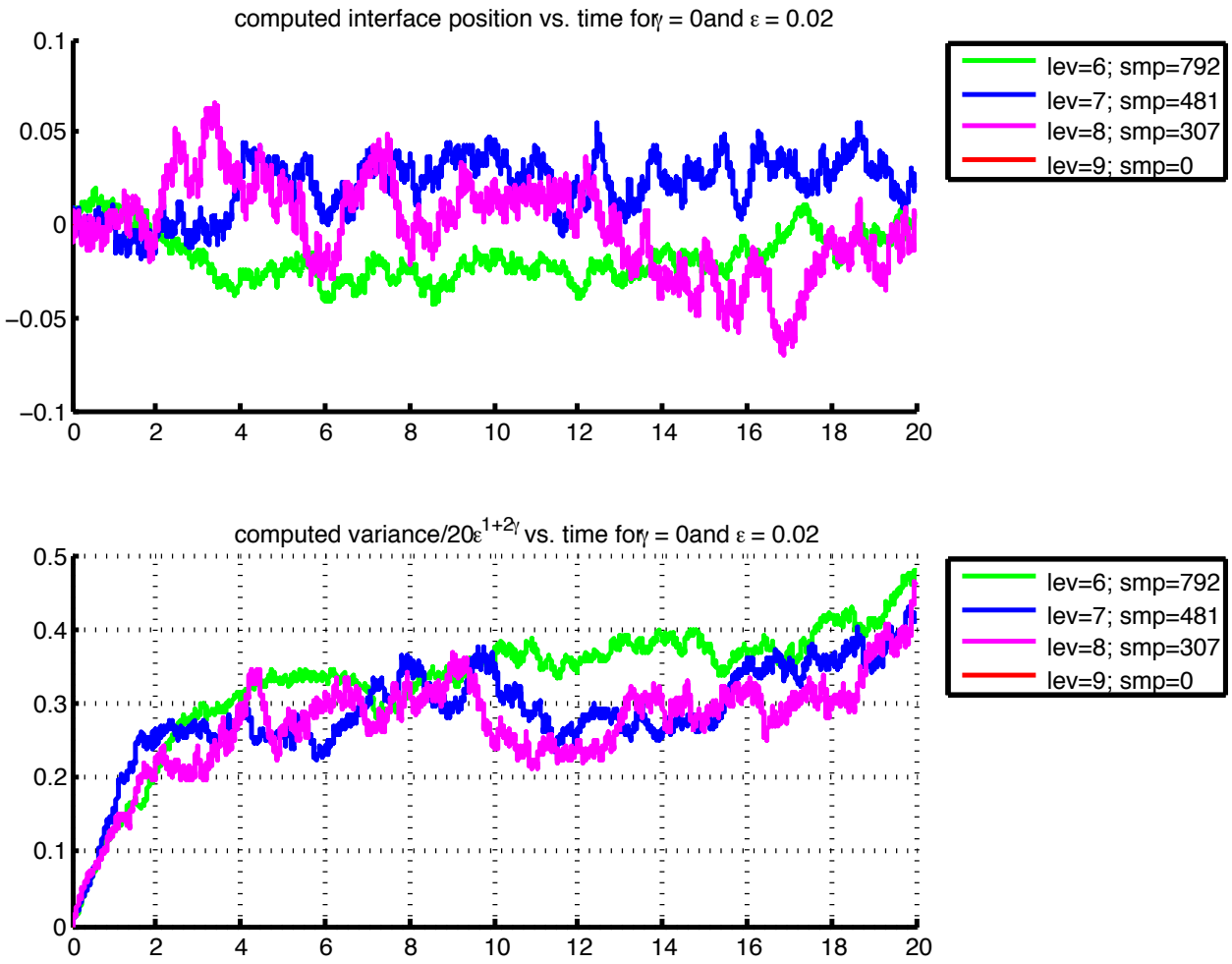
$\gamma = 0.0$ and $\epsilon = 0.02$



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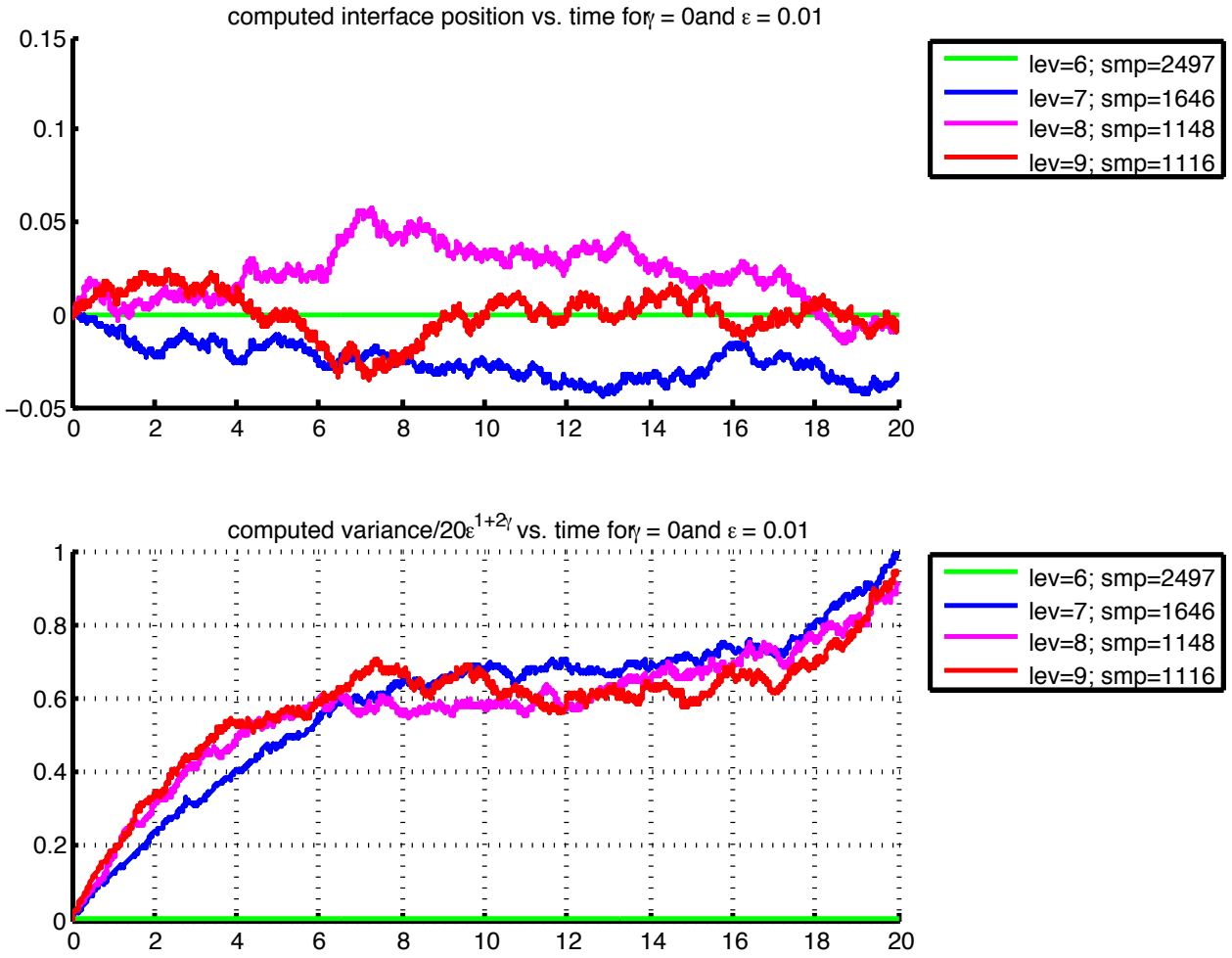

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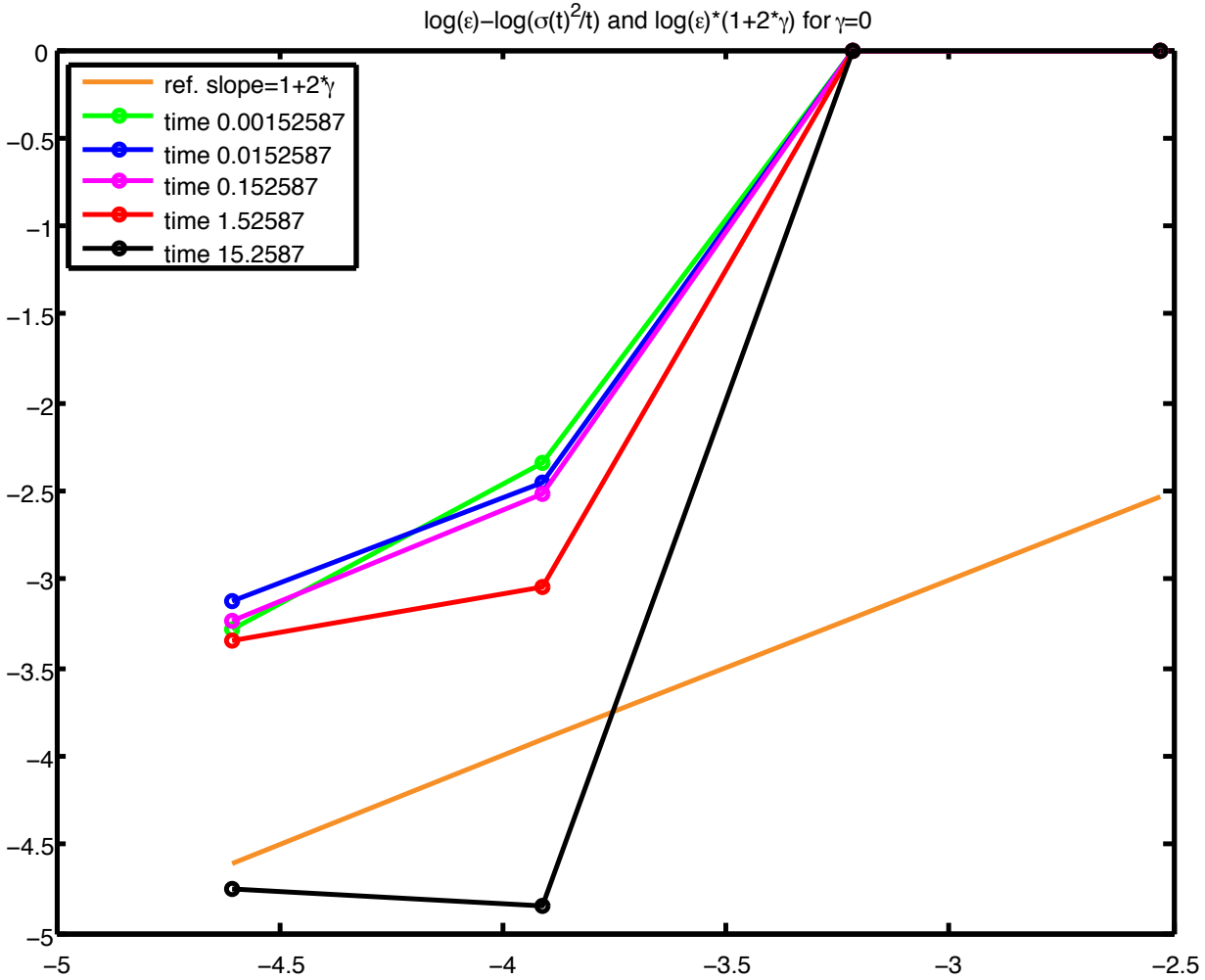

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Plotted in **orange** reference line

As $\epsilon \to 0$ slope improves.

Meshsize $h = 1/512$.

What we learned

- Stability and convergence numerical method for the stochastic Allen-Cahn in 1d.
- Monte-Carlo type simulations.
- Benchmarking via statistical comparison.
- Adaptivity mandatory (but different from "standard" phase-field approach) for higher dimensional study.

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What we're trying to learn

Current investigation [Kossioris et al., 2009]

- finite ϵ "exact" solutions
- multiple interface convergence
- structure of the stochastic solution
- colored noise in $d > 1$
- adaptivity

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What still needs to be learned

- Extension to Cahn-Hilliard type equations, Phase-Field, Dendrites.
- Other stochastic applications
- Chaos Expansion
- Kolmogorov (Fokker-Plank) equations
- Sparse methods

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