Universität Augsburg

Pattern size in Gaussian fields from spinodal decomposition

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(joint work with D. Blömker and P. Düren)

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Sit-rep

Cahn-Hilliard-Cook equation

The equation

$$\partial_t u = -\Delta(\varepsilon^2 \Delta u + f(u))$$

f derivative of a double well potential, (e.g. $f(u) = u - u^3$)

- A model for relative concentration of an alloy after quenching.
- Dynamics dominated by strongly unstable space.
- Initial condition: constant homogeneous concentration $u = m$.

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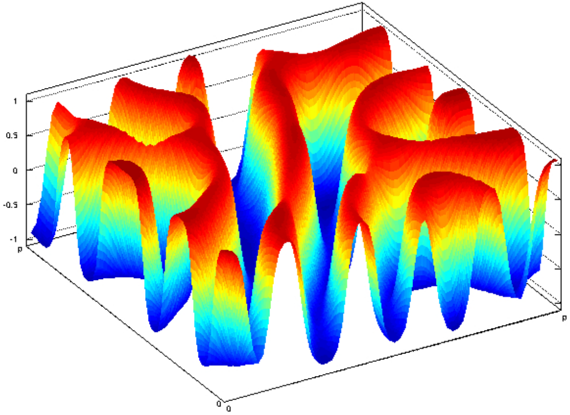
$$\partial_t u = -\Delta(\varepsilon^2 \Delta u + f(u)) + \partial_t W,$$

f derivative of a double well potential, (e.g. $f(u) = u - u^3$)
$\partial_t W$ derivative of a Q-Wiener process.

- A model for relative concentration of an alloy after quenching.
- Dynamics dominated by strongly unstable space.
- Initial condition: constant homogeneous concentration $u = m$.
- Perturbation causes spinodal decomposition.

Spinodal decomposition



Linearisation

- Good approximation by linearisation in m:

$$\partial_t u = -(\varepsilon^2 \Delta^2 + \Delta f'(m))u + \partial_t W \qquad \text{on } [0,1]^2.$$

L^2-basis: $e_{k,l}(x) = C\cos(k\pi x)\cos(l\pi y)$.

- Assuming $Qe_{k,l} = \alpha_{k,l}^2 e_{k,l}$, the solution is the stochastic convolution

$$\sum_{k,l\in\mathbb{N}} \alpha_{k,l} \int_0^t e^{(t-s)\lambda_{k,l}} \, \mathrm{d}B_{k,l}(s) e_{k,l}.$$

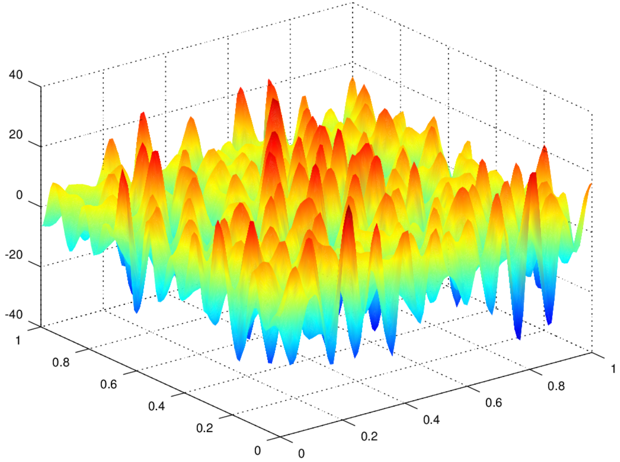
- The strong subspace is

$$R_\varepsilon^\gamma := \{(k,l) \in \mathbb{N}^2 : \lambda_{k,l} > \gamma\lambda_{\max}\}, \qquad \gamma \in (0,1).$$

- Projecting there we have that at fixed time t the solution is well approximated by

$$u(x,y) \approx \sum_{(k,l)\in R_\varepsilon} \alpha_{k,l} c_{k,l} \cdot \cos(k\pi x)\cos(l\pi y), \quad c_{k,l} \text{ Gaussians.}$$

Cosine series



Some comments

We have

$$R_\varepsilon = \{(k,l) \in \mathbb{N}^2 |\ \alpha_\ominus < \sqrt{(k\varepsilon)^2 + (l\varepsilon)^2} < \alpha_\oplus\},$$

(growing and moving with $\varepsilon \to 0$), with

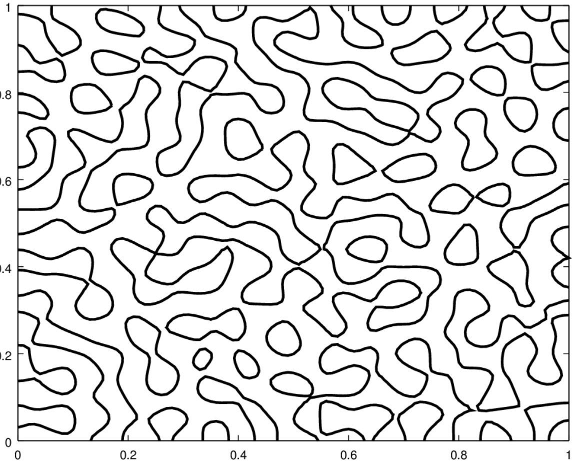
$$\alpha_\oplus = \sqrt{\frac{1+\sqrt{1-\gamma}}{2\pi^2}} \quad \text{and} \quad \alpha_\ominus = \sqrt{\frac{1-\sqrt{1-\gamma}}{2\pi^2}} \quad \text{with} \quad \gamma \in (0,1).$$

We restrict to

$$f(x,y) = \sum_{(k,l)\in R_\varepsilon} c_{k,l} \cdot \cos(k\pi x)\cos(l\pi y), \quad x,y \in [0,1]^2 \quad c_{k,l} \sim N(0,1) \text{ i.i.d.}$$

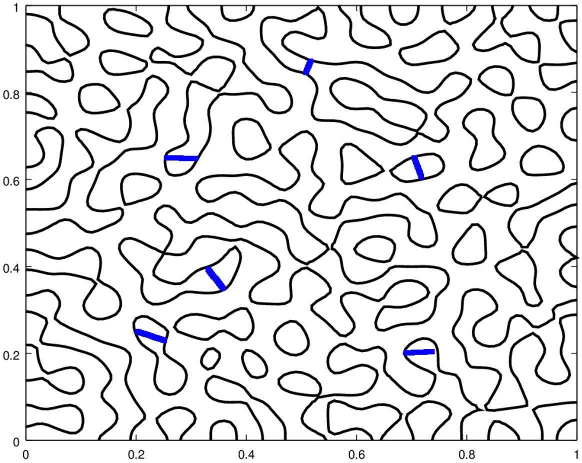
Note that f is neither stationary nor isotropic; its law might change under translation or rotation (as a function extended to $\mathbb{R}^2$).

Snake like pattern



Patterns like this one appear in other models (reaction-diffusions, . . .)

Snake like pattern



Sit-rep

The question

Question

What is the characteristic thickness of the pattern (i.e. snake-like structures) in our model $f(x, y)$, on the unit square $(x, y) \in [0, 1]^2$?

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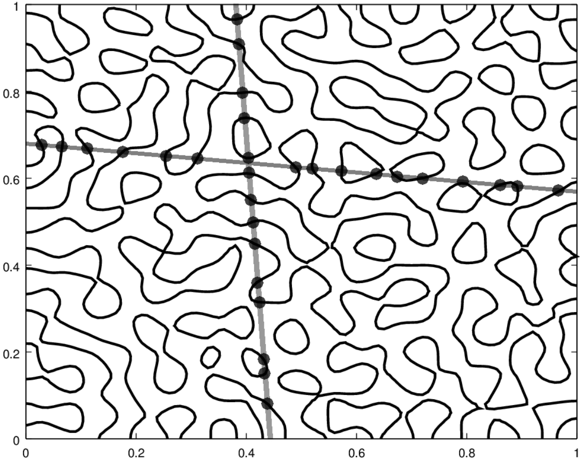
Strategy

To address the question, we draw a straight line across the unit square and we count the (average) number of zeros of f on that segment, then we divide the length of the segment by the number of zeros,.

Strategy



Count number of zeros on lines and divide length of segment by it.

Definition (Equator of a point and a curve)

Let $P \in \mathbb{S}^n$. $P_\perp$ is the hyperplane $\perp$ to PO in O, intersected with $\mathbb{S}^n$.
Let $\gamma(t)$ rectifiable, then $\gamma_\perp = \{P_\perp | P \in \gamma\}$.

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Definition (Multiplicity and area swept out)

Multiplicity of $Q \in \cup\gamma_\perp$ is $\#\{t \in \mathbb{R} | q \in \gamma(t)_\perp\}$ (equators containing Q).
$|\gamma_\perp|$ is the integral of multiplicity over $\cup\gamma_\perp$ (area swept out by equators).

How many zeros of a random polynomial?

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Lemma

If γ is a rectifiable curve, $|\gamma|$ its length,

$$\frac{|\gamma_\perp|}{\text{area } \mathbb{S}^n} = \frac{|\gamma|}{\pi}.$$

Where are the random polynomials?

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Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, and consider

$$a = (a_0, a_1, a_2, \ldots, a_n), \qquad v(t) = (1, t, t^2, \ldots, t^n).$$

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Let now E_n be the expected number of zeros. It is the portion of the surface swept out (with multiplicity), so

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Now we only need to compute the length of w, using Kac's formula.

A theorem by Edelman and Kostlan

Theorem (Edelman&Kostlan, 1995)

Let $v(x) = (g_0(x), \ldots, g_n(x))^T$ be any collection of differentiable functions and $c_0, \ldots, c_n$ be independent and identically distributed Gaussians centred in 0. Given the function

$$h(x) = \sum_{k=0}^{n} c_k \cdot g_k(x),$$

the density of real zeros of h on an interval I is

$$\delta(x) = \frac{1}{\pi} \left\| \frac{\mathrm{d}}{\mathrm{d}x} w(x) \right\|_{\mathbb{R}^n}, \qquad \textit{where} \qquad w(x) = \frac{v(x)}{\|v(x)\|_{\mathbb{R}^n}}.$$

The expected number of real zeros of h on I is then

$$\int_I \delta(x)\, \mathrm{d}x.$$

Main result

We need to introduce, in the spirit of the previous theorem, the following notation:

$$w_t(x) = \left(\frac{\cos(k\pi x)\cos(l\pi t)}{\sqrt{\sum_{m,n\in R_\varepsilon} \cos^2(m\pi x)\cos^2(n\pi t)}} \right)_{(k,l)\in R_\varepsilon}$$

$$W_t(x) = \left\| \left(\frac{\mathrm{d}}{\mathrm{d}x} w_t(x) \right)_{(k,l)\in R_\varepsilon} \right\|^2 = \frac{S_3}{S_1} - \left(\frac{S_2}{S_1} \right)^2$$

where we have

$$S_1 = \sum_{m,n\in R_\varepsilon} \cos^2(m\pi x)\cos^2(n\pi t)$$

$$S_2 = \sum_{m,n\in R_\varepsilon} m\pi\cos(m\pi x)\sin(m\pi x)\cos^2(n\pi t)$$

$$S_3 = \sum_{m,n\in R_\varepsilon} m^2\pi^2\sin^2(m\pi x)\cos^2(n\pi t).$$

Main result (cont.)

Let

$$L_t = \{(x,t) : x \in [0,1]\} \quad \text{for} \quad t \in [0,1].$$

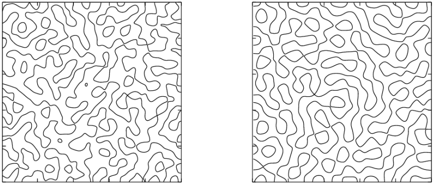
Theorem

For any $\gamma \in (0,1)$ and any horizontal line L_t for $x, t \in (0,1)$ the function $W_t(x)$ defined on L_t behaves asymptotically as $(2\varepsilon)^{-2}$ for $\varepsilon \to 0$.
This means that the average number of zeros is $(2\pi\varepsilon)^{-1}$, so the mean pattern size is $2\pi\varepsilon$.

Dependency on γ

Remark

The result is independent of γ, even if the number of Fourier modes involved is much smaller for $\gamma \approx 1$ than for $\gamma \approx 0$. As we can see , while the average asymptotic pattern size along lines remains the same, the domain with higher γ looks more organized. The pattern seems to be "more regular" in some sense.



($\gamma = 0.1$ and $\gamma = 0.9$)

Sit-rep

Ergodic Theory

Theorem (Birkhoff ergodic theorem)

Let (X, μ) be a probability space. If T is μ-invariant and ergodic and g is integrable, then for a.e. $z \in X$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g\left(T^k(z)\right) = \int_X g(\zeta)\,\mathrm{d}\mu(\zeta).$$

Moreover if T is continuous and uniquely ergodic with measure μ and if g is continuous, then the limit holds for all $z \in X$ (instead of a.e.).

Example

The map $z \mapsto z + \alpha$ on the unit circle is uniquely ergodic if and only if α is irrational. In this case the unique ergodic measure is the Lebesgue measure.

Weighted averaging condition

Requirement (Weighted averaging condition)

Let $([0,1]^d, \lambda)$ be the probability space with the Lebesgue-measure λ. We say that $(f, (a_m))$ with $f : [0,1]^d \to \mathbb{R}$ continuous and extended by periodicity to $\mathbb{R}^d$ and $a_m \in \mathbb{R}$ fulfils the weighted averaging condition, if for every $x^0 \in [0,1]^d$, every $\alpha \in \mathbb{N}^d$ and

$$Q_L = \bigotimes_{i=1}^{d} [1, \ldots, \alpha_i L] \cap \mathbb{N}^d,$$

the following holds:

$$\frac{1}{\sum_{m \in Q_L} a_m} \sum_{m \in Q_L} a_m \cdot f(m_1 x_1^0, m_2 x_2^0, \ldots, m_d x_d^0) \xrightarrow[L \to \infty]{} \int_{[0,1]^d} f(x)\,\mathrm{d}x \,.$$

For any open set $M \subset \mathbb{R}_+^d$ we define the (scaled) projection

$$M_L = (L \cdot M) \cap \mathbb{N}^d.$$

Weighted averaging condition (cont.)

Lemma

Let $(f,(a_m))$ fulfil the weighted averaging condition with the weights a generating a measure. Then for any open measurable set $S \subset \mathbb{R}_+^d$

$$\frac{1}{|S_L|_a} \sum_{m \in S_L} a_m \cdot f(m_1 x_1^0, m_2 x_2^0, \ldots, m_d x_d^0) \xrightarrow{L \to \infty} \int_{[0,1]^d} f(x)\,\mathrm{d}x\ .$$

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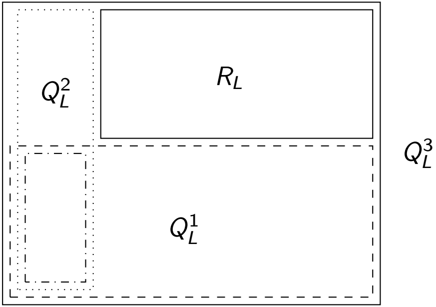

Weights generating measures

Requirement (Generation of measures)

We require that the weights $a = (a_m)$ generate a measure λ_a on $\mathbb{R}^d_+$ which is equivalent to the Lebesgue measure λ, i.e. there exists an $\alpha > 0$ such that for each set with open interior $M \subset \mathbb{R}^d_+$,

$$L^{-\alpha}|M_L|_a \xrightarrow{L\to\infty} \lambda_a(M).$$

Example

Consider $a_{k,l} = k^2$ in dimension $d = 2$.

$$L^{-4}|M_L|_a = L^{-4} \sum_{(k,l)\in L\cdot M} k^2 = \sum_{(k,l)\in M\cap\frac{1}{L}\mathbb{N}^2} k^2 L^{-2} \xrightarrow[L\to\infty]{} \int_M \xi^2 \,\mathrm{d}(\xi,\eta)\,.$$

The measure $\lambda_{(k^2,1)}$ has a Lebesgue-density $(\xi,\eta) \mapsto \xi^2$. The density is a. e. strictly positive, so the measures $\lambda_{(k^2,1)}$ and λ are equivalent.

Key Lemma

Lemma

For $x, t \in (0,1)$,

$$\lim_{\varepsilon \to 0} \frac{1}{|R_\varepsilon|} \cdot S_1 = \frac{1}{4}, \qquad \lim_{\varepsilon \to 0} \varepsilon^2 \cdot \frac{S_3}{S_1} = \frac{1}{4}, \qquad \lim_{\varepsilon \to 0} \varepsilon \cdot \frac{S_2}{S_1} = 0,$$

where the S_i are

$$S_1 = \sum_{m,n \in R_\varepsilon} \cos^2(m\pi x)\cos^2(n\pi t)$$

$$S_2 = \sum_{m,n \in R_\varepsilon} m\pi \cos(m\pi x)\sin(m\pi x)\cos^2(n\pi t)$$

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Proof of key lemma (sketch)

We consider only the case $x, t \notin \mathbb{Q}$.
First limit. We can use Birkhoff's ergodic theorem. Define $T_x(z) = z + x$: this is a measure-preserving and uniquely ergodic transformation (since $x \notin \mathbb{Q}$). Then

$$\frac{1}{N}\sum_{k=0}^{N-1}\cos^2(\pi k x) = \frac{1}{N}\sum_{k=0}^{N-1}\cos^2\left(\pi T_x^k(0)\right) \xrightarrow{N\to\infty} \int_0^1 \cos^2(\pi x)\,\mathrm{d}x = \frac{1}{2}.$$

All the coefficients a_m are 1 in this case and the function is multiplicative. Then the result follows immediately from previous lemma and the fact that

$$\int_{[0,1]^2} \cos^2(\pi x_1)\cos^2(\pi x_2)\,\mathrm{d}(x_1, x_2) = \frac{1}{4}.$$

Proof of key lemma (sketch)

Second limit. The asymptotic behaviour is (by previous limit)

$$\varepsilon^2 \frac{S_3}{S_1} \sim 4\pi^2 \frac{\varepsilon^2}{|R_\varepsilon|} \cdot \sum_{k,l \in R_\varepsilon} k^2 \sin^2(k\pi x)\cos^2(l\pi t).$$

By previous example

$$\lim_{\varepsilon\to 0} \varepsilon^2 \cdot \frac{|R_\varepsilon|_a}{|R_\varepsilon|} = \lim_{\varepsilon\to 0} \varepsilon^2 \cdot \frac{1}{|R_\varepsilon|} \sum_{k,l\in R_\varepsilon} k^2 = \frac{\lambda_a(R)}{\lambda(R)} = \frac{1}{4\pi^2},$$

where the rescaled domain is

$$R = \{(\eta,\xi) \in \mathbb{R}^2 |\ \alpha_\ominus < \sqrt{\xi^2+\eta^2} < \alpha_\oplus\},$$

We can check on rectangles and then on the quarter ring that

$$\varepsilon^2 \cdot \frac{S_3}{S_1} \sim 4\pi^2 \varepsilon^2 \cdot \frac{|R_\varepsilon|_a}{|R_\varepsilon|} \cdot \frac{1}{|R_\varepsilon|_a} \cdot \sum_{k,l\in R_\varepsilon} k^2 \sin^2(k\pi x)\cos^2(l\pi t) \xrightarrow[\varepsilon\to 0]{} \frac{1}{4}.$$

Proof of theorem

Proof of main theorem.

From key lemma:

$$W_t(x) = \frac{S_3}{S_1} - \left(\frac{S_2}{S_1}\right)^2 \sim \frac{1}{4\varepsilon^2} \qquad \text{as } \varepsilon \to 0.$$

By E-K's theorem, the number of expected zeros on the horizontal line L_t of length 1 is, for a given ε,

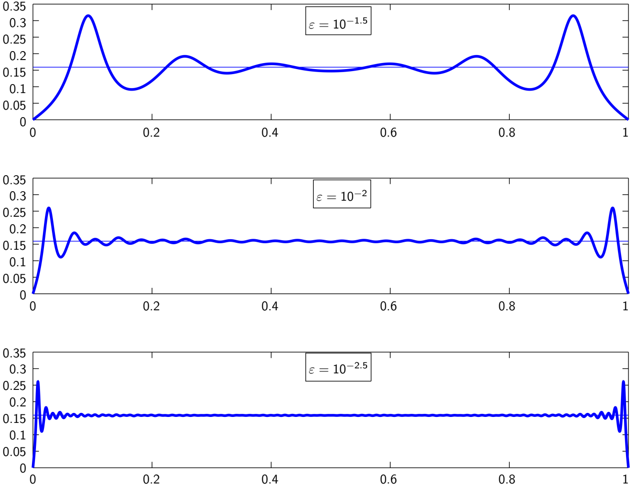
$$N = \frac{1}{\pi} \int_0^1 \sqrt{\frac{1}{4\varepsilon^2}}\, \mathrm{d}x = \frac{1}{2\pi \cdot \varepsilon}.$$

This is the same as saying that the average pattern size is $\frac{1}{N} = 2\pi\varepsilon$. □

Sit-rep

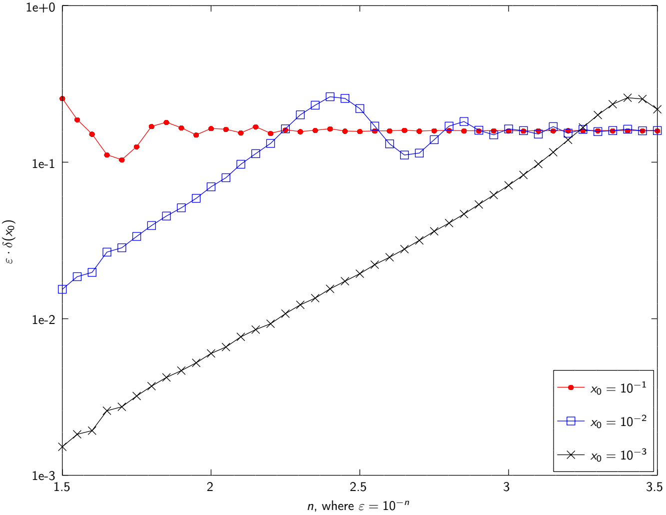
Fast convergence

To get an idea of the speed of convergence, let's consider the rescaled density of zeros $\varepsilon\delta(x)$.



Asymptotic result

Now, fix $x_0 \in (0, 1)$ and look at $\varepsilon\delta(x_0)$ as $\varepsilon \to 0$.



Sit-rep

Other Fourier domains

Lemma

Let $D_\varepsilon = \varepsilon^{-1} D \cap \mathbb{N}^2$ *be a scaled domain in Fourier space. Then*

$$\delta(x) \sim \frac{1}{2\pi\varepsilon} \cdot \sqrt{\frac{\lambda_{(k^2,1)}(D)}{\lambda(D)}}, \qquad \delta(x) \sim \frac{1}{2\pi\varepsilon} \cdot \sqrt{\frac{\lambda_{(1,l^2)}(D)}{\lambda(D)}},$$

respectively on horizontal and vertical lines.

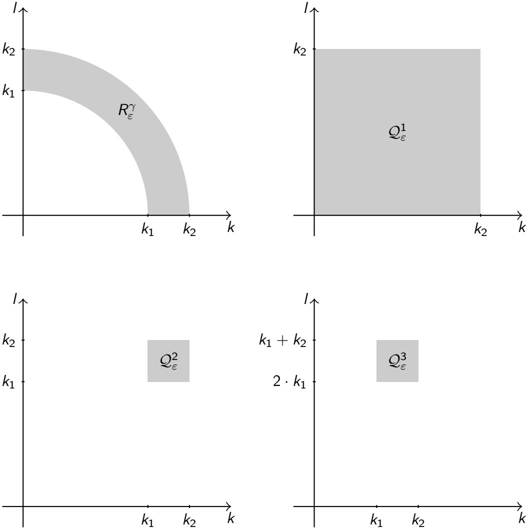
Proof.

Analogous to the one of the key lemma. We obtain the following asymptotic equivalences

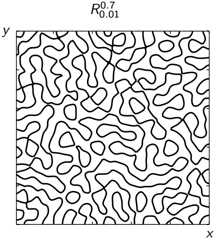
$$\delta(x)^2 \sim \frac{S_3}{S_1} \sim \frac{1}{4} \cdot \frac{|D_\varepsilon|_{(k^2,1)}}{|D_\varepsilon|} \sim \frac{1}{4\varepsilon^2} \cdot \frac{\lambda_{(k^2,1)}(D)}{\lambda(D)}.$$

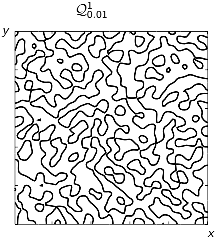
□

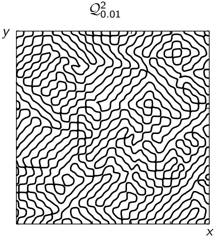
Examples of Fourier domains

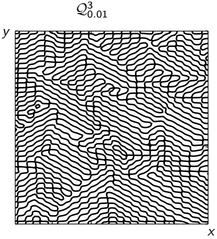


Corresponding patterns









Some numbers

Domain	Corr. coeff.	Avg. # 0s	Avg. # 0s (sampled)
$R^{0.7}_{0.01}$	1	15.915 ($\times$1)	16.413
$\mathcal{Q}^{1}_{0.01}$	1.032	16.167 ($\times$1.016)	16.984
$\mathcal{Q}^{2}_{0.01}$	1.891	21.887 ($\times$1.375)	21.931
$\mathcal{Q}^{3}_{0.01}$ (hor.)	1.891	21.887 ($\times$1.375)	21.931
$\mathcal{Q}^{3}_{0.01}$ (ver.)	5.374	36.894 ($\times$2.318)	37.315.

The correction coefficient depends on γ (the quarter ring is a special case!).

Further generalisations

- Relax iid and equal variance assumption on coefficients. (Free, using the fact that E-K result holds for a more general family of gaussian random coefficients).

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- Spaces of higher dimension?

Thank you for your attention!

Bibliography

- A. Edelman, and E. Kostlan.
  How many zeros of a random polynomial are real?
  Bulletin of the AMS, 32(1):1–37, 1995.
- L. A. Bianchi, D. Blömker, and P. Düren
  Pattern size in Gaussian fields from spinodal decomposition.
  arXiv:1510.01967v2, 2015