Aging of the Metropolis dynamics of the Random Energy Model

Jiří Černý
University of Vienna

with Tobias Wassmer

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Aim of the project:

- Understand aging in the dynamics of (real) spin glasses.
- Prove Bouchaud's aging heuristics.

Outline

1. Introduction
2. Bouchaud's trap model
3. History of proving Bouchaud's heuristics
4. Metropolis dynamics of the REM

Mean-field spin glasses

State space. $\Sigma_N = \{-1,+1\}^N$

Hamiltonians.

- *SK model.* For $x \in \Sigma_N$,

$$H_N(x) = N^{-1/2} \sum_{i,j} J_{ij} x_i x_j, \qquad J_{ij}\text{'s i.i.d. } \mathcal{N}(0,1).$$

  i.e. H_N is a Gaussian process on Σ_N with covariance

$$\mathbb{E}[H_N(x)H_N(y)] = (\tfrac{1}{N} x \cdot y)^2$$

- *p-spin SK model.*

$$\mathbb{E}[H_N(x)H_N(y)] = (\tfrac{1}{N} x \cdot y)^p$$

- *Random Energy Model (REM).* a formal $p \to \infty$ limit

$$H_N(x) \text{ are i.i.d.}$$

Gibbs measure. $\tau_x = e^{\beta\sqrt{N} H_N(x)}$

Dynamic rules

Desirable properties of the dynamics.

- Markov process $(X_t)_{t\geq 0}$ on Σ_N
- nearest-neighbour = single spin flip
- τ is reversible for X
- attracted to states with large τ

Possible transition rates.

- *Metropolis dynamics.*

$$w_{xy}^{M} = e^{-\beta\sqrt{N}(H_N(x)-H_N(y))^+} = 1 \wedge \frac{\tau_y}{\tau_x} \qquad \text{if } x \sim y.$$

- *Asymmetric Bouchaud' dynamics.* $\quad a \in [0,1]$

$$w_{xy}^{a} = \tau_x^{a-1}\tau_y^{a} \qquad \text{if } x \sim y.$$

- *Random Hopping Time (RHT) dynamics.*

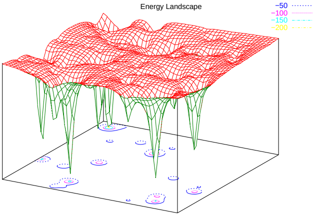
$$w_{xy}^{RHT} = \tau_x^{-1} \qquad \text{if } x \sim y.$$

Aim

Understand aging!

Remarks.

- We want to understand out-of-equilibrium behaviour of finite-state reversible Markov chains
- These chains have random transition rules = random environment
- The mixing time grows as $T_{\mathrm{mix}} \sim e^{cN}$.



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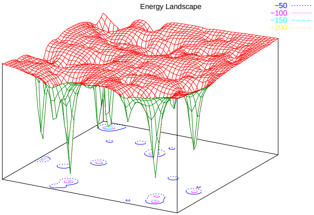

Bouchaud's trap model

= a toy model to explain the aging behaviour of (real) spin glasses

State space. $\{1, \ldots, n\}$

Hamiltonian. $(E_i)_{i=1,\ldots,n}$ i.i.d. standard exponential r.v.'s

Gibbs measure. $\tau_i = e^{\beta E_i}$. Then $\mathbb{P}[\tau_i \geq u] = u^{-1/\beta}$.

Transition rates. $w_{ij}^{BTM} = \frac{1}{(n-1)\tau_i}, \qquad i \neq j$

BTM is a time change of the simple random walk Y on the complete graph!

Theorem (Bouchaud 1992)

*If $\alpha := 1/\beta \in (0,1)$, $\theta > 1$, then for **a.e.** realisation of τ's*

$$P_{\text{unif}}^{BTM}[X(t) = X(\theta t)] \xrightarrow{n\to\infty, t\to\infty} \mathsf{Asl}_\alpha(\theta) \in (0,1).$$

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Proof of Bouchaud's theorem

- Take first $n \to \infty$: Y_i's are "i.i.d. uniform over $\mathbb{N}$"
- As consequence: τ_{Y_i}'s are i.i.d., $\mathbb{P}[\tau_{Y_i} \geq u] = u^{-\alpha}$.
- Let S_k be the time of the k-th jump of X. Then

$$S_k = \sum_{j=0}^{k-1} e_i \tau_{Y_i}$$

- By standard convergence results

$$k^{-1/\alpha} S_{kt} \xrightarrow{k\to\infty} V_\alpha(t), \qquad \text{where } V_\alpha \text{ is an } \alpha\text{-stable Lévy process}$$

- Conclusion:

$$P^{BTM}_{\text{unif}}[X(t) = X(\theta t)] = P^{BTM}_{\text{unif}}[\{S_j : j \geq 0\} \cap [t, \theta t] = \emptyset]$$
$$\xrightarrow{t\to\infty} \mathsf{Asl}_\alpha(\theta).$$

And now . . . ?

Simplifications of the BTM:

1. Hypercube Σ_N is replaced by the complete graph K_n
2. It considers the RHT dynamics
3. Hamiltonian is i.i.d.
4. (Energies are exponential instead of Gaussian.)

Question.

Can we confirm the aging heuristics based on the convergence to Lévy processes for a dynamics of a mean-field spin glass?

History of proving Bouchaud's heuristic

Ben Arous, Bovier, Gayrard (2003): REM (truncated at 0), RHT

Ben Arous, Č. (2008): REM, RHT

Let S_k be time of k-th jump, Y a SRW on Σ_N. Then

$$S_k = \sum_{j=0}^{k-1} e_i \tau_{Y_i}, \qquad X(t) = Y(S^{-1}(t)).$$

Theorem

$\mathbb{P}$-a.s. under P^{RHT}

$$\frac{1}{t(N)} S(sr(N)) \xrightarrow{N\to\infty} V_\alpha(s)$$

where $\alpha \in (0,1)$ and

$$t(N) = e^{\alpha\beta^2 N}, \quad r(N) = Q(N) e^{\alpha^2\beta^2 N/2} \ll 2^N.$$

Scales choice.

$$\mathbb{P}\big[\tau_x \ge ut(N)\big] = \mathcal{P}\big[e^{\beta\sqrt{N} H_N(x)} \ge ut(N)\big] \sim \frac{1}{r(N)} u^{-\alpha}$$

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History of proving Bouchaud's heuristic (2)

RHT dynamics:

- $\mathbb{Z}^d$, τ_x i.i.d., $\mathbb{P}[\tau_x \geq u] \sim u^{-\alpha}$, RHT:
  - Ben Arous-Č-Mountford (2006),
  - Mourrat (2011),
  - Gayrard-Švejda (2013),
  - Fontes-Mathieu (2014)
- REM, RTH, new techniques: Gayrard (2010,2012)
- p-spin model, RHT: Ben Arous-Bovier-Č.
- $\mathbb{Z}^d$, τ_x coming from GFF, RHT: Louidor et al. 2015+

Non-RHT dynamics:

- $\mathbb{Z}^d$, Asymmetric Bouchaud's dynamics, τ_x i.i.d.:
  - Barlow-Č. (2011) $d \geq 3$,
  - Č. (2011) $d = 2$.
  - Gayrard-Švejda (2014)
- K_n, Asymmetric Bouchaud's dynamics: Gayrard (2010,2012)

Non-RHT dynamics

Recall

$$w_{xy}^M = 1 \wedge \frac{\tau_y}{\tau_x}, \qquad w_{xy}^a = \tau_x^{a-1}\tau_y^a, \qquad \text{if } x \sim y.$$

The rate depend on the target vertex. $\implies$ X is not a time change of the SRW.

A similar trick can be done: replace the SRW by a Markov chain with same *transition probabilities* as X but whose equilibrium measure is flat.

Let Y by the chain with transition rates

$$q_{xy}^M = \tau_x \wedge \tau_y, \qquad q_{xy}^a = \tau_x^a \tau_y^a, \qquad \text{if } x \sim y.$$

Define

$$S(t) = \int_0^t \tau_{Y_s} \mathrm{d}s.$$

Then

$$X(t) = Y(S^{-1}(t)).$$

But . . . Y depends on τ. It is a RWRC.

Ingredients of the proof

Goal: Show for some $t(N)$, $r(N) \to \infty$ that

$$\frac{1}{t(N)} S(sr(N)) \xrightarrow{N\to\infty} V_\alpha(s).$$

Step 1. Ignore "small" traps: There is a scale $\rho(N) \to \infty$ such that for

$$\mathfrak{S}(t) = \int_0^t \tau_{Y_s} \mathbf{1}\{\tau_{Y_S} \geq \rho(N)\} \, \mathrm{d}s$$

the processes S and $\mathfrak{S}$ are very close, $\frac{S(r(N))}{\mathfrak{S}(r(N))} \to 1$.

And then . . .: For $T_N = \{x : \tau_x \geq \rho_N\}$ we should know how Y visits T_N.

- $E_x[H_{T_N}]$ for a "typical" x
- $E_x[H_{T_N \setminus \{x\}}]$ for $x \in T_N$
- rescaled hitting times are asymptotically exponential
- E_x["time spent in x before escaping"].
- Approximate $\mathfrak{S}$ by an i.i.d. sequence, compute Laplace transform . . .

Difficulties in the REM

'Singularity' of the Metropolis dynamics:

Let $x \in T_N$

- typically all its neighbours are not in T_N
- Let y_1, y_2 be the sites with the first and second maximal energy over the neighbours of x.

$$H_N(y_1) \sim \sqrt{2 \log N}, \qquad H_N(y_1) - H_N(y_2) \sim 1/\sqrt{2 \log N}.$$

- Recall $q^M_{xy} = \tau_x \wedge \tau_y$. So

$$\frac{q^M_{xy_2}}{q^M_{xy_1}} = \frac{\tau_x \wedge \tau_{y_2}}{\tau_x \wedge \tau_{y_2}} = \frac{\tau_{y_2}}{\tau_{y_2}} = \exp\{\beta\sqrt{N}(H_N(y_2) - H_N(y_1))\} \xrightarrow{N\to\infty} 0.$$

- Bouchaud's asymmetric dynamics has the same property if $a > 0$.

Y is very different from the SRW.

Recent works on asymmetric dynamics

- **Mathieu-Mourrat (2015):** REM with the Asymmetric Bouchaud's dynamics, but with $a = a_N \leq c\sqrt{\log(N)/N} \to 0$.

$$\frac{q^a_{xy_2}}{q^a_{xy_1}} = \frac{\tau^a_{y_2}}{\tau^a_{y_2}} = \exp\{\beta a\sqrt{N}(H_N(y_2) - H_N(y_1))\}$$

remains non-negligible as $N \to \infty$

- **Gayrard (2014):** Truncated REM with the Metropolis dynamics. Replace $H_N(x)$ by $H_N(x)\mathbf{1}\{H_N(x) \geq u_N\}$

$$\mathbb{P}[H_N(x) \neq 0] \leq cN^{-3}$$

As consequence, typically, all neighbours of $x \in T_N$ have the same energy.

Y recovers certain features of the SRW and (non-trivial) extensions of usual techniques apply, that is S_N converges to a stable process.

Metropolis dynamics of the REM

Theorem (Č-Wassmer (2015))

Let $\alpha \in (0,1)$ *such that* $\frac{1}{2} < \frac{\alpha^2\beta^2}{2\log 2} < 1$. *Then,*

$$\frac{1}{t(N)} S(sR_N) \xrightarrow{N\to\infty} V_\alpha(s)$$

in P^M*-distribution, in* $\mathbb{P}$*-probability where*

- $t(N) = e^{\alpha\beta^2 N}$ *as before.*
- R_N *are random,* $\sigma(\tau_x : x \in \Sigma_N)$*-measurable. But, as before,*

$$\frac{1}{N}\log R_N \xrightarrow{N\to\infty} \frac{\alpha^2\beta^2}{2\log 2}.$$

- *The process* Y_N *should be modified slightly.*

The theorem confirms BTM universality class for the Metropolis of the REM, at the level of convergence of the clock.

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Modified process Y

Natural choice of Y. Recall $q_{xy}^M = \tau_x \wedge \tau_y$.

- $+$ has the same transition probabilities as X
- $+$ has uniform invariant measure
- $-$ is trapped on sites with $\tau_x << 1$
- $-$ its mixing time grows exponentially with N

Modified Y. Set

$$q_{xy}^M = \frac{\tau_x \wedge \tau_y}{1 \wedge \tau_x}, \qquad \pi_x = 1 \wedge \tau_x$$

- $+$ has the same transition probabilities as X
- $\pm$ its invariant measure π is uniform on sites with large energy
- $+$ its mixing time $T_{\mathrm{mix}} = o(N^5)$

$$S(t) = \int_0^t (1 \vee \tau_{Y_s}) \mathrm{d}s, \qquad X(t) = Y(S^{-1}(t)).$$

Ingredients of the proof

- **Exponentiality of hitting times.** Aldous-Brown (1992), $A \subset \Sigma$,

$$P_\pi\Big[\frac{H_A}{E_\pi H_A} \geq u\Big] = e^{-u} + O\Big(\frac{T_{\mathrm{mix}}}{E_\pi H_A}\Big)$$

- $\log E_\pi H_{T_N} \sim cN$
- For $x \in T_N$, $E_x[H_{T_N \setminus \{x\}}]$???
- Staying time in $x \in T_N$???

Ideas of the proof.

Laplace transform computation.

- **Lemma.** $\mathbb{P}$-a.s. there is no $x, y \in T_N$ such that $x \sim y$.
- As consequence, $q^M_{xy} = \frac{\tau_x \wedge \tau_y}{1 \wedge \tau_x}$ do not depend on $\tau_x, x \in T_N$.
- We may average over those first.

$$\mathbb{E}^{\mathcal{T}}\Big[\exp\Big\{-\lambda\frac{\mathfrak{S}(sR(N))}{t(N)}\Big\}\Big] \sim \exp\Big\{-C\lambda^\alpha h(N)\sum_{x\in T_N}\ell_{sR(N)}(x)^\alpha\Big\},$$

  where $\ell_t(x)$ is the local time of Y at time t at site x, and $h(N)$ is explicit.
- Prove concentration

$$h(N)\sum_{x\in T_N}\ell_{tR(N)}(x)^\alpha \xrightarrow{N\to\infty} s$$

Open questions

- Can R_N be made deterministic?
- Asymmetric Bouchaud's dynamics?
- Aging in terms of the usual two-point functions?
- Correlated spin glasses?

Thank you for your attention.