Polarity of points for systems of linear spde's in critical dimensions

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Based on joint work with:

Carl Mueller and Yimin Xiao

Overview

- Introduction to the problem of polarity of points
- Existing results for Gaussian and non-Gaussian random fields
- The "standard method" for non-critical dimensions
- Talagrand's idea for handling critical dimensions (fBM)
- Our results for a class of Gaussian processes
- Application to systems of linear stochastic heat and wave equations in critical dimensions

Polarity of points for random fields

Let $U = (U(x),\ x \in \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous stochastic process.

Fix $I \subset \mathbb{R}^k$, compact with positive Lebesgue measure.

The range of U over I is the random compact set

$$U(I) = \{U(x),\ x \in I\}.$$

Question. Fix $z \in \mathbb{R}^d$. Is z hit by U, that is,

$$P\{\exists x \in I : U(x) = z\} > 0\,?$$

Polarity. If $P\{\exists x \in I : U(x) = z\} = 0$, then z is polar for U.

Typically, there is a critical value $Q(k)$ such that:

- if $d < Q(k)$, then points are not polar.

- if $d > Q(k)$, then points are polar.

- at the critical valued $d = Q(k)$: ???

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First example: the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}^k_+)$ denote an k-parameter $\mathbb{R}^d$-valued Brownian sheet, that is, a centered continuous Gaussian random field

$$W(x) = (W_1(x), \dots, W_d(x))$$

with covariance

$$E[W_i(x)W_j(y)] = \delta_{i,j} \prod_{\ell=1}^{k} \min(x_\ell, y_\ell), \qquad i,j \in \{1, \dots, d\},$$

where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$.

The case $k = 1$: Brownian motion $B = (B(t),\ t \in \mathbb{R}_+)$.

The case $k > 1$: multi-parameter extension of Brownian motion.

A few references: Orey & Pruitt (1973), R. Adler (1978), W. Kendall (1980), J.B. Walsh (1986), D. & Walsh (1992), Khoshnevisan & Shi (1999)

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Hitting probabilities for the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}_+^k)$ denote a k-parameter $\mathbb{R}^d$-valued Brownian sheet.

Theorem 1 (Khoshnevisan and Shi, 1999)

Fix $M > 0$ and $0 < a_\ell < b_\ell < \infty$ $(\ell = 1, \dots, k)$. Let

$$I = [a_1, b_1] \times \cdots \times [a_k, b_k] \quad (\subset \mathbb{R}^k).$$

There exists $0 < C < \infty$ such that for all compact sets $A \subset B(0, M)$ $(\subset \mathbb{R}^d)$,

$$\frac{1}{C}\ Cap_{d-2k}(A) \leqslant P\{W(I) \cap A \neq \emptyset\} \leqslant C\ Cap_{d-2k}(A).$$

(see also F. Hirsch and S. Song (1991, 1995).

Example. $A = \{z\}$.

$$\mathrm{Cap}_{d-2k}(\{z\}) = \begin{cases} 1 & \text{if } d < 2k, \\ 0 & \text{if } d \geqslant 2k, \end{cases}$$

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Anisotropic Gaussian random fields (Biermé, Lacaux & Xiao, 2007)

Let $(V(x),\ x \in \mathbb{R}^k)$ be a centered continuous Gaussian random field with values in $\mathbb{R}^d$ with i.i.d. components: $V(x) = (V_1(x), \dots, V_d(x))$. Set

$$\Delta(x, y) = \|V_1(x) - V_1(y)\|_{L^2}$$

Let I be a "rectangle". Assume the two conditions:

(C1) There exists $0 < c < \infty$ and $H_1, \dots, H_k \in]0, 1[$ such that for all $x \in I$,

$$c^{-1} \leqslant \Delta(0, x) \leqslant c,$$

and for all $x, y \in I$,

$$c^{-1} \sum_{j=1}^{k} |x_j - y_j|^{H_j} \leqslant \Delta(x, y) \leqslant c \sum_{j=1}^{k} |x_j - y_j|^{H_j}$$

(H_j is the Hölder exponent for coordinate j).

(C2) There is $c > 0$ such that for all $x, y \in I$,

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Anisotropic Gaussian fields

Theorem 2 (Biermé, Lacaux & Xiao, 2007)

Fix $M > 0$. *Set*

$$Q = \sum_{j=1}^{k} \frac{1}{H_j}.$$

Then there is $0 < C < \infty$ *such that for every compact set* $A \subset B(0, M)$,

$$C^{-1}\ Cap_{d-Q}(A) \leqslant P\{V(I) \cap A \neq \emptyset\} \leqslant C\mathcal{H}_{d-Q}(A).$$

Example. $A = \{z\}$

$$\mathrm{Cap}_{d-Q}(\{z\}) = \begin{cases} 1 & \text{if } d < Q, \\ 0 & \text{if } d = Q, \\ 0 & \text{if } d > Q, \end{cases} \qquad \mathcal{H}_{d-Q}(\{z\}) = \begin{cases} \infty & \text{if } d < Q, \\ 1 & \text{if } d = Q, \\ 0 & \text{if } d > Q. \end{cases}$$

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Funaki's random string

Let $(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ be an $\mathbb{R}^d$-valued random field such that

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \dot{W}(t,x), \qquad x \in \mathbb{R},\ t > 0,$$

$u(0,\cdot) : \mathbb{R} \to \mathbb{R}^d$ given, $\dot{W}(t,x)$ is space-time white noise.

Theorem 3 (Mueller & Tribe, 2002)

The critical dimension for hitting points is $d = 6$ and points are polar in this dimension.

Their proof uses the "stationary pinned string," then scaling and time reversal (method of Paul Lévy).

It does not apply to the wave equation, nor to heat equation with deterministic non-constant coefficients, such as

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(t,x)\dot{W}(t,x),$$

where $(t,x) \mapsto \sigma(t,x)$ is deterministic but not constant.

(They also treat the issue of double points for this random field)

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Systems $1d$ nonlinear wave equations

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The critical dimension for hitting points is $d = 4$ and points are polar in this dimension.

The proof uses Malliavin calculus and Cairoli's maximal inequality for multi-parameter martingales.

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Other nonlinear systems of spde's

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When $k = 1$, $\dot{W}$ can be space-time white noise: $E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\,\delta(x,y)$
When $k > 1$, $\dot{W}$ is spatially homogeneous: $E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\,\|x-y\|^{-\beta}$

Theorem 5 (D., Khoshnevisan & Nualart, 2007, 2013)

Fix $\eta > 0$. Then

$$c_\eta \, Cap_{d-Q+\eta}(A) \leqslant P\{u(I \times J) \cap A \neq \emptyset\} \leqslant C_\eta \mathcal{H}_{d-Q-\eta}(A)$$

Remarks. (a) This is similar to the result of Biermé, Lacaux and Xiao (2007).
(b) In the critical dimension $d = Q$ $(= \frac{4+2k}{2-\beta})$, this is not informative!
(c) For wave equations ($k \in \{1,2,3\}$): see D. & Sanz-Solé, Memoirs AMS 2015, lower bound is less sharp.

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Polarity of points in dimensions > the critical dimension

Case $k=1$, $d \geqslant 3$: let $(W(t), t \in \mathbb{R}_+)$ be a standard Brownian motion with values in $\mathbb{R}^3$. Want to explain why it does not hit points.

Explanation. A.a. points are polar $\longleftrightarrow \lambda_d(W([1,2])) = 0$ (Fubini)

Let $t_k = 1 + k2^{-2n}$. Then

$$W([1,2]) \subset \bigcup_{k=1}^{2^{2n}} B(W(t_k), \sup_{|t-t_k| \leqslant 2^{-2n}} |W(t) - W(t_k)|)$$

so

$$\begin{aligned}
\lambda_d(W([1,2])) &\leqslant \sum_{k=1}^{2^{2n}} \lambda_d(B(W(t_k), \sup_{|t-t_k| \leqslant 2^{-2n}} |W(t) - W(t_k)|)) \\
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\end{aligned}$$

We covered $W([1,2])$ using a uniform partition of the parameter space $[1,2]$.

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A non-uniform partition

Let $X = \{X(t),\ t \in \mathbb{R}^k\}$, be an $\mathbb{R}^d$-valued fractional Brownian motion:
$E[|X(t) - X(s)|^2] = d \cdot |t-s|^{2\alpha}$, where $0 < \alpha < 1$.

Theorem 6 (Talagrand, 1998)

There are constants $\delta > 0$ and $K < \infty$ with the following property: Given $r_0 \leqslant \delta$ and $t_0 \in \mathbb{R}^k$, we have

$$P\left\{\exists r \in [r_0^2, r_0] : \sup_{t:\,|t-t_0|\leqslant r} |X(t) - X(t_0)| \leqslant K \frac{r^\alpha}{\left[\log\log \frac{1}{r}\right]^{\frac{\alpha}{k}}}\right\} \geqslant 1 - \exp\left[-\left[\log \frac{1}{r_0}\right]^{\frac{1}{2}}\right]$$

Interpretation. It is quite likely that there will be an $r > 0$ such that increments of X in the ball centered at t_0 of radius r are smaller than is typical.

Utilization. Many points t_0 will have this property, so if $d = Q(k)$ $(= k/\alpha)$, then he can use a non-uniform partition and smaller balls to create a covering $\rightsquigarrow$ the d-dimensional Hausdorff measure of the range of $t \mapsto X(t)$ is 0.

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$$P\left\{\exists r \in [r_0^2, r_0] : \sup_{t:\,|t-t_0|\leqslant r} |X(t) - X(t_0)| \leqslant K \frac{r^\alpha}{\left[\log\log \frac{1}{r}\right]^{\frac{\alpha}{k}}}\right\} \geqslant 1 - \exp\left[-\left[\log \frac{1}{r_0}\right]^{\frac{1}{2}}\right]$$

Interpretation. It is quite likely that there will be an $r > 0$ such that increments of X in the ball centered at t_0 of radius r are smaller than is typical.

Utilization. Many points t_0 will have this property, so if $d = Q(k)$ $(= k/\alpha)$, then he can use a non-uniform partition and smaller balls to create a covering $\rightsquigarrow$ the d-dimensional Hausdorff measure of the range of $t \mapsto X(t)$ is 0.

A non-uniform partition

Let $X = \{X(t),\ t \in \mathbb{R}^k\}$, be an $\mathbb{R}^d$-valued fractional Brownian motion: $E[|X(t) - X(s)|^2] = d \cdot |t - s|^{2\alpha}$, where $0 < \alpha < 1$.

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Harmonizable representation of fBM

Talagrand makes essential use of:

Fact. Let $\dot W_1$ and $\dot W_2$ be independent white noises on $\mathbb{R}^k$. Then

$$Y(t) = \int_{\mathbb{R}^k} \frac{1-\cos\langle t,\xi\rangle}{|\xi|^{\alpha+\frac{k}{2}}}\,\dot W_1(d\xi) + \int_{\mathbb{R}^k} \frac{\sin\langle t,\xi\rangle}{|\xi|^{\alpha+\frac{k}{2}}}\,\dot W_2(d\xi)$$

is an fBM. (The ξ plays the role of a frequency.)

Another representation (that looks more like a solution of an spde), such as:

$$Y(t) := \int_{\mathbb{R}^k} (|t-x|^{\alpha-\frac{k}{2}} - |x|^{\alpha-\frac{k}{2}})\,\dot W(dx)$$

Passing from one to the other: set $f_t(x) := |t-x|^{\alpha-\frac{k}{2}} - |x|^{\alpha-\frac{k}{2}}$, so

$$Y(t) = \langle \dot W, f_t\rangle = \langle \mathcal{F}\dot W, \mathcal{F}f_t^{\vee}\rangle \overset{law}{=} \langle \dot W, \mathcal{F}f_t^{\vee}\rangle$$

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Use of the harmonizable representation

It has the form

$$Y(t) = \int_{\mathbb{R}^k} F_t(\xi) \dot{W}(d\xi),$$

where $\xi \mapsto F_t(\xi)$ has a specified decay as $\xi \to \infty$.
Define a white noise:

$$A \mapsto Y(A, t) := \int_A F_t(\xi) \dot{W}(d\xi)$$

When $F_t(\xi)$ is smooth and has appropriate decay as $\xi \to \infty$, it can happen that

$$|t - s| \sim 2^{-n/\beta} \Rightarrow Y(t) - Y(s) \sim Y([2^n, 2^{n+1}[, t) - Y([2^n, 2^{n+1}[, s)$$

"most of the increment of Y over an interval of length $2^{-n/\beta}$ comes from the contribution of $Y([2^n, 2^{n+1}[, t)$.

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Extension to a wide class of anisotropic Gaussian processes

Suppose that

$$\|v(x) - v(y)\|_{L^2} \leqslant C \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j} =: \Delta(x, y)$$

(the α_j bound the Hölder-exponents of $x \mapsto v(x)$)
$+$ Additional Assumptions (that include a kind of harmonizable representation).

Proposition (D., Mueller & Xiao)

Let

$$Q = \sum_{j=1}^{k} \frac{1}{\alpha_j}.$$

Under the above assumptions,

$$P\left\{\exists r \in [r_0^2, r_0]: \sup_{y:\Delta(y,x_0)<r} |v(y) - v(x_0)| \leqslant \tilde{K} \frac{r}{(\log\log\frac{1}{r})^{1/Q}}\right\} \geqslant 1 - \exp\left[-\left[\log\frac{1}{r_0}\right]^{\frac{1}{2}}\right].$$

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Main abstract result

Let $v = (v(x),\ x \in \mathbb{R}^k)$ be a centered continuous $\mathbb{R}^d$-valued Gaussian random field with i.i.d. components: $v(x) = (v_1(x), \dots, v_d(x))$.

Suppose in particular that

$$c \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j} \leqslant \|v(x) - v(y)\|_{L^2} \leqslant C \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j}$$

+ Additional Assumptions.
Recall that the critical dimension is:

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Theorem 1 (D., Mueller & Xiao)

Assume that $Q = d$. Then for any closed box J and for all $z \in \mathbb{R}^Q$,

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Linear systems of stochastic p.d.e.'s

Heat equations, spatial dimension 1, space-time white noise

Let $\hat{v} = (\hat{v}(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R})$ solve

$$\begin{cases} \frac{\partial}{\partial t}\hat{v}_j(t,x) = \frac{\partial^2}{\partial x^2}\hat{v}_j(t,x) + \dot{\hat{W}}_j(t,x), & j = 1,\dots,d, \\ v(0,x) = 0, & x \in \mathbb{R}^k. \end{cases} \tag{1}$$

Here, $\hat{v}(t,x) = (\hat{v}_1(t,x),\dots,\hat{v}_d(t,x))$

Corollary 1

Suppose $d = 6$ (critical dimension). Then points are polar for $\hat{v}$.

Proof. Check the Assumptions, using the harmonizable representation

$$v(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\xi\cdot x}\,\frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau}\,W(d\tau,d\xi).$$

(This representation also appears in R. Balan, 2012). Method also works with smooth deterministic non-constant coefficients.

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Explanation

Let $G(s,y) = (4\pi t)^{-1/2} \exp[-y^2/(4t)]$, so that

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Then

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Checking the Assumptions

For $A \subset \mathbb{R}$, set

$$v(A,t,x) := \iint_{\max(|\tau|^{\frac{1}{4}},\,|\xi|^{\frac{1}{2}})\in A} e^{-i\xi x}\,\frac{e^{-i\tau t}-e^{-t\xi^2}}{\xi^2-i\tau}\,W(d\tau,d\xi),$$

Need to check:

$$\|v([0,a[,t,x)-v([0,a[,s,y)\|_{L^2} \leqslant c_0\left[a^3|t-s|+a|x-y|\right]$$

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Proving the inequality requires estimating double integrals.

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Linear systems of stochastic heat equations with nonconstant coefficients

Let $\hat{v} = (\hat{v}(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^k)$ solve

$$\begin{cases} \frac{\partial}{\partial t}\hat{v}_j(t,x) = \Delta \hat{v}_j(t,x) + \sigma_j(t,x)\dot{\hat{W}}_j(t,x), & j = 1,\dots,d, \\ v(0,x) = 0, & x \in \mathbb{R}^k. \end{cases}$$

The harmonizable representation is:

$$v_j(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} W_j(d\tau, d\xi)(\mathcal{F}_{s,y}\tilde{G}_{t,x} * \mathcal{F}_{s,y}\sigma_j)(\tau,\xi).$$

Assumption. $\mathcal{F}_{s,y}\sigma_j$ is a measure μ_j with compact support (i.e. σ is much smoother than the noise).

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Linear systems of stochastic p.d.e.'s (2)

Heat equations, spatial dimension $k \geqslant 1$, spatially homogeneous noise:
$E[\dot W(t,x)\dot W(s,y)] = \delta(t-s)\,\|x-y\|^{-\beta}$ $(0<\beta<2\wedge k)$

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Corollary 2

Suppose $d = \frac{4+2k}{2-\beta}$ (critical dimension). Then points are polar for $\hat v$.

Proof. Check Assumptions, using the harmonizable representation

$$v(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}^k} e^{-i\xi\cdot x}\,\frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau}\,|\xi|^{(\beta-k)/2}\,W(d\tau,d\xi)$$

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Linear systems of stochastic p.d.e.'s (3)

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Corollary 3

Suppose $d = 4$ (critical dimension). Then points are polar for $\hat{v}$.

Proof. Check Assumptions, using the harmonizable representation

$$v(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i\xi\cdot x - i\tau t}}{2|\xi|} \left[\frac{1 - e^{it(\tau+|\xi|)}}{\tau + |\xi|} - \frac{1 - e^{it(\tau-|\xi|)}}{\tau - |\xi|} \right] W(d\tau, d\xi).$$

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Suppose $d = 4$ (critical dimension). Then points are polar for $\hat{v}$.

Proof. Check Assumptions, using the harmonizable representation

$$v(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left[\frac{1 - e^{it(\tau + |\xi|)}}{\tau + |\xi|} - \frac{1 - e^{it(\tau - |\xi|)}}{\tau - |\xi|} \right] W(d\tau, d\xi).$$

(This representation also appears in R. Balan, 2012).

Linear systems of stochastic p.d.e.'s (4)

Wave equations, spatial dimension $k \geqslant 1$, spatially homogeneous noise
$E[\dot W(t,x)\dot W(s,y)] = \delta(t-s)\,\|x-y\|^{-\beta}$ $(0 < \beta < 2)$
Let $\hat v = (\hat v(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R})$ solve

$$\begin{cases} \frac{\partial^2}{\partial t^2}\hat v_j(t,x) = \Delta \hat v_j(t,x) + \dot{\hat W}_j(t,x), & j = 1,\dots,d, \\ \hat v(0,x) = 0, \quad \frac{\partial}{\partial t}\hat v(0,x) = 0, & x \in \mathbb{R}^k. \end{cases}$$

Corollary 4

Suppose $1 < \beta < k \wedge 2$, and $d = \frac{2(k+1)}{2-\beta}$ (critical dimension). Then points are polar for $\hat v$,

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Let $\hat{v} = (\hat{v}(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R})$ solve

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Reference and ongoing work

References.

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Ongoing:

Multiple points in critical dimensions (linear systems of spde's).

Polarity of points in critical dimensions for nonlinear systems of spde's.

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