Paracontrolled KPZ equation

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Motivation: modelling of interface growth

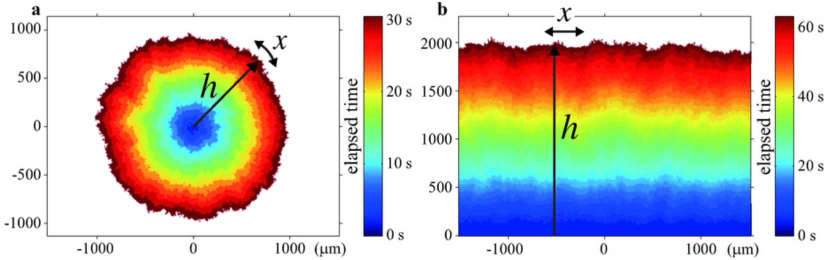

Figure: Takeuchi, Sano, Sasamoto, Spohn (2011, Sci. Rep.)

KPZ equation

- KPZ equation is a model for random interface growth: $h \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\partial_t h(t,x) = \underbrace{\kappa \Delta h(t,x)}_{\text{diffusion}} + \underbrace{\lambda |\partial_x h(t,x)|^2}_{\text{slope-dependence}} + \underbrace{\xi(t,x)}_{\text{space-time white noise}}$$

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- Kardar-Parisi-Zhang (1986): slope-dependent growth $F(\partial_x h)$;

$$F(\partial_x h) = F(\bar{h}) + F'(\bar{h})(\partial_x h - \bar{h}) + \frac{1}{2} F''(\bar{h})(\partial_x h - \bar{h})^2 + \dots$$

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Highly non-rigorous since $\partial_x h$ is a distribution. But: Hairer-Quastel (2015, unpublished) justify it via scaling.

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  Highly non-rigorous since $\partial_x h$ is a distribution. But: Hairer-Quastel (2015, unpublished) justify it via scaling.
- Fluctuations of $\varepsilon^{1/3} h(t\varepsilon^{-1}, x\varepsilon^{-2/3})$ should converge to KPZ fixed point. Only known for one-point distribution, special h_0 (Amir et al. (2011), Sasamoto-Spohn (2010), Borodin et al. (2014)).

Weak KPZ universality conjecture

$$\partial_t h = \Delta h + |\partial_x h|^2 + \xi.$$

- KPZ equation for $t \to \infty$ in KPZ universality class. For $t \to 0$ Gaussian (Edwards-Wilkinson class of symmetric "growth" models).

Weak KPZ universality conjecture

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- KPZ equation for $t \to \infty$ in KPZ universality class. For $t \to 0$ Gaussian (Edwards-Wilkinson class of symmetric "growth" models).
- Weak KPZ universality conjecture: KPZ equation is only growth model interpolating EW and KPZ. Mathematically: fluctuations of weakly asymmetrical models converge to KPZ.
- Example: Ginzburg-Landau $\nabla\varphi$ model

$$\mathrm{d}x^j = \big(pV'(r^{j+1}) - qV'(r^j)\big)\,\mathrm{d}t + \mathrm{d}w^j; \quad r^j = x^j - x^{j-1};$$

For $p = q$ convergence to $\partial_t \psi = \alpha\Delta\psi + \beta\xi$. For $p - q = \sqrt{\varepsilon}$ convergence to KPZ Diehl-Gubinelli-P. (2015, in preparation).

How to interpret KPZ?

$$\mathcal{L}h(t,x) = (\partial_t - \Delta)h(t,x) = |\partial_x h(t,x)|^2 + \xi(t,x).$$

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- Hairer (2013): series expansion and rough paths/regularity structures, defines $|\partial_x h(t,x)|^2$. So far on circle ($h\colon \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$), but certainly soon extended to $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

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- Martingale problem: Assing (2002), Gonçalves-Jara (2014), Gubinelli-Jara (2013) define "energy solutions" of equilibrium KPZ. Uniqueness long open, solved in Gubinelli-P. (2015).

Solution concepts and weak KPZ universality

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- **Pathwise approach:** needs precise control of regularity, so far only semilinear S(P)DEs: Hairer-Quastel (2015), Hairer-Shen (2015), Gubinelli-P. (2015).
- **Martingale problem:** powerful for universality of equilibrium fluctuations Gonçalves-Jara (2014), Gonçalves-Jara-Sethuraman (2015), Diehl-Gubinelli-P. (2015, in preparation). Before only tightness and martingale characterization of limits. Now: uniqueness proves convergence.

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- New stochastic optimal control formulation of the KPZ equation.
- Uniqueness of equilibrium KPZ martingale problem.

1. Paracontrolled formulation of the equation

2. KPZ as HJB equation

3. Uniqueness of the martingale solution

Formal expansion of the KPZ equation

$$\mathcal{L}h(t,x) = (\partial_t - \Delta)h(t,x) = |\partial_x h(t,x)|^2 - \infty + \xi(t,x),$$

- Perturbative expansion around linear solution: $h = Y + h^{\geq 1}$ with $Y \in C^{1/2-}$,

$$\mathcal{L}Y = \xi,$$

thus

$$\mathcal{L}h^{\geq 1} = \underbrace{|\partial_x Y|^2 - \infty}_{C^{-1-} = B^{-1-}_{\infty,\infty}} + 2\underbrace{\partial_x Y \partial_x h^{\geq 1}}_{C^{-1/2-}} + \underbrace{|\partial_x h^{\geq 1}|^2}_{C^{0-}}.$$

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- Continue expansion: set $\mathcal{L}Y^{\vee} = |\partial_x Y|^2 - \infty$ and then $\mathcal{L}Y^{\vee\!\!\!\vee} = \partial_x Y^{\vee} \partial_x Y$ and in general $\mathcal{L}Y^{(\tau_1 \tau_2)} = \partial_x Y^{\tau_1} \partial_x Y^{\tau_2}$. Formally:

  $$h = \sum_\tau c(\tau) Y^\tau.$$

Seems very difficult to make this rigorous.

Truncated expansion

Following Hairer (2013), truncate expansion and set

$$h = Y + Y^{\vee} + 2Y^{\text{\vee}} + h^P,$$

where h^P is paracontrolled by P with $\mathcal{L}P = \partial_x Y$, write

$$h^P = \underbrace{h' \prec P}_{C^{3/2-}} + \underbrace{h^\sharp}_{C^{2-}},$$

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where for $\Delta_k \equiv k$-th Littlewood-Paley block:

$$h' \prec P = \sum_{i<j-1} \Delta_i h' \Delta_j P.$$

Intuitively: h^P is frequency modulation of P plus smoother remainder; more intuitively: on small scales h^P "looks like" P.

Paracontrolled differential equation

Paracontrolled ansatz: $h \in \mathcal{D}_{\mathrm{rbe}}$ if $h = Y + Y^{\vee} + 2Y^{\vee\!\!\cdot} + h^P$ with $h^P = h' \prec P + h^\sharp$.

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Theorem (Gubinelli, Imkeller, P. (2015))

For paracontrolled $h \in \mathcal{D}_{\mathrm{rbe}}$ the square $|\partial_x h|^2 - \infty$ is well defined, depends continuously on h and $(Y, Y^{\vee}, Y^{\vee\!\vee}, Y^{\vee\!\vee\!\vee}, Y^{\vee\vee}, \partial_x P \partial_x Y)$, and we have

$$|\partial_x h|^2 - \infty = \lim_{\varepsilon \to 0} (|\partial_x(\delta_\varepsilon * h)|^2 - c_\varepsilon).$$

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Theorem (Gubinelli, P. (2015))

Local-in-time existence and uniqueness of paracontrolled solutions. Solution depends locally Lipschitz continuously on extended data $(Y, Y^{\vee}, Y^{\vee\!\vee}, Y^{\vee\!\vee\!\vee}, Y^{\vee\vee}, \partial_x P \partial_x Y)$. Agrees with Hairer's solution.

Formal derivation

- Cole-Hopf: $h = \log w$, where

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- Feynman-Kac:

$$w(t,x) = \mathbb{E}_x\Big[\exp\Big(h_0(B_t) + \int_0^t (\xi(t-s, B_s) - \infty)\mathrm{d}s\Big)\Big].$$

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- Boué-Dupis (1998):

$$\log \mathbb{E}[e^{F(B)}] = \sup_v \mathbb{E}\Big[F(B + \int_0^\cdot v_s \mathrm{d}s) - \frac{1}{4}\int_0^t v_s^2 \mathrm{d}s\Big].$$

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- Thus (see also E-Khanin-Mazel-Sinai (2000)):

$$h(t,x) = \sup_v \mathbb{E}_x\Big[h_0(\gamma_t^v) + \int_0^t (\xi(t-s,\gamma_s^v) - \infty)\mathrm{d}s - \frac{1}{4}\int_0^t v_s^2 \mathrm{d}s\Big],$$

where $\gamma_s^v = x + B_s + \int_0^s v_r \mathrm{d}r$. (But of course nothing was rigorous!)

Let's make it rigorous

Regularize ξ:

$$\mathcal{L}h_\varepsilon = |\partial_x h_\varepsilon|^2 - c_\varepsilon + \xi_\varepsilon.$$

Then

$$h_\varepsilon(t,x) = \sup_v \mathbb{E}_x\Big[h_0(\gamma_t^v) + \int_0^t (\xi_\varepsilon(t-s,\gamma_s^v) - c_\varepsilon)\mathrm{d}s - \frac{1}{4}\int_0^t v_s^2 \mathrm{d}s\Big].$$

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Fix singular part of optimal control:

$$\mathrm{d}\zeta_s^v = 2\partial_x(Y_\varepsilon + Y_\varepsilon^{\mathbf{V}})(t-s,\zeta_s^v)\mathrm{d}s + v_s\mathrm{d}s + \mathrm{d}B_s,$$

Then Itô gives

$$\begin{aligned} h_\varepsilon(t,x) = {} & (Y_\varepsilon + Y_\varepsilon^{\mathbf{V}} + Y_\varepsilon^R)(t,x) \\ & + \sup_v \mathbb{E}_x\Big[h_0(\zeta_t^v) + \int_0^t \Big(\partial_x Y_\varepsilon^R(t-s,\zeta_s^v)v_s - \frac{1}{4}|v_s|^2\Big)\mathrm{d}s\Big], \end{aligned}$$

where Y_ε^R solves a linear paracontrolled equation.

Singular control problem

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- Result independent of Cole-Hopf, only used to abbreviate derivation.

Burgers generator

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- Invariant measure $\mu = \mathrm{law}$(white noise).
- Formally: generator $L_0 + B$, L_0 symmetric in $L^2(\mu)$ and B antisymmetric. L_0 is generator of OU process $\partial_t \psi = \Delta \psi + \partial_x \xi$.
- So for $u(0) \sim \mu$, backward process $\hat{u}(t) = u(T - t)$ should solve

$$\partial_t \hat{u} = \Delta \hat{u} - \partial_x \hat{u}^2 + \partial_x \hat{\xi}$$

for new white noise $\hat{\xi}$. Difficult to make rigorous.

Gubinelli-Jara controlled processes

Gubinelli-Jara (2013): u is called controlled by the OU process if

1. $u_t \sim \mu$ for all t;
2. for all $\varphi \in \mathcal{S}$

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\Delta\varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) + M_t(\varphi),$$

   $M(\varphi)$ martingale with $\langle M(\varphi)\rangle_t = 2t||\partial_x\varphi||_{L^2}$ and $\langle \mathcal{A}(\varphi)\rangle \equiv 0$;
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Define $\int_0^T \partial_x u_s^2 \mathrm{d}s$ via martingale trick:

$$F(u_T) = F(u_0) + \int_0^T L_0 F(u_s)\mathrm{d}s + \int_0^T \mathrm{D}F(u_s)\mathrm{d}\mathcal{A}_s + M_T^F,$$

$$F(\hat{u}_T) = F(\hat{u}_0) + \int_0^T L_0 F(u_s)\mathrm{d}s + \int_0^T \mathrm{D}F(\hat{u}_s)\mathrm{d}\hat{\mathcal{A}}_s + \hat{M}_T^F,$$

so $2\int_0^T L_0 F(u_s)\mathrm{d}s = -M_T^F - \hat{M}_T^F$.

Uniqueness of energy solutions I

Call controlled u energy solution if $\mathcal{A} = \int_0^\cdot \partial_x u_s^2 \mathrm{d}s$. Gubinelli-Jara (2013): existence.

- Uniqueness difficult because energy formulation gives little control.
- Easy: uniqueness of paracontrolled energy solutions.

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Then we read Funaki-Quastel (2014) who study invariant measure for KPZ via Sasamoto-Spohn discretization:

- Mollify discrete model to safely pass to continuous limit

$$\partial_t h^\varepsilon = \Delta h^\varepsilon + \delta_\varepsilon * \delta_\varepsilon * (\partial_x h^\varepsilon - c_\varepsilon)^2 + \delta_\varepsilon * \xi;$$

- Cole-Hopf: $w^\varepsilon = e^{h^\varepsilon}$ solves

$$\partial_t w^\varepsilon = \Delta w^\varepsilon + w^\varepsilon \left(\delta_\varepsilon * \delta_\varepsilon * \left(\frac{\partial_x w^\varepsilon}{w^\varepsilon} \right)^2 - \left(\frac{\partial_x w^\varepsilon}{w^\varepsilon} \right)^2 \right) + w^\varepsilon (\delta_\varepsilon * \xi).$$

- Use Boltzmann-Gibbs principle to show convergence of nonlinearity.

Uniqueness of energy solutions II

Implement Funaki-Quastel strategy for energy solutions:

- $u^\varepsilon = \delta_\varepsilon * u$. Itô: $w^\varepsilon = e^{\partial_x^{-1} u^\varepsilon}$ solves

$$\mathrm{d}w_t^\varepsilon = \Delta w^\varepsilon \mathrm{d}t + w_t^\varepsilon \partial_x^{-1}(\mathrm{d}M_t^\varepsilon + \mathrm{d}\mathcal{A}_t^\varepsilon) - w_t^\varepsilon (u_t^\varepsilon)^2 \mathrm{d}t + w_t^\varepsilon c_\varepsilon \mathrm{d}t.$$

- $\partial_x^{-1}\partial_t M_t^\varepsilon \longrightarrow \xi$. If rest converges to $c \in \mathbb{R}$, then $\partial_t w = \Delta w + w(\xi + c)$. Since $\partial_x \log w^{c_1} = \partial_x \log w^{c_2}$, u is unique.

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Implement Funaki-Quastel strategy for energy solutions:

- $u^\varepsilon = \delta_\varepsilon * u$. Itô: $w^\varepsilon = e^{\partial_x^{-1} u^\varepsilon}$ solves

$$\mathrm{d}w_t^\varepsilon = \Delta w^\varepsilon \mathrm{d}t + w_t^\varepsilon \partial_x^{-1}(\mathrm{d}M_t^\varepsilon + \mathrm{d}\mathcal{A}_t^\varepsilon) - w_t^\varepsilon (u_t^\varepsilon)^2 \mathrm{d}t + w_t^\varepsilon c_\varepsilon \mathrm{d}t.$$

- $\partial_x^{-1}\partial_t M_t^\varepsilon \longrightarrow \xi$. If rest converges to $c \in \mathbb{R}$, then $\partial_t w = \Delta w + w(\xi + c)$. Since $\partial_x \log w^{c_1} = \partial_x \log w^{c_2}$, u is unique.
- Remains to study $(\mathrm{d}\partial_x^{-1}\mathcal{A}_t^\varepsilon - (u_t^\varepsilon)^2 \mathrm{d}t + c_\varepsilon \mathrm{d}t)$.

Uniqueness of energy solutions III

Convergence of $(\mathrm{d}\partial_x^{-1}\mathcal{A}_t^\varepsilon - (u_t^\varepsilon)^2\mathrm{d}t + c_\varepsilon \mathrm{d}t)$:

- $\mathcal{A}^\varepsilon = \delta_\varepsilon * \mathcal{A} = \int_0^\cdot \delta_\varepsilon * \partial_x u_s^2 \mathrm{d}s$, so

$$\begin{aligned}(\mathrm{d}\partial_x^{-1}\mathcal{A}_t^\varepsilon &- (u_t^\varepsilon)^2\mathrm{d}t + c_\varepsilon \mathrm{d}t) \\ &= \Pi_0(\delta_\varepsilon * (u_t^2) - (\delta_\varepsilon * u_t)^2)\mathrm{d}t \\ &\quad + (c_\varepsilon - \int_{\mathbb{T}} (\delta_\varepsilon * u_t)^2 \mathrm{d}x)\mathrm{d}t\end{aligned}$$

  where $\Pi_0\varphi = \varphi - \int_{\mathbb{T}} \varphi \mathrm{d}x$.

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  where $\Pi_0\varphi = \varphi - \int_{\mathbb{T}} \varphi \mathrm{d}x$.

- Remains to control integrals like $\int_0^T F(u_s)\mathrm{d}s$. Kipnis-Varadhan extends to controlled processes, so

$$\mathbb{E}\Big[\Big|\sup_{t\le T}\int_0^t F(u_s)\mathrm{d}s\Big|^2\Big] \lesssim \sup_G\{2\mathbb{E}[F(u_0)G(u_0)] - \mathbb{E}[G(u_0)(-L_0 G)(u_0)]\},$$

  where L_0 is OU generator, $u_0 \sim$ white noise.

Uniqueness of energy solutions IV

- Control $\sup_G \{2\mathbb{E}[F(u_0)G(u_0)] - \mathbb{E}[G(u_0)(-L_0 G)(u_0)]\}$, where L_0 is OU generator, $u_0 \sim$ white noise.
- For us: F in second chaos of white noise. Use Gaussian IBP to reduce to deterministic integral over explicit kernel.

Uniqueness of energy solutions IV

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Theorem (Gubinelli, P. (2015))

There exists a unique controlled process u which is an energy solution to Burgers equation.

Thank you